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Basic Structure of Algebraic Varieties

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The aims of this paper are twofold. In part 1, we shall give definitions of some basic concepts in the theory of Kodaira dimension, which will be referred to in the papers of Kawamata and Viehweg in this volume. In part 2, using some results in part 1, we shall develop a theory of pairs of reduced divisors and normal varieties, that is a new development of the theory of Cremona transformations from the viewpoint of Kodaira dimension.

In what follows, we fix the base field k, which is algebraically closed and is of characteristic zero. Occasionally, k is assumed to be the field of complex numbers. Varieties mean irreducible reduced separated schemes algebraic over k.

Part 1. Kodaira dimension of Varieties

§ 1.

In order to introduce the notion of Kodaira dimension, we begin by proving rather elementary facts. For systematic description of these results, we refer to [3] or [4].

Proposition 1. *Let W, V be normal varieties and E a locally principal effective divisor on W. Let $f: W \to V$ be a proper surjective morphism with connected fibers. If $\mathrm{codim}_V f(E) \geq 2$ and $\mathscr{L}$ is a locally free $\mathscr{O}_V$-module of finite rank, then*

$$f_*(f^*\mathscr{L} \otimes \mathscr{O}(E)) \cong \mathscr{L}.$$

In particular,

$$H^0(V, \mathscr{L}) \cong H^0(W, f^*\mathscr{L} \otimes \mathscr{O}(E)).$$

Here, $\mathscr{O}(E)$ denotes the invertible sheaf associated with E.

Proof. By projection formula, it suffices to prove that $f_*\mathscr{O}(E) \cong \mathscr{O}_V$.

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Let $F=f(E)$, $Z=W-f^{-1}(F)$ and $g=f|_Z$. Then $g_*\mathcal{O}_Z=\mathcal{O}_{V-F}$ (see [3, Chap 2]). Denoting the natural inclusions $Z\subset W$ and $V-F\subset V$ by j and i, respectively, we have $i_*\mathcal{O}_{V-F}=i_*g_*\mathcal{O}_Z=f_*j_*\mathcal{O}_Z$. Since $j_*\mathcal{O}_Z\supseteq\mathcal{O}_W(E)$, it follows that $i_*\mathcal{O}_{V-F}=f_*j_*\mathcal{O}_Z\supseteq f_*\mathcal{O}_W(E)$. By hypothesis, codim $(F)\geq 2$ and V is normal. Hence $i_*\mathcal{O}_{V-F}=\mathcal{O}_V$. Thus $f_*\mathcal{O}_W(E)\subseteq\mathcal{O}_V$. Clearly, $f_*\mathcal{O}_W(E)\supseteq\mathcal{O}_V$ and so $f_*\mathcal{O}_W(E)=\mathcal{O}_V$. Q.E.D.

Now, assume V to be complete and $\mathcal{L}$ to be an invertible sheaf. If $H^0(V, \mathcal{L})\neq 0$, then taking a base $\varphi_0, \cdots, \varphi_l$ of the vector space, we define a rational map $[\varphi_0: \cdots : \varphi_l] : V\to \boldsymbol{P}^l$, which is denoted by $\Phi_{\mathcal{L}}$. In general, let $N(\mathcal{L})=\{m \in N \mid H^0(V, \mathcal{L}^{\otimes m})\neq 0\}$. If $N(\mathcal{L})\neq\phi$, then the image of V by the rational map associated to $\mathcal{L}^{\otimes m}$, where $m\in N(\mathcal{L})$, is denoted by W_m.

Definition. *If $N(\mathcal{L})\neq\phi$, then the $\mathcal{L}$-dimension of V is defined to be* $\max\{\dim W_m \mid m\in N(\mathcal{L})\}$, *which is denoted by $\kappa(\mathcal{L}, V)$ or $\kappa(V, \mathcal{L})$. Further, if $N(\mathcal{L})=\phi$, put $\kappa(\mathcal{L}, V)=-\infty$.*

§ 2.

If V is a non-singular complete variety of dimension n, the sheaf of germs of regular n-forms is an invertible sheaf, denoted by ω_V or Ω^n_V.

Definition. $\kappa(\omega_V, V)$ is called the *Kodaira dimension* of V, which is indicated by $\kappa(V)$.

Following the traditional convention, we introduce the plurigenera of V by putting $P_m(V)=\dim_k H^0(V, \omega_V^{\otimes m})$ for any $m>0$. $P_m(V)$ is said to be the *m-genus* of V. $P_1(V)$ is often called the geometric genus and denoted also by $p_g(V)$. $P_2(V)$ is occasionally very important, which we call the *bigenus* of V.

Let ψ be a non-zero rational section of ω_V over V. Then taking a suitable open coordinate cover $\{U_\lambda\}$ of V with local coordinate systems $(z^1_\lambda, \cdots, z^n_\lambda)$ on U_λ, $\psi|_{U_\lambda}$ is written as $\psi_\lambda\, dz^1_\lambda\wedge\cdots\wedge dz^n_\lambda$ for some non-zero rational functions ψ_λ on U_λ. Gluing local divisors defined by ψ_λ, we obtain a divisor, that is denoted by div (ψ) or just (ψ). div (ψ) is often written as $K(V)$ and is called a canonical divisor on V. By definition, all canonical divisors are linearly equivalent. In other words, invertible sheaves associated to canonical divisors on V are isomorphic to each other.

Let W be a non-singular complete variety of dimension n and f: $W\to V$ a surjective morphism. Then there exists an effective divisor R_f, called the ramification divisor, such that

$$K(W)\sim f^*K(V)+R_f,$$

where $\sim$ means the linear equivalence of divisors. Moreover, if f is birational, then codim $f(R_f)\geq 2$. Hence, $\omega_W\cong f^*(\omega_V)\otimes\mathcal{O}(R_f)$. By Proposition 1, $H^0(W, \omega_W^{\otimes m})\cong H^0(V, \omega_V^{\otimes m})$ for all m, in other words, $P_m(W)=P_m(V)$ if W and V are birationally equivalent. Thus, $P_m(V)$ and also $\kappa(V)$ are birational invariants.

§ 3.

The following four results are basic in the theory of Kodaira dimension. These are easily proven, even though not trivial (see [3, Ch. 10], [4], [5]).

Theorem 1. *If $P_{m_0}(V)>0$, then there exist α, $\beta>0$ and η such that*

$$\alpha m^{\kappa}\leq P_{mm_0}(V)\leq \beta m^{\kappa} \quad \textit{for all } m>\eta, \quad \textit{where } \kappa=\kappa(V).$$

Theorem 2 (*Covering theorem*). *If $V_1\to V_2$ is an étale covering, then $\kappa(V_1)=\kappa(V_2)$.*

Theorem 3 (*Easy addition theorem*). *Let f: $V\to W$ be a surjective proper morphism with connected fibers. For a general point x of W,*

$$\kappa(V)\leq\kappa(f^{-1}(x))+\dim W.$$

Theorem 4 (*Fibering theorem*). *If $\kappa(V)\geq 0$, then there exist a complete non-singular variety V^* birationally equivalent to V, a complete variety W, and a proper surjective morphism f: $V^*\to W$ such that*

(1) $\dim W=\kappa(V)$, (2) *f has connected fibers,*

(3) *for each m, there exists an open subset $W_{(m)}$ such that f is smooth over $W_{(m)}$ and $P_{mm_0}(f^{-1}(x))=1$ for all $y\in W_{(m)}$, where $P_{m_0}(V)>0$.*

In the above statement, if $x\in\bigcap W_{(m)}$, then $\kappa(f^{-1}(x))=0$. Here, k is assumed to be $\boldsymbol{C}$ and thus $\bigcap_m W_{(m)}$ has infinitely many closed points.

§ 4.

Basic birational structure of varieties can be revealed by their Kodaira dimensions. Indeed, if $n=\dim V=1$, then the following three classes are certainly the most fundamental classification of curves: let g denote the genus of V, i.e. $g=P_1(V)$.

$g=0\Leftrightarrow\kappa(V)=-\infty\Leftrightarrow V$ is a rational curve.

$g=1\Leftrightarrow\kappa(V)=0\Leftrightarrow V$ is an elliptic curve.

$g\geq 2\Leftrightarrow\kappa(V)=1\Leftrightarrow$the universal covering manifold of V is a complex upper half plane.

In the case where $n=2$, a similar classification tells us that surfaces can be classified into the four classes according to their Kodaira dimensions.

If $\kappa(V)=1$, then by Theorem 4, we have a curve W and a surjective morphism $f: V \to W$ such that general fibers are elliptic curves. Such a fiber space structure $f: V \to W$ is called an elliptic surface.

Remark. Only in the case $n=2$, we can suppose $V^*=V$. In general, given a complete non-singular variety V, consider the Albanese map $\alpha_V: V \to \mathrm{Alb}(V)$, where $\mathrm{Alb}(V)$ denotes the abelian variety called the Albanese variety of V. It is well-known that $\dim \mathrm{Alb}(V)$ equals the number of linearly independent regular 1-forms on V, which we call the irregularity of V, denoted by $q(V)$.

(A) If $n=\dim V=2$ and $\kappa(V)=0$, Enriques proved that α_V is surjective. Moreover, if $q(V)=2$, then α_V is birational. If $q(V)=1$, then general fibers are elliptic curves.

(B) On the other hand, if $n=2$, $\kappa(V)=-\infty$ and $q(V)\geq 1$, then the image $\alpha_V(V)$ is a curve B and general fibers are rational curves. This was proved also by Enriques.

One of the aims in sight is to generalize these two facts into higher-dimensional cases. Classical proofs of these are based on the intersection theory of curves on surfaces; in particular the theory of minimal models.

However, in general case, we cannot rely on similar minimal models. It seemed risky to build classification theory of surfaces without using minimal models. It is K. Ueno who first succeeded in untieing the classification theory from minimal models.

At any way, the following conjecture seems very natural.

Conjecture C_n. *Let V be a variety of dimension n, W a variety of dimension r and $f: V \to W$ a proper surjective morphism with connected fibers. Then there exist a countable set of open dense subsets $W_{(m)}$ such that*

$$\kappa(V) \geq \kappa(f^{-1}(w)) + \kappa(W)$$

for all $w \in \bigcap_m W_{(m)}$.

Conjecture C_n is more precisely referred to as Conjecture $C_{n,r}$. Until now, the following cases have been verified:

(1) $C_{n,n-1}$ by Viehweg in [13].

(2) $C_{n,1}$ where $p_g(V)\geq 1$ and $g(W)\geq 2$ by Ueno [10].

(3) $C_{n,1}$ where $p_g(f^{-1}(w))\geq 1$ and $g(W)\geq 2$ by Fujita [2].

(4) $C_{n,m}$ where the $f^{-1}(w)$ are abelian varieties by Ueno [11].

(5) $C_{3,1}$ by Viehweg [14].
(6) $C_{n,n-2}$ where $\kappa(f^{-1}(w))\leq 1$ by Kawamata [7].
(7) $C_{n,m}$ where $\kappa(V)\geq 0$ and $\kappa(W)=\dim W$ by Kawamata [5].
(8) $C_{n,m}$ where $\kappa(W)=\dim W$ by Viehweg [15].
(9) $C_{n,1}$ by Kawamata [6].
(10) $C_{n,n-2}$ by Viehweg in this volume.

Note that (1)+(5) implies C_3, and (1)+(9)+(10) implies C_4.

§5.

Now, let V be a non-singular complete variety with $q=q(V)>0$ and let $X=\alpha_V(V)$. Let $V\to W\to X$ be the Stein factorization of $\alpha_V: V\to X$. One wishes to apply partial solutions of $C_{n,r}$ to this fiber space $V\to W$. The structures of X and W have been studied in detail by Ueno and Kawamata.

Theorem 5 (Ueno [9]). *Let X be a closed subvariety of an abelian variety. Then $\kappa(X)\geq 0$ and there exist a closed subvariety Z of an abelian variety and a proper surjective morphism $h: X\to Z$ such that* (i) $\dim Z=\kappa(Z)$, (ii) *all fibers $h^{-1}(z)$ are translations of some abelian subvariety. In particular, if $\kappa(X)=0$ then X is a translation of some abelian subvariety. Moreover, if $\kappa(X)>0$, then there exists an étale covering $\tilde{X}\to X$ such that $\tilde{X}\cong B\times Z$, where B is an abelian variety and Z is of general type.*

Theorem 6 ([8]). *Let $g: W\to A$ be a finite morphism, where A is an abelian variety.*

(1) *If W is normal and $\kappa(W)=0$, then $g(W)$ is a translation of some abelian subvariety and $g: W\to g(W)$ is étale; hence W is also an abelian variety.*

(2) *If $\kappa(W)>0$, then there exist an étale covering $\hat{W}\to W$ and a proper surjective morphism $h: \hat{W}\to Z$ such that* (i) $\dim Z=\kappa(Z)$, *and* (ii) *all fibers $h^{-1}(z)$ are isomorphic to some abelian variety.*

For a proof, we refer to [8] and [5].

Let $f: V\to W$ be a morphism derived from the Stein factorization of the Albanese map $\alpha_V: V\to X=\alpha_V(V)$. Then by Theorem 6, we have a morphism $h: W\to Z$. Thus $g=h\circ f: V\to Z$ satisfies the case (8) of $C_{n,m}$. Hence $\kappa(g^{-1}(z))+\kappa(Z)\leq\kappa(V)\leq\kappa(g^{-1}(z))+\dim Z$. However, since $\kappa(Z)=\dim Z$, it follows that

$$\kappa(V)=\kappa(g^{-1}(z))+\kappa(Z).$$

Whenever $\kappa(V)=-\infty$, we have $\kappa(g^{-1}(z))=-\infty$. If one can apply $C_{n,m}$

to the fiber space $g^{-1}(z)\to h^{-1}(z)$, one obtains $\kappa(f^{-1}(w))=-\infty$. But, except for the case where $\dim h^{-1}(z)\leq 2$, we do not have affirmative solutions to this case of $C_{n,m}$.

Next, assume $\kappa(V)=0$. Then $\kappa(g^{-1}(z))=\kappa(Z)=0$. Hence Z is a point, which implies W is itself an abelian variety. Hence, $W\to \mathrm{Alb}\,(V)$ is étale. By the universal mapping property of the Albanese map, we conclude that $W=\mathrm{Alb}\,(V)$, hence α_V: $V\to \mathrm{Alb}\,(V)$ is surjective and has connected fibers. In particular, $q(V)\leq \dim V$. Moreover, $q(V)=\dim V$ implies α_V is birational. Thus, the following beautiful result is obtained.

Theorem 7 (Kawamata [5]). *If $\kappa(V)=0$, then the Albanese map α_V: $V\to \mathrm{Alb}\,(V)$ is surjective and has connected fibers.*

Corollary. *V is birationally equivalent to an abelian variety, if and only if $\kappa(V)=0$ and $q(V)=\dim V$.*

For a proof, see [5].

§ 6.

For the convenience of the reader, we collect a couple of basic results on logarithmic Kodaira dimension.

Let $\bar{V}$ be a non-singular complete variety of dimension n and D a divisor with simple normal crossings on $\bar{V}$. For each positive integer $m>0$, we have a vector space $H^0(\bar{V}, (\Omega^n(D))^{\otimes m})$, which depends only on $\bar{V}-D$. Thus, letting $V=\bar{V}-D$, define the *logarithmic m-genus* of V to be $\dim_k H^0(\bar{V}, (\Omega^n(D))^{\otimes m})$, which is denoted by $\bar{P}_m(V)$. For simplicity, let $H^i(V, D)$ denote $H^i(V, \mathcal{O}(D))$ for any i and any divisor D. Thus, $\dim_k H^0(\bar{V}, m(K(\bar{V})+D))=\bar{P}_m(V)$.

Let $\bar{W}$ be a non-singular complete variety of dimension n and B a divisor with simple normal crossings on $\bar{W}$. Let f: $\bar{W}\to\bar{V}$ be a surjective morphism such that $f^{-1}(D)\subseteq B$. Then, there exists an effective divisor $\bar{R}_f$ called the logarithmic ramification divisor which satisfies

$$K(\bar{W})+B\sim f^*(K(\bar{V})+D)+\bar{R}_f.$$

Let $W=\bar{W}-B$. If $f|_W$: $W\to V$ is a proper birational morphism, then $f^{-1}(D)=B$ and $\bar{R}_f$ is exceptional with respect to f, i.e. $\mathrm{codim}\,(f(\bar{R}_f))\geq 2$. Thus by Proposition 1,

$$H^0(\bar{W}, m(K(\bar{W})+B))\cong H^0(\bar{V}, m(K(\bar{V})+D)),$$

i.e. $\bar{P}_m(W)=\bar{P}_m(V)$ for arbitrary positive integer m. Therefore $\bar{P}_m(V)$ is a proper birational invariant.

Definition. *$\kappa(K(\bar{V})+D, \bar{V})$ is said to be the logarithmic Kodaira dimension of V.*

Indeed, there exist α, $\beta>0$ such that

$$\alpha m^{\kappa}\leq \bar{P}_{mm_0}(V)\leq \beta m^{\kappa} \qquad \text{for } m\gg 0,$$

where $\kappa=\kappa(K(\bar{V})+D, \bar{V})$ and $\bar{P}_{m_0}(V)\neq 0$. Hence, κ depends only on V and is a proper birational invariant. Let $\bar{\kappa}(V)$ denote $\kappa(K(\bar{V})+D, \bar{V})$.

By using the logarithmic Kodaira dimension, one can develop the classification theory of not necessarily complete varieties in a way similar to that of complete varieties. One of the results is the finiteness of proper birational automorphism groups, which is stated as follows. For a variety V, let PBir (V) denote the group of proper birational maps of V into itself. Further, let SBir (V) be the group generated by strictly birational maps of V into itself.

Theorem 8. *If $\bar{\kappa}(V)=\dim V$, then* SBir (V) *is a finite group and* SBir $(V)=$ PBir (V).

For an algebraic proof, we refer to [3, Ch. 11].

Part 2. Cremona Transformations and Logarithmic Kodaira Dimension

§ 1.

As in part 1, varieties mean irreducible reduced separated schemes algebraic over a base field k, which is an algebraically closed field of characteristic zero.

Let V be a normal variety and D a reduced divisor on V. In this paper, such a pair of D and V is denoted by the symbol $(D\,\&\,V)$, which we call a *birational pair* or simply *a pair*. We shall develop birational geometry for pairs. First of all, we have to introduce the notion of a proper rational map of pairs. Let W be a normal variety and f a dominant proper rational map from W into V. Supposing that $\dim W=\dim V$, denoted by n, we say that f is *pure with respect to D*, whenever for the generic point ξ_i of each irreducible component Γ_i of D, $f^{-1}(\xi_i)$ consists of a finite set of points whose closures are of codimension 1, in other words, the closure of $f^{-1}(\xi_i)$ is (the support of) a reduced divisor F_i. In this case, a reduced divisor $\sum F_i$ is called the *strict* (or proper) *inverse image* of D by f, which we denote by the bracket $f^{-1}[D]$. Furthermore, if $\Delta\geqq f^{-1}[D]$, we say that f is a *proper rational map from* $(\Delta\,\&\,W)$ to $(D\,\&\,V)$.

Note that if f is a proper morphism from W into V, f is always pure with respect to any reduced divisor. Moreover, if f is birational, f^{-1} is a proper rational map from (D, V) into $(f^{-1}[D], W)$. In general, if f: $(\Delta \& W) \to (D \& V)$ is a proper birational map and $\Delta = f^{-1}[D]$, then f^{-1} is, in fact, a proper rational map from $(D \& V)$ into $(\Delta \& W)$. In this case, $(\Delta \& W)$ is said to be *proper birationally equivalent to* $(D \& V)$.

For a pair $(D \& V)$, define a subgroup PBir $(D \& V)$ of PBir (V) (see [3, p. 136]) by

$$\mathrm{PBir}\,(D \& V) = \{\varphi \in \mathrm{PBir}\,(V) \mid \varphi^{-1}[D] = D\}.$$

In the case where W and V are complete, let $\xi_1, \cdots, \xi_r$ be the generic points of the irreducible components of D. When f is pure with respect to D, for a couple of generic points η and ξ of $f^{-1}[D]$ and D such that $f(\eta) = \xi$, a local homomorphism $f^*: \mathcal{O}_{V,\xi} \to \mathcal{O}_{W,\eta}$ is induced. Denote by $\mathcal{O}_{V,[D]}$ the intersection of all $\mathcal{O}_{V,\xi}$ where the ξ are generic points of the irreducible components of D. Similarly, for $\Delta = f^{-1}[D]$, we define $\mathcal{O}_{W,[\Delta]}$. Then $f^*: \mathcal{O}_{V,\xi} \to \mathcal{O}_{W,\eta}$ induces a ring homomorphism: $\mathcal{O}_{V,[D]} \to \mathcal{O}_{W,[\Delta]}$, which we denote again by f^*.

In the case when V is complete, let Bir $(D \& V)$ stand for PBir $(D \& V)$. Then Bir $(D \& V)$ is anti-isomorphic to $\mathrm{Aut}_k\,(\mathcal{O}_{V,[D]})$. Furthermore, let $(\Delta \& W)$ be a pair where W is complete. Then $(D \& V)$ is birationally equivalent to $(D \& V)$ if and only if $\mathcal{O}_{V,[D]} \cong \mathcal{O}_{W,[\Delta]}$ as a k-algebra.

We wish to develop birational geometry for pairs in the following way:

(1) we define a non-singular model of a given pair,

(2) we introduce the m-ple regular forms of the pairs,

(3) we define the plurigenera and Kodaira dimension of the pairs,

(4) we classify the pairs according to the values of their Kodaira dimensions.

We shall prove a finiteness of PBir $(D \& V)$ if the Kodaira dimension of the pair $(D \& V)$ equals dim V (Theorem 2).

§ 2.

When W is a non-singular variety and Δ is a disjoint union of non-singular prime divisors, we say that $(\Delta \& W)$ is a *non-singular pair*. Furthermore, let $(D \& V)$ be a pair and f: $W \to V$ a proper birational morphism such that $(f^{-1}[D] \& W)$ is a non-singular pair. Then $(f^{-1}[D] \& W)$ is said to be a *non-singular model* of $(D \& V)$.

By Hironaka's famous result, given a pair, there exists a non-singular model of it.

Let $(\Delta \& W)$ be a non-singular pair. Take a non-singular completion $\bar{W}$ of W such that $B = \bar{W} - W$ is a divisor with simple normal crossings.

Let $\bar{\Delta}$ denote the closure of Δ in $\bar{W}$. After performing certain monoidal transformations, we can also assume that $B+\bar{\Delta}$ has only normal crossings and $\bar{\Delta}$ is a disjoint union of non-singular prime divisors.

For any $m>0$, we have a finite-dimensional vector space $H^0(\bar{W}, \Omega^n(\bar{\Delta}+B)^{\otimes m})$, where $n=\dim W$. Letting $K(\bar{W})$ be a canonical divisor on $\bar{W}$, $\Omega^n(\bar{\Delta}+B)$ is an invertible sheaf associated to $K(\bar{W})+\bar{\Delta}+B$. Hence,

$$H^0(\bar{W}, \Omega^n(\bar{\Delta}+B)^{\otimes m}) \cong H^0(\bar{W}, m(K(\bar{W})+\bar{\Delta}+B)).$$

We shall show that this vector space is independent of the choices of a non-singular model $(\Delta \,\&\, W)$ of $(D \,\&\, V)$ and a completion $\bar{W}$ of W.

Proposition 1. *Let Δ_1+B_1 and Δ_2+B_2 be divisors with simple normal crossings on non-singular complete varieties $\bar{W}$ and $\bar{Y}$, respectively. Let f: $\bar{Y}\to\bar{W}$ be a generically finite surjective morphism such that $f^{-1}(B_1)\leqq B_2$ and $f^{-1}[\Delta_1]\leqq\Delta_2$. Suppose that Δ_1 is a disjoint union of non-singular prime divisors. Then there exists an effective divisor $R\&_f$ such that*

$$K(\bar{Y})+B_2+\Delta_2 \sim f^*(K(\bar{W})+B_1+\Delta_1)+R\&_f.$$

Proof. There exists a closed proper subset S of $\bar{Y}$ of codimension $\geqq 2$ such that if $p \notin S\cup B_2$, then $f(\Delta_2)$ is non-singular at $f(p)$. For any point p of $\bar{Y}$, we choose systems of local coordinates $(z_1, \cdots, z_n)$ around p and $(w_1, \cdots, w_n)$ around $f(p)$ such that $z_1\cdots z_a=0$ defines Δ_2+B_2 at p and $w_1\cdots w_b=0$ defines Δ_1+B_1 at $f(p)$. By pulling back, we have a rational function ψ defined by

$$f^*\left(\frac{dw_1}{w_1}\wedge\cdots\wedge\frac{dw_b}{w_b}\wedge dw_{b+1}\wedge\cdots\wedge dw_n\right)$$
$$=\psi\cdot\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_a}{z_a}\wedge dz_{a+1}\wedge\cdots\wedge dz_n.$$

We shall show that ψ is holomorphic at p. It suffices to show it under the assumption $p\in\Delta_2$. Since $\operatorname{codim} S\geqq 2$, we can assume $p\notin S$. Moreover, if $p\in B_2$, then ψ is holomorphic at p by the logarithmic ramification formula (§ 6 in Part 1). Hence we can assume $p\notin B_2$. Then by the choice of S, $f(\Delta_2)$ is non-singular at $f(p)$; thus $b=1$. By choosing S in an appropriate way, p is assumed to be a non-singular point of Δ_2. Hence $a=1$. If $\operatorname{codim}_{f(p)}(f(\Delta_2))=1$, then $f^*f(\Delta_2)=\nu\Delta_2$ around p for some $\nu>0$. Thus $w_1=z_1^{\nu}\varepsilon$ for some non-vanishing holomorphic function ε at p. Since $dw_1/w_1=\nu(dz_1/z_1)+$a holomorphic form, it follows that ψ is holomorphic at p. Next, if $\operatorname{codim}_{f(p)} f(\Delta_2)\geqq 2$, then after renumbering $w_1, \cdots, w_n$ we have holomorphic functions φ_1 and φ_2 at p such that $w_1=z_1\varphi_1$ and $w_2=z_1\varphi_2$. Factorizing into prime factors in the local ring $\mathcal{O}_{\bar{Y},p}$, we obtain holomorphic functions ε_1 and ε_2 at p such that

$$w_1 = z_1^{\nu}\varepsilon_1 \quad \text{and} \quad w_2 = z_1^{\lambda}\varepsilon_2$$

for some ν, $\lambda > 0$ where $\varepsilon_1(0, z_2, \cdots, z_n) \neq 0$ and $\varepsilon_2(0, z_2, \cdots, z_n) \neq 0$. Then

$$\begin{aligned} f^*\left(\frac{dw_1}{w_1} \wedge dw_2 \wedge \cdots \wedge dw_n\right) &= \left(\nu\frac{dz_1}{z_1} + \frac{d\varepsilon_1}{\varepsilon_1}\right) \wedge (z_1^{\lambda}\varepsilon_2)\left(\lambda\frac{dz_1}{z_1} + \frac{d\varepsilon_2}{\varepsilon_2}\right) \wedge f^*(dw_3 \wedge \cdots \wedge dw_n) \\ &= \left(\nu z_1^{\lambda-1} dz_1 \wedge d\varepsilon_2 + \lambda z_1^{\lambda-1}\frac{\varepsilon_2}{\varepsilon_1} d\varepsilon_1 \wedge dz_1 + \frac{z_1^{\lambda}}{\varepsilon_1} d\varepsilon_1 \wedge d\varepsilon_2\right) \\ &\quad \wedge f^*(dw_3 \wedge \cdots \wedge dw_n). \end{aligned}$$

Therefore, ψ does not have Δ_1 as a pole. Hence, ψ is holomorphic at p.

Gluing local effective divisors defined by ψ, we have an effective divisor $R\&_f$, which satisfies

$$K(\bar{Y}) + B_2 + \Delta_2 \sim f^*(K(\bar{W}) + B_1 + \Delta_1) + R\&_f. \qquad \text{Q.E.D.}$$

From the above proof, we know that the pull-back of $\omega \in H^0(\bar{W}, (\Omega^n(B_1+\Delta_1))^{\otimes m})$ by f belongs to $H^0(\bar{Y}, (\Omega^n(B_2+\Delta_2))^{\otimes m})$. Hence, we have a linear map $f^*: H^0(\bar{W}, (\Omega^n(B_1+\Delta_1))^{\otimes m}) \to H^0(\bar{Y}, (\Omega^n(B_2+\Delta_2))^{\otimes m})$ for each $m > 0$, which is injective, because f is dominant.

Since $H^0(\bar{W}, (\Omega^n(B_1+\Delta_1))^{\otimes m}) \cong H^0(\bar{W}, m(K(\bar{W})+B_1+\Delta_1))$ and $H^0(\bar{Y}, (\Omega^n(B_2+\Delta_2))^{\otimes m}) \cong H^0(\bar{Y}, m(K(\bar{Y})+B_2+\Delta_2))$, we have

$$\dim_k H^0(\bar{Y}, m(K(\bar{Y})+B_2+\Delta_2)) \geqq \dim_k H^0(\bar{W}, m(K(\bar{W})+B_1+\Delta_1)).$$

If f is birational and $f^{-1}(B_1) = B_2$ and $f^{-1}[\Delta_1] = \Delta_2$, then $f^{-1}(B_1+\Delta_1) \supseteqq B_2+\Delta_2$ and so $R\&_f \leqq \bar{R}_f$, which is exceptional with respect to f. Here $\bar{R}_f$ is the logarithmic ramification divisor for f: $\bar{Y} - f^{-1}(B_1+\Delta_1) \to \bar{W} - (B_1+\Delta_1)$ (see [3, § 11.4]). Thus, by Proposition 1 in Part 1, we have

$$l(m(K(\bar{Y})+B_2+\Delta_2)) = l(m(K(\bar{W})+B_1+\Delta_1)).$$

Hence, the following invariance is established.

Lemma 1. *Under the same hypothesis as in the last proposition, suppose further that f is birational and $f^{-1}(B_1) = B_2$, $f^{-1}[\Delta_1] = \Delta_2$ Then,*

$$\dim_k H^0(\bar{Y}, m(K(\bar{Y})+B_2+\Delta_2)) = \dim_k H^0(\bar{W}, m(K(\bar{W})+B_1+\Delta_1)),$$

for all $m > 0$.

Remark. This invariance was *first* proved by S. Suzuki in the case when $B_1=B_2=0$, [17].

§ 3.

Given any pair $(D\,\&\,V)$, we take a non-singular model $(\varDelta\,\&\,W)$ of $(D\,\&\,V)$ and choose a non-singular completion $\bar{W}$ of W such that $B=\bar{W}-W$ is a divisor with simple normal crossings, the closure $\bar{\varDelta}$ of $\varDelta$ is a disjoint union of non-singular prime divisors and that $B+\bar{\varDelta}$ has only normal crossings.

By Proposition 1 and the argument after it, $H^0(\bar{W}, m(K(\bar{W})+B+\bar{\varDelta}))$ depends only on $(D\,\&\,V)$, which we denote by $T_{\{m\}}(D\,\&\,V)$. Moreover, we put $P_m(D\,\&\,V)=\dim_k\ T_{\{m\}}(D\,\&\,V)$ and $\kappa(D\,\&\,V)=\kappa(K(\bar{W})+B+\bar{\varDelta}, \bar{W})$, which is said to be the Kodaira dimension of the pair $(D\,\&\,V)$.

Let f: $(D_1\,\&\,V_1)\to(D\,\&\,V)$ be a proper rational map of pairs. Let $(\varDelta\,\&\,W)$ and $(\varDelta_1\,\&\,W_1)$ be non-singular models of $(D\,\&\,V)$ and $(D_1\,\&\,V_1)$, respectively. Then the rational map h: $W_1\to W$ defined by f gives rise to a proper birational map from $(\varDelta_1\,\&\,W_1)$ into $(\varDelta\,\&\,W)$. Take non-singular completions $\bar{W}_1$ of W_1 and $\bar{W}$ of W such that (1) $B_1=\bar{W}_1-W_1$ and $B=\bar{W}-W$ are divisors with simple normal crossings, (2) $B_1+\bar{\varDelta}_1$ and $B+\bar{\varDelta}$ have only simple normal crossings and that (3) $\bar{\varDelta}_1$ and $\bar{\varDelta}$ are disjoint unions of prime divisors. Moreover, we can assume that the rational map $\bar{h}$: $\bar{W}_1\to\bar{W}$ defined by h is a morphism. Then by Proposition 1, we have an injective linear map from $H^0(\bar{W}, m(K(\bar{W})+B+\bar{\varDelta}))$ into $H^0(\bar{W}_1, m(K(\bar{W}_1)+B_1+\bar{\varDelta}_1))$ for all $m>0$. Via natural isomorphisms, we have a corresponding injective linear map: $T_{\{m\}}(D\,\&\,V)\to T_{\{m\}}(D_1\,\&\,V_1)$, which we denote by f^*. Thus, $P_m(D\,\&\,V)\leqq P_m(D_1\,\&\,V_1)$ for all $m>0$.

§ 4.

Let $(D\,\&\,V)$ be a pair and f: $V_1\to V$ be a generically finite dominant strictly rational map. (For the definition of strictly rational map, see [3, p. 134]). As in the case of proper rational maps, we say that f is *pure with respect to D* if $f^{-1}(\xi)$ is empty or consists of generic points of prime divisors for generic points ξ of the irreducible components of D. In this case, the strict inverse image of D by f is also defined and is indicated by the bracket $f^{-1}[D]$. In this case, whenever $D_1\geqq f^{-1}[D]$, we say that f is a *strictly rational map* from $(D_1\,\&\,V_1)$ into $(D\,\&\,V)$. Moreover, for all $m>0$, the injective linear map f^*: $T_{\{m\}}(D\,\&\,V)\to T_{\{m\}}(D_1\,\&\,V_1)$ is derived. Thus, we obtain the following theorem.

Theorem 1. *Let f: $(D_1\,\&\,V_1)\to(D\,\&\,V)$ be a strictly rational map Then for all $m>0$, an injective linear map f^*: $T_{\{m\}}(D\,\&\,V)\to T_{\{m\}}(D_1\,\&\,V_1)$*

is derived and so $P_m(D\,\&\,V) \leqq P_m(D_1\,\&\,V_1)$. *In particular,* $\kappa(D\,\&\,V) \leqq \kappa(D_1\,\&\,V_1)$.

Furthermore, if f *is proper birational and* $f^{-1}[D] = D_1$, *then* $P_m(D\,\&\,V) = P_m(D_1\,\&\,V_1)$ *and* $\kappa(D\,\&\,V) = \kappa(D_1\,\&\,V_1)$.

§ 5.

By Theorem 1, PBir $(D\,\&\,V)$ has a group representation with $T_{\{m\}}(D\,\&\,V)$ as a representation space. Moreover, let $f\colon (D\,\&\,V) \to (D\,\&\,V)$ be a strictly birational map. Then $f^*\colon T_{\{m\}}(D\,\&\,V) \to T_{\{m\}}(D\,\&\,V)$ is injective and so f^* is an isomorphism. Hence, letting SBir $(D\,\&\,V)$ be a group generated by all strictly birational maps $f\colon (D\,\&\,V) \to (D\,\&\,V)$, it has $T_{\{m\}}(D\,\&\,V)$ as a representation space, too.

We shall prove that SBir $(D\,\&\,V)$ is a finite group, if $\kappa(D\,\&\,V) = \dim V$.

If $(\Delta\,\&\,W)$ is a non-singular model of $(D\,\&\,V)$, SBir $(\Delta\,\&\,W) \cong$ SBir $(D\,\&\,V)$ and PBir $(\Delta\,\&\,W) \cong$ PBir $(D\,\&\,V)$. Thus, we can assume that $(D\,\&\,V)$ is already a non-singular pair.

As in the last section, we take a non-singular completion $\bar{V}$ of V such that both $B = \bar{V} - V$ and $B + \bar{D}$ have only simple normal crossings. Further, we assume that $\bar{D}$ is a disjoint union of non-singular prime divisors. Fix m such that the rational map Φ_m associated with $|m(K(\bar{V}) + B + \bar{D})|$ gives rise to a birational map $\Phi_m\colon V \to \Phi_m(\bar{V})$. Here $\Phi_m(\bar{V})$ is defined to be the closure of $\Phi_m(*)$, where $*$ denotes the generic point of $\bar{V}$. After performing suitable monoidal transformations with non-singular centers, we can suppose that Φ_m is a morphism onto $\Phi_m(\bar{V})$.

Let $\varphi\colon (D\,\&\,V) \to (D\,\&\,V)$ be a strictly birational map such that $\varphi^{-1}[D] = D$. Then $\varphi[\varphi^{-1}[D]] = \varphi[D]$ and so $D = \varphi[D]$. The isomorphism $\varphi^*\colon T_{\{m\}}(D\,\&\,V) \to T_{\{m\}}(D\,\&\,V)$ induces an automorphism φ_1 of $\boldsymbol{P}^l$, where $l = P_m(D\,\&\,V) - 1$, such that $\varphi_1(\Phi_m(\bar{V})) = \Phi_m(\bar{V})$. Letting $\bar{\varphi}$ denote the rational map: $\bar{V} \to \bar{V}$ determined by φ, we have the following commutative square:

$$\begin{array}{ccc} \bar{V} & \xrightarrow{\ \bar{\varphi}\ } & \bar{V} \\ {\scriptstyle \Phi_m}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_m} \\ \Phi_m(\bar{V}) & \xrightarrow[\ \varphi_1\ ]{} & \Phi_m(\bar{V}). \end{array}$$

By an argument in [3, p. 335] we have $\varphi_1(\Phi_m(B)) = \Phi_m(B)$, and since $\varphi[D] = D$, we have

$$\Phi_m(D) = \Phi_m(\varphi[D]) = \varphi_1 \Phi_m(D).$$

Thus, letting F be the closed subset $\Phi_m(B) \cup \Phi_m(D)$, we have $\varphi_1(F) = F$ and

$\Phi_m^{-1}(F) \supseteqq B \cup D$. Hence $\kappa(D \& V) = \bar{\kappa}(\bar{V} - B \cup D) \leqq \bar{\kappa}(\bar{V} - \Phi_m^{-1}(F)) = \kappa(\Phi_m(\bar{V}) - F)$. By hypothesis, $\dim V = \kappa(D \& V)$ and $\dim \Phi_m(\bar{V}) = \dim V$. Hence, $\Phi_m(\bar{V}) - F$ is of hyperbolic type. Since $\varphi: (D \& V) \to (D \& V)$ induces $\varphi_1 \in \mathrm{Aut}\,(\Phi_m(\bar{V}))$ such that $\varphi_1(F) = F$, φ_1 belongs to $\mathrm{Aut}\,(\Phi_m(\bar{V}) - F)$. By Theorem 8 in part 1, $\mathrm{Aut}\,(\Phi_m(\bar{V}) - F)$ is a finite group. Hence the subgroup generated by strictly birational maps $\varphi: (D \& V) \to (D \& V)$ with $\varphi^{-1}[D] = D$ is a finite group.

Now, let $\psi: (D \& V) \to (D \& V)$ be a strictly birational map. Then $\psi^{-1}[D] \leqq D$. Let $D_1 = \psi^{-1}[D]$. Since $P_m(D \& V) \geqq P_m(\psi^{-1}[D] \& V) \geqq P_m(D \& V)$, it follows $P_m(D \& V) = P_m(D_1 \& V)$. Repeating this process, we have a reduced divisor D' such that (1) $D' \leqq D$ (2) $P_m(D' \& V) = P_m(D \& V)$ and (3) $\psi^{-1}[D'] = D'$. Therefore $\psi \in \mathrm{SBir}\,(D' \& V)$ and so $\psi^s = id$ for some $s > 0$. Let $r(D)$ denote the number of irreducible components of D. Then $r(\psi^{-i}[D]) \leqq r(D)$ for all $i \geqq 0$ and if $r(\psi^{-1}[D]) < r(D)$, we have $r(D) = r(\psi^{-s+1}[\psi^{-1}[D]]) \leqq r(\psi^{-1}[D]) < r(D)$. This is a contradiction. Hence, $\psi^{-1}[D] = D$.

Furthermore, given any strictly birational map $\psi: (D \& V) \to (D \& V)$, there exists $s > 0$ such that $\psi^s = id$. Then ψ is also proper birational and $\psi^{-1}[D] = D$. Hence $\psi \in \mathrm{PBir}\,(D \& V)$. Thus, we complete the proof of the following result.

Theorem 2. *If* $\kappa(D \& V) = \dim V$, *then* $\mathrm{SBir}\,(D \& V)$ *is a finite group and* $\mathrm{SBir}\,(D \& V) = \mathrm{PBir}\,(D \& V)$.

§ 6.

Let D be a reduced divisor on $\boldsymbol{P}^n$. Then $\mathrm{Bir}\,(D \& \boldsymbol{P}^n)$ is a subgroup of $\mathrm{Bir}\,(\boldsymbol{P}^n)$, which we denote by Cr_n. Cr_n is called the group of *Cremona transformations* of $\boldsymbol{P}^n$.

Example 1. Let Γ be a non-singular hypersurface of degree d in $\boldsymbol{P}^n$. If $d > n+1$, then $\kappa(\Gamma \& \boldsymbol{P}^n) = n$. Let H be a hyperplane of $\boldsymbol{P}^n$. Then $K(\boldsymbol{P}^n) + \Gamma \sim (d-n-1)H$ and so Φ_1 is a birational morphism from $\boldsymbol{P}^n$ onto the Veronese imbedding of $\boldsymbol{P}^n$ into $\boldsymbol{P}^N$, N being $\binom{n+d}{n} - 1$. Hence, $\mathrm{Bir}\,(\Gamma \& \boldsymbol{P}^n)$ is a finite linear group. Thus, if φ is a Cremona transformation preserving Γ, then φ is linear. The following example will be proven in the forthcoming paper of the author [16].

Example 2. Let C be an irreducible curve on $\boldsymbol{P}^2$ of degree d. Suppose that C has only double points including infinitely near singular points. Then if $d \geqq 7$, the group $\mathrm{Bir}\,(C \& \boldsymbol{P}^2)$ is a linear subgroup of $PGL\,(3, k)$. If the genus of C is 12, then $P_2(C \& \boldsymbol{P}^2) = 33 + (d-5)(d-4)/2 \neq 35$. On the other hand, let Γ be a general member of $|7H|_{3p}$ which is a linear system of curves of degree 7 which has the multiplicity $\geqq 3$ at p. Then Γ is irreducible, the genus of Γ is 12 and $P_2(\Gamma \& \boldsymbol{P}^2)=35$. Hence, $(C \& \boldsymbol{P}^2)$ cannot be birationally equivalent to $(\Gamma \& \boldsymbol{P}^2)$. In other words, Γ cannot be transformed into a curve with double points by any Cremona transformation.

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