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On the Structure of Compact Complex Manifolds in $\mathscr{C}$

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Notations and conventions.

We use the following convention and terminology (cf. Ueno [43]). A complex manifold is always assumed to be *connected.* *A complex variety* is a reduced and irreducible complex space. A *fiber space* is a proper surjective morphism of complex spaces with general fiber irreducible.

a) Let X be a compact complex manifold and L a line bundle on X. Then:

$a(X)$: the algebraic dimension of X
$q(X)$ $:=\dim H^1(X, \mathcal{O}_X)$, the irregularity of X
K_X : the canonical bundle of X
$\kappa(L, X)$: the L-dimension of X
$\kappa(X)$ $:=\kappa(K_X, X)$ the Kodaira dimension of X

b) Let X be a compact complex variety, $\tilde{X}$ a nonsingular model of X, $A \subseteqq X$ an analytic subspace, E a holomorphic vector bundle on X. Then:

$a(X)$ $=a(\tilde{X})$
$q(X)$ $=q(\tilde{X})$
Aut X : the complex Lie group of biholomorphic automorphisms of X
$\mathrm{Aut}_0 X$: the identity component of Aut X
Aut (X, A) $:=\{g \in \mathrm{Aut}\, X; g(A)=A\}$
$\mathrm{Aut}_0 (X, A)$: the identity component of Aut (X, A)
$\boldsymbol{P}(E)$ $:=(E-\{0\})/\boldsymbol{C}^*$ ($\{0\}=$the zero section of E)
Θ_X : the sheaf of germs of holomorphic vector fields on X

A compact complex variety X' is called a bimeromorphic model of X if it is bimeromorphic to X.

c) Let Y be a complex variety. Then

$\pi_1(Y)$: the fundamental group of Y with respect to some reference point.
$\mathfrak{Q}(Y)$: the class of subsets of Y which is a complement of an at most countable union of proper analytic subvarieties of X, or

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equivalently, an at most countable intersection of nonempty Zariski open subsets of Y.

d) Let $f: X \to Y$ be a fiber space of complex varieties, $\tilde{Y} \to Y$ a morphism of complex varieties, $U \subseteq Y$ a Zariski open subset, and $y \in Y$ a point. Then

$$X_{\tilde{Y}} := X \times_Y \tilde{Y} \text{ and } f_{\tilde{Y}} := f_Y \times_{\mathrm{id}_Y} \mathrm{id}_{\tilde{Y}} : X_{\tilde{Y}} \longrightarrow \tilde{Y}.$$

In particular

$f_U \ : \ X_U \to U$ the restriction of f to X_U
$X_y := X_{\{y\}}$ the fiber over y
$\dim f = \dim X - \dim Y$
$q(f) = q(X_y)$ for general X_y if f is generically smooth
$\mathrm{Aut}\,(X/Y)$: the group of biholomorphic automorphisms $g: X \to X$ with $fg = f$.
$D_{X/Y}$: the relative Douady space associated to f
$\Theta_{X/Y}$: the sheaf of germs of relative holomorphic vector fields.

In this paper we have to distinguish two notions of 'general fibers': Let (P) be a property of a complex space. Then we say that X_y has the property (P) for *general* (resp. '*general*') $y \in Y$ if there exists a Zariski open subset $U \subseteq Y$ (resp. a subset $M \subseteq Y$ with $M \in \mathfrak{Q}(Y)$) such that X_y has property (P) for $y \in U$ (resp. $y \in M$).

e) Let $g: X \to Y$ be a meromorphic map of complex varieties, $\Gamma \subseteq X \times Y$ the graph of g, $p: \Gamma \to X$, $p': \Gamma \to Y$ the natural projections. Then we say that g is *surjective* if $p'(\Gamma) = Y$. The *general fiber* of g is $p(\Gamma_y) \subseteq X$ for general $y \in Y$. g is called a *meromorphic fiber space* if p' is a fiber space, i.e., g is surjective with general fiber irreducible. *A* (*meromorphic*) $\boldsymbol{P}^1$*-fiber space* is a (meromorphic) fiber space with general fiber isomorphic to the complex projective line $\boldsymbol{P}^1$.

A meromorphic map $g': X' \to Y$ is a *bimeromorphic model* of g if there exists a bimeromorphic Y-map $\varphi: X \to X'$. If g' is holomorphic, we call g' simply a *holomorphic model of* g. More generally given a diagram of meromorphic maps of complex spaces we can speak of its bimeromorphic model or holomorphic model in the obvious sense.

f) In this paper we are concerned with the structure of compact complex manifolds up to bimeromorphic equivalences. Thus if X is a given compact complex manifold and if we are given a meromorphic map $f: X \to Y$ into a complex variety Y, then passing to another bimeromorphic model X^* of X such that the resulting meromorphic map $f^*: X^* \to Y$ is holomorphic and then considering X^* instead of X we may assume from the beginning that f is holomorphic.

Contents

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§ 1. Introduction

The introduction contains a review of the known methods and results, the definition and the fundamental properties of the manifolds in the class $\mathscr{C}$, and a typical application of our main results.

1.1. We first review the general methods for studying nonalgebraic compact complex manifolds, together with some known results due mainly to K. Ueno. (See Ueno [43] for the detail.)

Let X be a compact complex manifold. Then the first bimeromorphic invariant of X we consider is the *algebraic dimension* $a(X)$ of X, which is by definition the transcendence degree tr. deg $\boldsymbol{C}(X)$ of the meromorphic function field $\boldsymbol{C}(X)$ of X; a$(X)=$tr. deg $\boldsymbol{C}(X)$. (Recall that $\boldsymbol{C}(X)$ is in general an algebraic function field over the complex number field $\boldsymbol{C}$.) A good geometric interpretation of $a(X)$ is provided by considering algebraic reduction of X; an *algebraic reduction* of X is a meromorphic fiber space $f\colon X\to Y$ such that Y is projective and f induces an isomorphism $f^*\colon \boldsymbol{C}(Y)\cong\boldsymbol{C}(X)$ of meromorphic function fields of X and Y. An algebraic reduction is unique up to bimeromorphic equivalences and we have of course

$a(X)=\dim Y$. In particular $0 \leqq a(X) \leqq n := \dim X$. When $a(X)=n$, X is by definition a Moishezon manifold and is bimeromorphic to a projective manifold. We call this case algebraic and exclude from our consideration in this paper.

a) When $0<a(X)<n$, then an algebraic reduction defines a nontrivial meromorphic fibering structure on X. So our first aim should be to study the structure of f. For this purpose, however, we may assume that $f: X \to Y$ is a (holomorphic) fiber space (cf. Convention f)). Then the next proposition already impose a strong restriction of the possible fibers of f. (See Ueno [43, 12.1] and Lieberman-Sernesi [34].)

Proposition 1.1. *For any line bundle L on X, $\kappa(L_y, X_y) \leqq 0$ for 'general' $y \in Y$. In particular $\kappa(X_y) \leqq 0$ for 'general' $y \in Y$.*

Example. 1) When $\dim f=1$, X_y is an elliptic curve. 2) When $\dim f=2$, $\kappa(X_y) \leqq 0$ for general $y \in Y$ (cf. [43]). Furthermore, the general fiber of f cannot be bimeromorphic to a ruled surface of genus $g \geqq 2$ (Kawai when $\dim X=3$ (cf. [43]) and Kuhlmann in general).

Unfortunately, however, the above seems to be all what is known on the general structure of f.

b) When $a(X)=0$, the general procedure for studying X is to take the Albanese map $\alpha: X \to \mathrm{Alb}\, X$ of X. The following observation due to Ueno is fundamental for the study of the structure of α.

Proposition 1.2. *Under the above assumption that $a(X)=0$, α is necessarily a fiber space. In particular* $\dim \mathrm{Alb}\, X \leqq \dim X$.

Proof. See Ueno [43, § 13].

However, nothing more seems to be known about the general structure of α, except the case of $\dim \alpha=1$ and 2 where Ueno [43] proved the following:

Proposition 1.3. 1) *Suppose that* $\dim \alpha=1$. *Then for general $a \in$* Alb X, *X_a is either isomorphic to $\boldsymbol{P}^1$ or an elliptic curve. Moreover there exists a Zariski open subset $U \subseteqq$* Alb *X such that α is a holomorphic fiber bundle over U.* 2) *Suppose that* $\dim \alpha=2$. *Then $\kappa(X_a) \leqq 0$ for general $a \in$* Alb X. *Moreover X_a cannot be bimeromorphic to a ruled surface of genus $\geqq 2$.*

Proof. See Ueno [43, 13.8 and 13.11]

c) When $a(X)=q(X)=0$, no general method is known.

1.2. Now our starting point in this paper is the observation that the class of nonalgebraic manifolds is not uniformly nonalgebraic, but there exists a special subclass of compact complex manifolds containing the algebraic ones which enjoys a number of properties in common with algebraic manifolds. Namely we recall from Fujiki [12] (cf. also [13]) the following:

Definition. A compact complex variety X is said to be in (the class) $\mathscr{C}$ if there exist a compact Kähler manifold Y and a surjective meromorphic map $h: Y \to X$. By Chow's lemma [28] one can always take h to be holomorphic if one likes (cf. [12, Lemma 4.6]).

Most typical properties of the varieties in $\mathscr{C}$ are summarized as follows. See also [43a].

A. *Functorial properties.* Let X be a compact complex variety in $\mathscr{C}$. Then:

1) Any subvariety of X is again in $\mathscr{C}$.

2) Any meromorphic image of X is again in $\mathscr{C}$.

3) Let Y be a compact complex variety and $h: Y \to X$ a proper and Kähler (e.g. projective) morphism (cf. [12]) or a Moishezon morphism (cf. [14] and the definition in 2.1 below). Then $Y \in \mathscr{C}$.

B. *Hodge decomposition* [13]. Let X be a compact complex manifold in $\mathscr{C}$. Then for any $k \geqq 0$ we have the natural decomposition

$$H^k(X, \boldsymbol{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \qquad H^{p,q}(X) = \overline{H}^{q,p}(X) \tag{1}$$

of $H^k(X, \boldsymbol{C})$ into the subspaces $H^{p,q}(X)$ of elements of type (p, q) where $\overline{}$ denotes the complex conjugate. In particular any odd dimensional Betti number of X is even. Moreover, in connection with this, we have $q(X) = \dim \operatorname{Alb} X$ where $\operatorname{Alb} X$ is the Albanese variety of X.

Essential point in the proof is contained in the following lemma which will be used in the sequel.

Lemma 1.4. *Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Then $f^*: H^k(Y, \boldsymbol{R}) \to H^k(X, \boldsymbol{R})$ is injective for any k.*

C. *Closedness of the Douady space D_X of X* [12]. For any compact complex variety X in $\mathscr{C}$, any irreducible component of the Douady space D_X is again compact and belongs to $\mathscr{C}$ (cf. [14]).

More concretely, the akinness of manifolds in $\mathscr{C}$ to algebraic manifolds are typically seen in the case $\dim X = 2$, by the following classification table due to Kodaira [32] in its roughest form:

Classification of compact analytic surfaces

	$a(X)$	X	$\kappa(X)$
$X\in\mathscr{C}$	2	projective	$-\infty, 0, 1, 2$
	1	elliptic surface, b_1 even	0, 1
	0	$\sim$ complex torus or $K3$ surface	0
$X\notin\mathscr{C}$	1	elliptic surface, b_1 odd	0, 1
	0	surface of class VII [32]	$-\infty$
		(non-Kähler $K3$ surface)	(0)

b_1 first Betti number, $\sim$ bimeromorphic to

Remark 1.1. 1) When $\dim X=2$, $X\in\mathscr{C}$ if and only if X is Kähler (Fujiki [21]). 2) We put parentheses on the last row since the existence of a non-kähler $K3$ surface is suspected. 3) Except for this possible exception, $X\in\mathscr{C}$ if and only if X is a deformation of a projective surface.

1.3. Now the purpose of this paper is to establish some general structure theorems (Theorems 1 and 2 to be formulated and stated in the next section) for an algebraic reduction $f: X\to Y$ of X (including the case where $a(X)=0$ so that f is a constant map) under the assumption that X is in the class $\mathscr{C}$.

In fact, these structure theorems together with some specific consideration yield as a consequence the following generalization of the above classification table in the three-dimensional case (in case $X\in\mathscr{C}$).

Theorem. *Let X be a compact complex manifold in $\mathscr{C}$ with* $\dim X=3$. *Then X falls into one of the following classes.*

1) $a(X)=3$ *and X is Moishezon*

2) $a(X)=2$ *and X is an elliptic threefold*

3) $a(X)=1$; *there are two cases to be distinguished.*

I. *For any bimeromorphic model X^* of X an algebraic reduction f^*: $X^*\to Y$ is always a morphism. Let X_y^* be any smooth fiber of f^*. Then X_y^* is isomorphic either to a complex torus or a holomorphic $\boldsymbol{P}^1$-bundle over an elliptic curve.*

II. *X is bimeromorphic to a quotient variety $(C\times S)/G$ where C is a compact Riemann surface, S is either a complex torus or a $K3$ surface, with $a(S)=0$, and G is a finite group acting on both C and S and acting on $C\times S$ diagonally.*

4) $a(X)=0$; *there are three cases to be distinguished.*

I. *X is Kummer*

II. *X is a $\boldsymbol{P}^1$-fiber space over a normal compact analytic surface S with* $a(S)=0$.

III. *X is simple and its Kummer dimension* $k(X)=0$.

Relevant definitions are: 1) A compact complex manifold X is said to be *Kummer* if X is bimeromorphic to a quotient variety T/G of a complex torus T by a finite group G (cf. [43]). 2) A compact complex manifold is said to be *simple* if there exists no (analytic) covering family $\{A_t\}_{t\in T}$ of proper analytic subvarieties A_t of X with dim $A_t>0$. ('covering' means that $\bigcup_{t\in T} A_t=X$.) 3) '$k(X)=0$' means that there is no surjective meromorphic map of X onto a Kummer manifold. In particular then $q(X)=0$.

Roughly, the content of Theorem may be summarized in the following table:

$a(X)$	X
3	Moishezon
2	elliptic threefold
1	I. f: $X\to Y$ (algebraic reduction) is holomorphic α. $X_y\cong$complex torus β. $X_y\cong\boldsymbol{P}^1$-bundle over an elliptic curve II. quasi-trivial type (cf. 10.2)
0	I. Kummer II. $\boldsymbol{P}^1$-fiber space over a surface III. simple and $k(X)=0$

See Sections 10–13 for more detailed information on the individual classes with $a(X)\leq 1$. On the other hand, see Viehweg [44] for the case $a(X)=3$ where remarkable progress has been made recently by the work of Ueno, Fujita, Kawamata and Viehweg.

§ 2. Formulation and statement of Theorems 1 and 2

2.1. First we recall from [18] the theory of relative algebraic reduction, together with some relevant definitions which is of constant use in this paper. This theory was also developed by Campana [6] independently. We refer to [6] and [18] for the more detail.

Definition. Let f: $X\to Y$ be a proper morphism of complex spaces. Then: 1) f is a called *Moishezon* if f is bimeromorphic to a projective morphism, and 2) f is called *locally Moishezon* if for any point $y \in Y$ there exists a neighborhood $y \in U$ (in the usual topology) such that the induced morphism $f_U: X_U \to U$ is Moishezon.

Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $U \subseteqq Y$ be a Zariski open subset over which f is smooth. For any integer $k \geqq 0$ we set $A_k := \{y \in U; a(X_y) \geqq k\}$. Then we have a descending sequence $U = A_0 \supseteqq A_1 \supseteqq \cdots A_k \supseteqq \cdots$ of subsets of U. Actually it is known that A_k is at most a countable union of analytic subvarieties of U whose closures in Y are analytic [18, Proposition 3].

Definition. We set $a(f) := \max\{k; A_k = U\}$ and call it the *relative algebraic dimension* of X over Y or simply an *algebraic dimension* of f. The definition is independent of the choice of U by the remark preceding to the definition. The remark also shows that $a(f) = k$ for an integer $k \geqq 0$ if and only if $a(X_y) = k$ for 'general' $y \in Y$. Clearly if f is locally Moishezon, then $a(f) = \dim f$.

Definition. Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Then a *relative algebraic reduction* of f is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle f_1}\searrow \quad \nearrow{\scriptstyle f_2} & \\ & X_1 & \end{array}$$

where X is a compact complex manifold in $\mathscr{C}$, f_1 is a meromorphic fiber space and f_2 is a (holomorphic) fiber space, such that 1) $a(f) = \dim f_2$ and 2) f_2 is locally Moishezon. We also say that $(f_1: X \to X_1, f_2: X_1 \to Y)$, or simply, f_1 itself, is a relative algebraic reduction of f.

Remark 2.1. 1) and 2) together are also equivalent to: For 'general' $y \in Y$ f_1 induces a meromorphic fiber space $f_{1,y}: X_y \to X_{1,y}$ which is an algebraic reduction of X_y.

In [18, Proposition 4] (cf. also Proposition 8) we have proved the following:

Proposition 2.1. *For any fiber space $f: X \to Y$ of compact complex manifolds in $\mathscr{C}$ a relative algebraic reduction of f exists and it is up to bimeromorphic equivalences uniquely determined by f.*

2.2. Now in general let X be a compact complex manifold in $\mathscr{C}$. Let $f: X \to Y$ be an algebraic reduction of X. By f) of Notations and Conventions we may assume that f is holomorphic. Let $(f_1: X \to X_1, f':$

$X_1 \to Y$) be a relative algebraic reduction of f. By the same convention as above we may assume that f is holomorphic. Then take a relative algebraic reduction (f_2: $X \to X_2$, f_2': $X_2 \to X_1$) of f_1 where we may assume that f_2 is holomorphic as above. Continuing analogously we finally obtain a commutative diagram of fiber spaces

$$(2) \qquad \begin{array}{ccccccccc} X & & & \xrightarrow{\qquad\qquad f \qquad\qquad} & & & & & Y \\ & \searrow^{f_m} & & \searrow & & \searrow^{f_1} & & & \| \\ & & X_m & \xrightarrow[f_m']{} \cdots \longrightarrow & X_2 & \xrightarrow[f_2']{} & X_1 & \xrightarrow[f_1']{} & Y \end{array}$$

where (f_i, f_i') are relative algebraic reductions of f_{i-1} for $1 \leqq i \leqq m$ ($f_0 = f$), $a(f_{i-1}) = \dim f_i' > 0$, $1 \leqq i \leqq m$, and $a(f_m) = 0$. Changing the notation we set $g = f_m$, $h = f_1' \cdots f_m'$ and $\bar{X} = X_m$. Then we get the following commutative diagram of fiber spaces

$$(3) \qquad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {}_g\searrow \quad \nearrow_h & \\ & \bar{X} & \end{array} \qquad a(g) = 0$$

which is up to bimeromorphic equivalences canonically associated to X. In fact there exists a characterization of (3) by a certain universal property (cf. Proposition 8.4 below). In any case the diagram (3) reduces in a certain extent the study of the structure of f to that of g and h. Note that from our construction we might say that $\bar{X}$ is composed of algebraic (Moishezon) manifolds (cf. 9.5), and its structure is expected to be very close to algebraic manifolds. In fact, the structure of (the fibers of) h turns out to be surprisingly simple.

Theorem 1. *The general fiber $\bar{X}_y$ of h is a holomorphic fiber bundle over its Albanese torus* Alb $\bar{X}_y$ *via the Albanese map α_y: $\bar{X}_y \to$* Alb $\bar{X}_y$ *whose typical fiber F_y is an almost homogeneous unirational Moishezon manifold. Moreover if* $\dim h > 0$ (*or equivalently* $a(f) > 0$), *then* $q(\bar{X}_y) = \dim \text{Alb}\, \bar{X}_y > 0$. *In particular* $0 < q(\bar{X}_y) \leqq \dim h$ *if* $\dim h > 0$.

Remark 2.2. A compact complex manifold is said to be unirational if it is a meromorphic image of some complex projective space $\boldsymbol{P}^N$. Thus it is necessarily Moishezon. Then F_y being 'almost homogeneous unirational' is equivalent to saying that there exists a linear algebraic group G_y acting holomorphically and algebraically on F_y with a (dense) Zariski open orbit (cf. [13]).

Example. 1) If $\dim h = 1$, $q(\bar{X}_y) = 1$ and $\bar{X}_y$ is an elliptic curve.

2) If dim $h=2$, $q(\bar{X}_y)=2$ or 1. If $q(\bar{X}_y)=2$, $\bar{X}_y$ is a complex torus and if $q(\bar{X}_y)=1$, $\bar{X}_y$ is a holomorphic $\boldsymbol{P}^1$-bundle over an elliptic curve.

2.3. Theorem 1 largely reduces the study of the structure of f to that of g. So we are led to the study of a fiber space $g: X \to Y$ with $a(g)=0$ in general. This includes as a special case the study of a compact complex manifold X with $a(X)=0$ (the case Y is a point). We start with this absolute case. In this case it turns out to be more reasonable to consider the Kummer reduction of X instead of the Albanese map (torus reduction) of X as in 1.2.

Definition. Let X be a compact complex manifold. Then a *Kummer reduction* is a meromorphic fiber space $\beta: X \to B$ over a Kummer manifold B such that if $\beta': X \to B'$ is any surjective meromorphic map of X onto a Kummer manifold B' there exists a unique meromorphic map $\gamma: B \to B'$ such that $\beta'=\gamma\beta$. Obviously a Kummer reduction is up to bimeromorphic equivalences unique if one exists. In this case we call $k(X) :=$ dim B the *Kummer dimension* of X.

Starting from Proposition 1.2 we can prove in Section 7 easily the following:

Proposition 2.2. *For any compact complex manifold X with* $a(X)=0$ (*not necessarily in* $\mathscr{C}$) *a Kummer reduction of X exists and it is unique up to bimeromorphic equivalences.*

As we shall see, even if we start from the absolute case it becomes necessary also to consider a relative version of the above proposition.

First we need the following:

Definition. Let $g: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$, with $a(g)=0$. Then a *relative Kummer reduction* of f is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ & {\scriptstyle\beta}\searrow \quad \nearrow{\scriptstyle h} & \\ & B & \end{array}$$

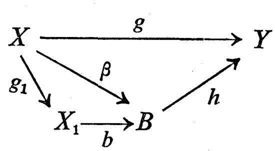
where B is a compact complex manifold, g is a meromorphic fiber space and h is a (holomorphic) fiber space such that for 'general' $y \in Y$, X_y is smooth, $a(X_y)=0$, and β induces a meromorphic fiber space $\beta_y: X_y \to B_y$ which is a Kummer reduction of X_y.

Then we can prove the following:

Proposition 2.3. *For any fiber space $g: X \to Y$ of compact complex manifolds in $\mathscr{C}$ with $a(g)=0$ a relative Kummer reduction of g exists and is unique up to bimeromorphic equivalences.*

In this case we call $k(f) := \dim h$ (where h is as in the above definition) the *relative Kummer dimension* of X over Y or the *Kummer dimension* of f.

2.4. Now in general suppose that we are given a fiber space $g: X \to Y$ of compact complex manifolds in $\mathscr{C}$ with $a(g)=0$. Take a relative Kummer reduction ($\beta: X \to B$, $h: B \to Y$) of g according to Proposition 2.3. As in f) of Notations and Conventions we may assume that β is holomorphic. Let ($g_1: X \to X_1$, $b: X_1 \to B$) be a relative algebraic reduction of β where we may assume that g_1 is holomorphic as above. (Note that the algebraic dimension $a(b)$ of b is positive in general, cf. Proposition 1.3). Then we obtain the following commutative diagram of fiber spaces

(4) 

canonically associated to g. Theorem 2 then concerns the structure of this diagram.

Theorem 2. *There exist Zariski open subsets $V \subseteqq X$, $U \subseteqq Y$ with $h(V) \subseteqq U$ such that* 1) *for any $y \in U$ $X_{1,y}$, B_y are both smooth and the induced morphism $b_y: X_{1,y} \to B_y$ is a holomorphic fiber bundle over the Zariski open subset $V_y \subseteqq B_y$ with typical fiber an almost homogeneous unirational Moishezon manifold, and* 2) $a(g_1)=k(g_1)=0$.

Thus the general fiber of $hb: X_1 \to Y$ is a fiber space over a Kummer manifold which is almost a holomorphic fiber bundle as is described in 1). So Theorem 2 reduces our original problem considerably to the study of a fiber space $g': X \to Y$ with $a(g')=k(g')=0$ in general. Note that in a special case where Y is a point, this amounts to considering the manifolds with $a(X)=k(X)=0$ (in particular $q(X)=0$).

In this case our method is to take a relative semi-simple reduction of g' (to be developed in the subsequent paper [20]) to obtain again a canonical decomposition of g'.

In any case we shall here remark that the same proof as for Theorem 2 gives also the following: Let $g': X \to Y$ be as above with $a(g')=k(g')$

$=0$. Let (β': $X \to B'$, $B' \to Y$) be any decomposition of f into two fiber spaces and (g_1': $X \to X_1'$, b': $X_1' \to B'$) be a relative algebraic reduction of β', so that we get a commutative diagram

$$(4)' \qquad \begin{array}{ccccc} X & & \xrightarrow{\quad g' \quad} & & Y \\ & {\scriptstyle g_1'}\searrow & & \nearrow{\scriptstyle h'} & \\ & & X_1' \xrightarrow{\ b'\ } B' & & \end{array}$$

analogous to (4). Then the same conclusion as 1) and 2) holds also for this diagram. In particular $a(g_1')=k(g_1')=0$ so that 'Kummer (in particular torus) part' never again appears in the study of such a morphism. Thus the reduction by Theorem 2 to the case $a(g')=k(g')=0$ mentioned above is in this very strong sense.

2.5. For later reference here we recall the existence theorem of a relative Albanese map for a locally Moishezon morphism and an immediate consequence of it which is of frequent use in this paper.

Definition. Let f: $X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Then a *relative Albanese map for f* is a commutative diagram

$$\begin{array}{ccccc} X & & \xrightarrow{\quad f \quad} & & Y \\ & {\scriptstyle \alpha}\searrow & & \nearrow{\scriptstyle \eta} & \\ & & \mathrm{Alb}^* X/Y & & \end{array}$$

where $A:=\mathrm{Alb}^* X/Y$ is a compact complex manifold in $\mathscr{C}$, η is a (holomorphic) fiber space, and α is a meromorphic map with the property that there exists a Zariski open subset $U \subseteqq Y$ such that η and f are both smooth over U and α induces a holomorphic map α_U: $X_U \to A_U$ with α_y: $X_y \to A_y$, $\alpha_y=\alpha_U|_{X_y}$, being an Albanese map for X_y for $y \in U$. We also call α itself the relative Albanese map for f.

The following results are shown in [18].

Theorem 2.4. *Let f: $X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Suppose that f is locally Moishezon. Then a relative Albanese map for f exists and it is unique up to bimeromorphic equivalence. Moreover α is a Moishezon map, i.e., any holomorphic model of α is Moishezon.*

As an immediate consequence of the last assertion we get the following:

Proposition 2.5. *Let f: $X \to Y$ be a fiber space of compact complex* *manifolds in $\mathscr{C}$. Suppose that f is locally Moishezon and $q(f)=0$. Then f is Moishezon.*

For a later purpose we also introduce the notion of relative algebraic Albanese map; let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$ and $(g: X \to X_1, h: X_1 \to Y)$ be a relative algebraic reduction of X. Let $(\alpha: X_1 \to \mathrm{Alb}^* X_1/Y, \eta: \mathrm{Alb}^* X_1/Y \to Y)$ be a relative Albanese map for h which is locally Moishezon. Then we shall call the composite map $\varphi = \alpha g: X \to \mathrm{Alb}^* X_1/Y$, or the pair $(\alpha g, \eta)$, a *relative algebraic Albanese map* for f.

When Y is a point, $\varphi: X \to \mathrm{Alb}\, X_1$ is simply called an algebraic Albanese map for X. An algebraic Albanese map has the universal property among the morphisms of X into an abelian variety in analogy with the usual Albanese map. In particular, if φ is smooth, the fiber of φ is connected. We set $a\text{-}q(X) = \dim \mathrm{Alb}\, X_1$ and call it the algebraic irregularity of X. For a fiber space $f: X \to Y$ as above we define *the algebraic irregularity $a\text{-}q(f)$* of f by $a\text{-}q(f) := a\text{-}q(X_y)$ for 'general' $y \in Y$.

§ 3. A preliminary proposition

The purpose of this section is to prove Proposition 3.2 below.

3.1. *Terminology.* Let Y be a complex space and X, X' be complex spaces over Y with X reduced. Let $Z \subseteq X \times_Y X'$ be a subspace. Then by Frisch (cf. [9, 3.18]) there exists a dense Zariski open subset $U \subseteq X$ such that Z is flat over U. Let $\tau_0: U \to D_{X'/Y}$ be the associated universal morphism into the relative Douady space $D_{X'/Y}$. Then τ_0 extends to a unique meromorphic map $\tau: X \to D_{X'/Y}$ which is independent of the choice of U as above (cf. [12, Lemma 5.1]). We call τ the *universal meromorphic map associated to the inclusion $Z \subseteq X \times_Y X'$.*

Lemma 3.1. *Let $f: X \to Y$, $f': X' \to Y$ be fiber spaces of complex varieties. Let $g: X \to X'$ be a surjective meromorphic Y-map. Then there exists a unique compact subvariety $X'_0 \subseteq D_{X/Y}$ such that the universal subspace $Z_0 \subseteq X \times_Y X'_0$ restricted to X'_0 is the graph of a meromorphic Y-map $g_0: X \to X'_0$ which is bimeromorphic to g.*

Proof. Let $\Gamma \subseteq X \times_Y X'$ be the graph of g. Let $\tau: X' \to D_{X/Y}$ be the universal meromorphic Y-map defined by Γ. Let $X'_0 = \tau(X')$. We show that τ gives a bimeromorphic map of X' onto X'_0. Let $V \subseteq \Gamma$ be a Zariski open subset such that the natural projection $q: \Gamma \to X$ gives an isomorphism of V and $q(V)$. Let $U \subseteq X'$ be a Zariski open subset such that the natural projection $p: \Gamma \to X'$ is flat over U and that $\Gamma_{x'}$ is the closure of $V_{x'}$ for $x' \in U$. Then it is immediate to see that τ is holomorphic and injective on U. So τ is bimeromorphic and the following commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\tilde{\tau}} & Z_0 \\ {\scriptstyle p}\downarrow & & \downarrow \\ X' & \xrightarrow{\tau} & X'_0 \end{array}$$

gives a bimeromorphic map of Γ and Z over τ. Hence $Z_0 \subseteqq X \times_Y X'_0$ gives a meromorphic map bimeromorphic to f. Next we show the uniqueness. Let $X''_0 \subseteqq D_{X/Y}$ be another compact subvariety having the same property as X'_0. Let $g'_0: X \to X''_0$ be the associated meromorphic map. There exists a bimeromorphic Y-map $\tau': X' \to X''_0$ with $g'_0 = \tau' g$. Then returning to the construction of τ above we see immediately that the image of τ is in fact X''_0 and $\tau' = \tau$. q.e.d.

We call g obtained in the lemma the canonical model of g. Of course g_0 depends only on the bimeromorphic equivalence class of g (with X fixed). It is easy to see that in the situation of the above lemma there exists a Zariski open subset $U \subseteqq Y$ such that for any $y \in U$ g_0 (resp. g) induces a meromorphic map $g_{0,u}: X_{0,u} \to X'_{0,u}$ (resp. g_u; $X_u \to X'_u$) with $g_{0,u}$ the canonical model of g_u.

3.2. Let $f: X \to Y$ be a fiber space of complex spaces. Let $M \subseteqq Y$ be a subset. Suppose that for each $y \in M$ we are given a surjective meromorphic map $\psi_y: X_y \to \overline{X(y)}$. We set $\mathfrak{S} = \{\psi_y\}_{y \in M}$ and call $\mathfrak{S}$ a family of meromorphic maps parametrized by M. Let $\nu: \tilde{Y} \to Y$ be any morphism. Then we set $\tilde{M} = \nu^{-1}(M)$ and $\mathfrak{S}_{\tilde{Y}} = \{\psi_{\tilde{y}}\}_{\tilde{y} \in \tilde{M}}$ where $\psi_{\tilde{y}} = \psi_{\nu(\tilde{y})}: (X \times_Y \tilde{Y})_{\tilde{y}} = X_y \to \overline{X(y)}$, $y = \nu(\tilde{y})$. $\mathfrak{S}_{\tilde{Y}}$ is called the pull-back of $\mathfrak{S}$ to $\tilde{Y}$. In particular for any open subset $W \subseteqq Y$ we can speak of the restriction $\mathfrak{S}_W$ of $\mathfrak{S}$ to W.

Let $\mathfrak{S} = \{\psi_y\}_{y \in M}$ be as above. Then we say that f is *good with respect to* $\mathfrak{S}$ if there exist a subset $N_0 = N_0(f, \mathfrak{S}) \subseteqq M \subseteqq Y$ with $N_0 \in \mathfrak{Q}(Y)$ (cf. Notation) and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \bar{X} \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & Y & \end{array}$$

where $\bar{X} \to Y$ is a fiber space and φ is a meromorphic Y-map, such that for any $y \in N_0$, φ induces a meromorphic map $\varphi_y: X_y \to \bar{X}_y$ which is bimeromorphic to ψ_y. In this case we call φ a good meromorphic map with respect to $\mathfrak{S}$. We say that f is *very good with respect to* $\mathfrak{S}$ if we can take φ to be holomorphic in the above diagram.

Proposition 3.2. *Let $f: X \to Y$ be a fiber space of compact complex varieties in $\mathscr{C}$. Let $U \subseteqq Y$ be a Zariski open subset. Let $M \subseteqq Y$ be a subset and $\mathfrak{S} = \{\psi_y\}_{y \in M}$ be a family of meromorphic maps parametrized by M.*

1) *Suppose that for any $y \in Y$ there exist a neighborhood $y \in N$ and a finite covering $n: \tilde{N} \to N$ such that n is isomorphic if $y \in U$ and that $f_{\tilde{N}}: X_{\tilde{N}} \to \tilde{N}$ is good with respect to $\mathfrak{S}_{\tilde{N}}$. Then there exist a fiber space $h: \bar{X} \to Y$ and a surjective meromorphic Y-map $\varphi: X \to \bar{X}$ such that for any $\tilde{N} \cong N$ with $N \subseteqq U$ as above $\varphi_{\tilde{N}}: X_{\tilde{N}} \to \bar{X}_{\tilde{N}}$ is bimeromorphic to some good meromorphic map $\varphi(\tilde{N})$ with respect to $\mathfrak{S}_{\tilde{N}}$, which exists by our assumption on $f_{\tilde{N}}$.*

2) *Suppose that for any $y \in U$ there exists a neighborhood $y \in N$ such that $f_N: X_N \to N$ is very good with respect to $\mathfrak{S}_N$. Then there exist $h: \bar{X} \to Y$ and $\varphi: X \to \bar{X}$ as in* 1) *such that for any N as above $\varphi_N: X_N \to \bar{X}_N$ is bimeromorphic to a good meromorphic map $\varphi(N)$ with respect to $\mathfrak{S}_N$.*

Proof. Suppose first that we are under the assumption of 1). Let $y \in U$ and $y \in N$ be as in 1). Let $\varphi(N): X_N \to \bar{X}(N)$ be a good meromorphic map with respect to $\mathfrak{S}_N$. Let $\tilde{\varphi}(N): X_N \to B(N) \subseteqq D_{X_N/N}$ be the canonical model of $\varphi(N)$. Let N' be the Zariski open subset of N such that $\varphi_u: X_u \to X'_u$, $\tilde{\varphi}_u: X_u \to B_u$ are meromorphic maps and $\tilde{\varphi}_u$ is the canonical model of φ_u for $u \in N'$ where $\tilde{\varphi}_u = \tilde{\varphi}(N)_u$ and $\varphi_u = \varphi(N)_u$. Then for each $u \in N' \cap N_0 \in \mathfrak{Q}(N)$, $\tilde{\varphi}_u$ is the canonical model of $\psi(u)$ where $N_0 = N_0(f_N, \mathfrak{S}_N)$. Let $y_1 \in N_1$ be another such point and its neighborhood. Let $\tilde{\varphi}(N_1): X_{N_1} \to B(N_1) \subseteqq D_{X_{N_1}/N_1}$ be the canonical model of φ_{N_1}. Then $B(N)$ and $B(N_1)$ coincide over $N \cap N_1$ as a subspace of $D_{X_{N \cap N_1}/N \cap N_1}$. (Note that for any open subset $W \subseteqq Y$, $D_{X_W/W}$ is naturally identified with an open subset of $D_{X/Y}$.) In fact, let $M = N' \cap N'_1 \cap N_0 \cap (N_1)_0 \in \mathfrak{Q}(N \cap N_1)$ where N'_1 is defined as N' and $(N_1)_0 = N_0(f_{N_1}, \mathfrak{S}_{N_1})$. Then for any $u \in M$, $B(N)_u = B(N_1)_u$ in D_{X_u} since both are the image of the canonical model of $\psi(u)$. Hence $B(N)$ must coincide with $B(N_1)$ over $N \cap N_1$. This already implies that there exists an analytic subvariety $B(U) \subseteqq D_{X_U/U}$ such that for each N as above, $B(U)|_N = B(N)$. Let $Z(U)$ $(\subseteqq X_U \times_U B(U)) \to B(U)$ be the universal family restricted to $B(U)$. Then the natural projection $Z(U) \to X_U$ is bimeromorphic since it is so over each N as above. Hence $Z(U)$ defines a meromorphic map $\tilde{\varphi}(U): X_U \to B(U)$ over U which is bimeromorphic to $\varphi(N)$ on each N.

Suppose now that f are even very good with respect to $\mathfrak{S}_N$. In this case each $B(N)$ turns out to be an irreducible component of $D_{X_N/N}$ (cf. the proof of [18, Lemma 9]). Hence $B(U)$ also is an irreducible component of $D_{X_U/U}$. Let B be the irreducible component of $D_{X/Y}$ which restricts to $B(U)$. B is proper over Y since $X\in\mathscr{C}$. Then by the same argument as above we get a meromorphic Y-map φ: $X\to B$ which restricts to $\varphi(U)$ over U. This shows 2).

To finish the proof of 1) it suffices as above to show that the closure B of $B(U)$ in $D_{X/Y}$ is analytic. The problem is local with respect to Y. Namely we have only to show that if $y\in Y$ is any point and if $y\in N$ is as in 1) then the closure $\bar{B}_{U\cap N}$ of $B_{U\cap N}$ in $D_{X_N/N}$ is analytic. Let $\tilde{\varphi}(\tilde{N})$: $X(\tilde{N})\to B(\tilde{N})\subseteq D_{X_{\tilde{N}}/\tilde{N}}$ be the canonical model of a good meromorphic map $\varphi(\tilde{N})$. Set $\tilde{U}=n^{-1}(U\cap N)$. Then we have $(B(\tilde{N}))_{\tilde{U}}=B_{U\cap N}\times_U\tilde{U}$ in $D_{X_{\tilde{U}}/\tilde{U}}$ with respect to the natural identification $D_{X_{\tilde{U}}/\tilde{U}}=D_{X_{U\cap N}/U\cap N}\times_U\tilde{U}$. In fact, for 'general' $\tilde{u}\in\tilde{N}$, $B(\tilde{N})_{\tilde{u}}=B_{U\cap N,u}$ in $D_{X_{\tilde{u}}}=D_{X_u}$ since both are the image of the canonical model of $\psi(u)$ in D_{X_u} where $u=n(\tilde{u})$. Hence $B(\tilde{N})$ is the closure of $B_{U\cap N}\times_U\tilde{U}$ in $D_{X_{\tilde{U}}/\tilde{U}}$. It follows that $\bar{B}_{U\cap N}$ is the image of $B(\tilde{N})$ by the finite morphism $D_{X_{\tilde{N}}/\tilde{N}}=D_{X_N/N}\times_N\tilde{N}\to D_{X_N/N}$ and hence is analytic. q.e.d.

§ 4. Consequences of the Hodge decomposition

In this section we derive two important consequences of the Hodge decomposition (1). (See Propositions 4.1 and 4.5 below.)

4.1. *Period map.* a) Let f: $X\to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $U\subseteq Y$ be a Zariski open subset over which f is smooth. Take a compact Kähler manifold Z and a surjective morphism h: $Z\to X$. Fix a Kähler class ω on Z. Let $V\subseteq U$ be a Zariski open subset of Y over which fh is smooth. Given these data we can construct naturally a variation of real Hodge structure over V in analogy with the case of polarized family of algebraic manifolds (Griffiths [24]). However, here we shall explain this only in the simplest case of weight $k=1$ since it is the only case we need in this paper. (In this case, the situation becomes much simpler since we need not take account of primitive classes, and the variation is actually defined over U.) We refer for the precise definition of variation of Hodge structure to Schmid [38, p. 220], and follow the notations there. Then in the notations there we set

a) $M=U$

b) $\boldsymbol{H}_{\boldsymbol{K}}=R^1f_{U*}\boldsymbol{K}$, $\boldsymbol{K}=\boldsymbol{Z}$, $\boldsymbol{R}$ or $\boldsymbol{C}$ which is a local system on U, and

c) $k=1$. Moreover,

d) a flat nondegenerate bilinear form S on $\boldsymbol{H}_{\boldsymbol{R}}$ is defined as follows;

for $y\in V$ and α, $\beta\in\boldsymbol{H}_{\boldsymbol{R},y}=H^1(X_y,\boldsymbol{R})$ we set $S_y(\alpha,\beta)=\int_{Z_y}\omega_y^{r-1}\wedge h_y^*\alpha\wedge h_y^*\beta$ where h: $Z_y\to X_y$, ω_y is the restriction of ω to Z_y and $r=\dim fh=$

dim Z_y. Since $h_y^*: H_1(X_y, \boldsymbol{R}) \to H^1(Z_y, \boldsymbol{R})$ is injective (Lemma 1.4), S_y defines a non-degenerate skew-symmetric form on $\boldsymbol{H}_{\boldsymbol{R},y}$. Moreover since $S_V = \{S_y\}_{y \in V}$ is flat and $\boldsymbol{H}_{\boldsymbol{R}}|_V$ extends to a local system on U, S_V also extends to a unique flat bilinear form S on $\boldsymbol{H}_{\boldsymbol{R}}$ over the whole U.

e) A holomorphic subbundle $\boldsymbol{F}_y^1 \subseteq \boldsymbol{H}_{\boldsymbol{C}}$ is defined as usual by $\boldsymbol{F}_y^1 := H^{1,0}(X_y) \subseteq \boldsymbol{H}_{\boldsymbol{C},y} = H^1(X_u, \boldsymbol{C})(\boldsymbol{F}^0 = \boldsymbol{H}_{\boldsymbol{C}}$ and $\boldsymbol{F}^2 = \{0\})$ where $\boldsymbol{H}_{\boldsymbol{C}}$ is considered as a holomorphic vector bundle with constant transition functions.

It is immediate to see that these data, denoted symbolically by $(Y, \boldsymbol{H}_{\boldsymbol{C}}, \boldsymbol{F}^p)$, actually satisfy the conditions i) and ii) of [39]. An important remark, however, is that in the definition of variation of Hodge structure we do not here require the bilinear form S to take rational values on $\boldsymbol{H}_{\boldsymbol{Z}}$. So we call $(U, \boldsymbol{H}_{\boldsymbol{C}}, \boldsymbol{F}^p)$ the variation of *real* Hodge structure.

b) Now to any such variation of Hodge structure $\{U, \boldsymbol{H}_{\boldsymbol{C}}, \boldsymbol{F}^p\}$ we can associate just as in Griffiths [24], or [38] the period map $\Phi: U \to D/\Gamma$ where D is the corresponding period matrix domain, i.e., the classifying space of Hodge structures (in our case of $k=1$ D is isomorphic to the Siegel upper half space, cf. [24, § 1]) and Γ is a discrete group acting properly discontinuously on D. More precisely, let $H_{\boldsymbol{K}}$, $\boldsymbol{K} = \boldsymbol{Z}$ or $\boldsymbol{R}$, be the fiber of the canonical Hodge bundle on D at $o \in D$ (cf. [38, § 3]). Let $G_{\boldsymbol{K}} := \{g \in GL(H_{\boldsymbol{K}}); S(gu, gv) = S(u, v)\}$, S being the corresponding bilinear form on $H_{\boldsymbol{K}}$. Then $G_{\boldsymbol{R}}$ acts transitively on D with compact stabilizer at o. Thus if $\rho: \pi_1(Y, o) \to G$ is the monodromy representation with image Γ, then Γ clearly is discrete and act properly discontinuously on D. Here an important thing to note is that since S is not required to take rational values on $H_{\boldsymbol{Z}}$, Γ is in general not arithmetic. For instance D/Γ has in general no compactification like the Baily-Borel compactification in the arithmetic case.

4.2. *Period map and algebraic reduction.* Roughly speaking we shall show that the period map constructed in 3.1 associated to $f: X \to Y$ factors through an algebraic reduction of Y at least when D is a bounded symmetric domain. To be more precise we make the following:

Definition. Let Y be a compact complex variety and U a Zariski open subset of Y. Let $\Phi: U \to Z$ be a morphism of U into a complex space Z. We say that Φ is *generically factored by an algebraic reduction* of Y if for any algebraic reduction $f: Y \to \bar{Y}$ of Y (defined in the same way as the smooth case), there exist a Zariski open subset V of Y contained in U, a Zariski open subset W of $\bar{Y}$ and a morphism $\bar{\Phi}: W \to Z$ such that f is defined on V, $f(V) \subseteq W$ and $\bar{\Phi} f|_V = \Phi|_V$.

Then we shall prove the following:

Proposition 4.1. *Let Y be a compact complex manifold and U a Zariski open subset of Y parametrizing an (abstract) variation of real Hodge structure. Let D (resp. $\Gamma \subseteqq G_Z$) be the associated period matrix domain (resp. a discrete group) and Φ: $U \rightarrow D/\Gamma$ be the period map. If D is a bounded symmetric domain, then Φ is generically factored by an algebraic reduction of Y.*

Remark 4.1. If Γ is arithmetic, this is well-known (Borel, Kobayashi-Ochiai). Our main interest is in the case of non-algebraic family of complex tori and $K3$ surfaces.

We first prove the following:

Lemma 4.2. *Let f: $U \rightarrow \bar{U}$ and Φ: $U \rightarrow Z$ be morphisms of normal complex varieties. Suppose that f is open and there exists a map $\bar{\Phi}$: $\bar{U} \rightarrow Z$ such that $\bar{\Phi} f = \Phi$. Then $\bar{\Phi}$ is holomorphic.*

Proof. Since f is open, $\bar{\Phi}$ is continuous. Take any $\bar{u} \in \bar{U}$ and $u \in U$ with $f(u) = \bar{u}$. Since f is open, $\dim_u f^{-1} f(u)$ is independent of $u \in U$ (cf. [9, 3.10]). Hence we can find an analytic subvariety B defined in a neighborhood of u and passing through u such that the induced map $(B, u) \rightarrow (U, u)$ of germs is finite and surjective (cf. [9, 3.7]). Since the problem is local, replacing U by B, Φ by $\Phi|_B$ etc. we may assume from the beginning that f is finite and surjective. Then f is locally biholomorphic on a dense Zariski open subset of U. The lemma immediately follows from this and the Riemann extension theorem in view of the normality of U.

Proof of Proposition 4.1. Since $\Gamma \subseteqq G_Z$, there exists a torsion free subgroup $\Gamma' \subseteqq \Gamma$ of finite index [4, p. 118]. In particular the action of Γ' on D is free. Let ρ: $\pi_1(U) \rightarrow \Gamma \subseteqq G_Z$ be the monodromy representation where $\pi_1(U)$ is the fundamental group of U with respect to some reference point. Let π: $U' \rightarrow U$ be the finite unramified covering of U corresponding to $\rho^{-1}(\Gamma') \subseteqq \pi_1(U)$. Extend π to a finite covering π': $Y' \rightarrow Y$ by a theorem of Grauert and Remmert where Y' is a normal complex variety containing U' as a dense Zariski open subset. Then pulling back the variation of Hodge structure to U' we obtain the associated period map Φ'; $U' \rightarrow D/\Gamma'$ such that $\Phi \pi = \Phi' \tilde{\omega}$ where $\tilde{\omega}$: $D/\Gamma' \rightarrow D/\Gamma$ is the natural projection.

We first show that it suffices to show that Φ' is generically factored by an algebraic reduction of Y'. In fact let f: $Y \rightarrow \bar{Y}$ be an algebraic reduction of Y. Passing to a suitable bimeromorphic model of Y we may assume that f is holomorphic. Let $f\pi' = \bar{\pi}' f'$, (f': $Y' \rightarrow \bar{Y}'$, $\bar{\pi}'$: $\bar{Y}' \rightarrow \bar{Y}$), be the Stein factorization of $f\pi'$. We see readily that f' is an algebraic reduction of Y' with $\bar{Y}'$ normal and the natural map F: $Y' \rightarrow \bar{Y}' \times_{\bar{Y}} Y$ is surjective. Suppose now that there exist Zariski open subsets $W'\subseteqq \bar{Y}'$, $V'\subseteqq U'\subseteqq Y'$ and a holomorphic map $\bar{\Phi}'$: $W'\to D/\Gamma'$ such that $f'(V')\subseteqq W'$, and $\bar{\Phi}'f'=\Phi'$ on V'. Combining this with the relation $\tilde{\omega}\Phi'=\Phi\pi$ together with the surjectivity of F it follows that $\tilde{\omega}\bar{\Phi}'$ and Φ coincide when they are considered as holomorphic maps from $V\times_W W'$ via the natural projections $V\times_W W'\to W'$ and $V\times_W W'\to V$ where $V=\pi(V')\subseteqq Y$ and $W=\pi'(W')\subseteqq \bar{Y}$. Then we see readily that if we restrict V' and W' $\Phi|_V$ is factored by some holomorphic map $V\to W$ (cf. Lemma 4.2), and hence Φ is factored generically by an algebraic reduction of Y. Thus we may assume from the beginning that the action of Γ is free so that D/Γ is a manifold.

Now since D is a bounded symmetric domain, by a theorem of Borel [5], for each $q\in D/\Gamma$ there exist meromorphic functions $g_1,\cdots,g_m$, $m=\dim D$, on D/Γ such that g_i are holomorphic at q, $g_i(q)=0$, and give local coordinates of D/Γ at q. Moreover g_i can be expressed as a quotient of two holomorphic sections of $K_{D/\Gamma}^{\otimes b}$ for some sufficiently large b where $K_{D/\Gamma}$ is the canonical bundle of D/Γ. On the other hand, as was shown by Sommese [40, p. 254ff] $\Phi^*K_{D/\Gamma}$ extends to a holomorphic line bundle L on Y (after passing to a suitable bimeromorphic model of Y), and moreover the pullback to $\Phi^*K_{D/\Gamma}$ of the canonical metric of $K_{D/\Gamma}$ induced by a G-invariant metric of K_D has L^2-poles at infinity in the sense of Sommese [39], so that for any holomorphic section h of $K_{D/\Gamma}^{\otimes b}$, its pull-back Φ^*h extends to a meromorphic section of $L^{\otimes b}$ on Y (cf. [39, Lemma I-F]). In particular for the above g_i, Φ^*g_i extends to a meromorphic function on the whole Y.

From this we can deduce the proposition as follows. Let f: $Y\to\bar{Y}$ be any algebraic reduction of Y which we may assume to be holomorphic as above. Restricting U if necessary we may assume that $f|_U$ is an open map and that $U\cap f^{-1}(\bar{y})$ is irreducible for $\bar{y}\in\bar{Y}$. Let $\bar{U}=f(U)$. Then by Lemma 4.2 it suffices to show that for any $\bar{u}\in\bar{U}$, Φ maps $U_{\bar{u}}:=U\cap f^{-1}(\bar{u})$ to a point. Take any $u\in U_{\bar{u}}$ and let $q=\Phi(u)$. Let $\tilde{g}_i=\Phi^*g_i$ with g_i as above. Then Φ is given locally at u by m meromorphic functions $\tilde{g}_1,\cdots,\tilde{g}_m$ which are holomorphic at u. On the other hand, since $\tilde{g}_i$ extends to a meromorphic function of Y as we have remarked above, by the definition of f we can find meromorphic functions $\bar{g}_i$ on Y such that $f^*\bar{g}_i=\tilde{g}_i$. This implies that $U_{\bar{u}}$ is contained in the fiber $\Phi^{-1}(q)$ near u and hence in the whole U since $U_{\bar{u}}$ is irreducible. q.e.d.

4.3. As another consequence of the Hodge decomposition (1) we obtain a result which relates the irregularities of the total space, the base space and of the general fiber of a given fiber space (Proposition 4.5 below). First we remark that in view of the Hodge decomposition and the functoriality of the class $\mathscr{C}$ it follows that Deligne's theory of mixed Hodge structure [8] is applicable with obvious modifications to Zariski open subsets of compact complex manifolds in $\mathscr{C}$. Especially the following proposition corresponding to Théorème 4.1.1 (ii) of [8] is important for us.

Proposition 4.3. *Let $f\colon X\to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $U\subseteqq Y$ be a Zariski open subset over which f is smooth. Then the composite map $H^k(X, \boldsymbol{C})\to H^k(X_U, \boldsymbol{C})\to \Gamma(U, R^k f_{U*}\boldsymbol{C})$ is surjective for any k.*

Proof. When X is Kähler, the Leray spectral sequence $E_2^{p,q} := H^p(X_U, R^q f_{U*}\boldsymbol{C})\Rightarrow H^{p+q}(X, \boldsymbol{C})$ degenerates at $E_2^{p,q}$ terms [8, 2.6.2]. Further for any compact smooth subspace $B\subseteqq X_U$ the images of $H^k(X_U, \boldsymbol{C})$ and $H^k(X, \boldsymbol{C})$ in $H^k(B, \boldsymbol{C})$ coincide (cf. [8, 3.2.18]). From this the proposition follows as in [8]. In the general case take a compact Kähler manifold Z and a surjective holomorphic map $h\colon Z\to X$. As in the proof of [8, 4.1.1] we may assume that $g=fh$ is smooth over U. Then the rest of the argument is the same as in [8] where the general case of complete varieties is reduced to the projective case.

We shall apply the above proposition in the following situation. Let $f\colon X\to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $U\subseteqq Y$ be a Zariski open subset over which f is smooth. Then from the Leray spectral sequence we get the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(Y, \boldsymbol{C}) & \longrightarrow & H^1(X, \boldsymbol{C}) & \xrightarrow{\lambda} & H^0(Y, R^1 f_*\boldsymbol{C}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow u & & \\ 0 & \longrightarrow & H^1(U, \boldsymbol{C}) & \longrightarrow & H^1(X_U, \boldsymbol{C}) & \xrightarrow{\lambda_U} & H^0(U, R^1 f_{U*}\boldsymbol{C}) & \longrightarrow & 0 \end{array}$$

where the surjectivity of λ_U (resp. λ) follows from Proposition 4.3 (resp. Lemma 1.4).

Lemma 4.4. *u is isomorphic. In particular if $R^1 f_{U*}\boldsymbol{C}$ is a constant system, then $q(X)=q(Y)+q(f)$.*

Proof. By Proposition 4.3 $u\lambda$ is surjective. Hence we have only to show that u is injective. Note first that the first two vertical arrows are injective since $H^1_A(Y, \boldsymbol{C})=H^1_{\tilde{A}}(X, \boldsymbol{C})=0$ where $A=Y-U$ and $\tilde{A}=X-X_U$. So we regard these as inclusions as well as the first two horizontal arrows. The above diagram then shows that the injectivity of u is equivalent to the equality $H^1(X, \boldsymbol{C})\cap H^1(U, \boldsymbol{C})=H^1(Y, \boldsymbol{C})$ in $H^1(X_U, \boldsymbol{C})$. Now take any holomorphic 1-form ω on X with $\lambda_U(\omega)=u\lambda(\omega)=0$, where ω is identified with the cohomology class it defines. This implies that ω, restricted to each fiber X_u, $u\in U$, vanishes identically. It follows that there exists a holomorphic 1-form $\bar{\omega}_U$ on U such that $\omega_U = f^* \bar{\omega}_U$. Then by Mabuchi [35, Cor. 2.2.3], $\bar{\omega}_U$ extends to a holomorphic 1-form $\bar{\omega}$ on Y such that $\omega = f^* \bar{\omega}$. Taking complex conjugate we also see that any anti-holomorphic 1-form ω' with $\lambda_U(\omega')=0$ is a pull-back of an anti-holomorphic 1-form on Y. Since $H^1(U, \boldsymbol{C}) \cap H^1(X, \boldsymbol{C})$ is a sub-Hodge structure of $H^1(X, \boldsymbol{C})$ [8], any element of it is expressed as a sum of holomorphic 1-form and an anti-holomorphic 1-form belonging to it. Thus by what we have shown above, we get $H^1(U, \boldsymbol{C}) \cap H^1(X, \boldsymbol{C}) = H^1(Y, \boldsymbol{C})$ as was desired. The last assertion then follows from the equalities $\dim H^0(U, R^1 f_* \boldsymbol{C}) = \dim H^1(X_u, \boldsymbol{C})$, $u \in U$, and $b_1(Z) = 2q(Z)$ for all $Z \in \mathscr{C}$. q.e.d.

Now we come to the main assertion of this subsection.

Proposition 4.5. *Let $f\colon X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $U \subseteqq Y$ be a Zariski open subset over which f is smooth. Suppose that the period map $\Phi\colon U \to D/\Gamma$ associated to f_U as in* 3.1 *is constant. Then there exist a normal compact complex variety $\tilde{Y}$ and a finite covering $\nu\colon \tilde{Y} \to Y$ which is unramified over U such that if $\tilde{X} := X \times_Y \tilde{Y}$ then $q(\tilde{X}) = q(f) + q(\tilde{Y})$.*

Proof. Our assumption is equivalent to saying that the variation of Hodge structure $(U, \boldsymbol{H}_{\boldsymbol{C}}, \boldsymbol{F}^p)$ associated to f_U is locally constant, i.e., a holomorphic fiber bundle over U. Let $\hat{\Phi}\colon \hat{U} \to D$ be any lift of Φ to $\hat{U}$ where $\hat{U}$ is the universal covering space of U (cf. [25, Lemma 9.6]). $\hat{\Phi}$ also is a constant map. Let $\hat{u} = \hat{\Phi}(\hat{U})$. Then the structure group of this bundle is clearly contained in the stabilizer $\Gamma_{\hat{u}}$ of Γ at $\hat{u}$, which is finite. Hence there exists a finite unramified covering $\nu_0\colon \tilde{U} \to U$ such that $R^1 f_{\tilde{U}*} \boldsymbol{R} = R^1 f_{U*} \boldsymbol{R} \times_U \tilde{U}$ is a constant system on $\tilde{U}$, where $f_{\tilde{U}}\colon X \times_U \tilde{U} \to \tilde{U}$ is the natural morphism. Let $\nu\colon \tilde{Y} \to Y$ be the unique finite covering which completes ν_0 over Y with $\tilde{Y}$ normal. Let $\tilde{f}_1\colon \tilde{X}_1 \to \tilde{Y}_1$ be any nonsingular model of $\tilde{X} \to \tilde{Y}$. Then by Lemma 4.4 we have $q(\tilde{X}) = q(\tilde{X}_1) = q(\tilde{f}_1) + q(\tilde{Y}_1) = q(f) + q(\tilde{Y})$. q.e.d.

As an easy application we note the following:

Proposition 4.6. *Let $\beta\colon X \to T$ be a smooth fiber space of compact complex manifolds in $\mathscr{C}$. Suppose that T is a complex torus and every fiber of β is a complex torus. Then X is hyperelliptic, i.e., it is isomorphic to a complex torus divided by a finite group acting fixed point freely on X.*

Proof. By the construction in 4.1 the period map $\Phi\colon T \to D/\Gamma$ is defined on the whole T. Then by Griffiths-Schmid [25, Cor. 9.7] Φ must be constant. By Proposition 4.5 there exists a finite unramified Galois covering $\tilde{T}\to T$ such that if we set $\tilde{X}:=X\times_T\tilde{T}$ then $q(\tilde{X})=q(\tilde{T})+q(\beta)=\dim\tilde{T}+\dim\beta=\dim\tilde{X}$. Moreover the proof shows that in the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\quad\tilde{\alpha}\quad} & \mathrm{Alb}\ \tilde{X} \\ & {}_{\beta}\searrow \quad \nearrow & \\ & \tilde{T}=\mathrm{Alb}\,\tilde{T} & \end{array}$$

$\tilde{\alpha}$ induces on each fiber $\tilde{X}_t$ of β a morphism $\tilde{\alpha}_t$: $\tilde{X}_t\to(\mathrm{Alb}\,\tilde{X})_t$ which is isogenous. This implies that $\tilde{\alpha}$ is unramified and hence $\tilde{X}$ is a complex torus. Therefore X is hyperelliptic. q.e.d.

§ 5. Structure of a projective morphism

In the statement of both Theorems 1 and 2 we have encountered a holomorphic fiber bundle with an almost homogeneous unirational Moishezon manifold as a typical fiber. This is of course not accidental; in this section we give some results on the structure of a projective morphism with respect to an algebraic reduction of its base space, which leads to the structure as above. These will be given in Propositions 5.1, 5.2, 5.3 below and will play an important role in this paper.

5.1. The first one is the following:

Proposition 5.1. *Let $g: X\to X_1$, $h: X_1\to Y$ be fiber spaces of compact complex manifolds in $\mathscr{C}$. Let $A=(A_1, \cdots, A_m)$ be a sequence of analytic subspaces of X. Suppose that g is projective and $a(X_1)=a(Y)$. Then there exist Zariski open subsets $U\subseteqq Y$, $V\subseteqq X_1$ with $h(V)\subseteqq U$ such that for any $u\in U$, X_u, X_{1u} are both smooth and $g_u:(X_u, A_u)\to X_{1u}$ is a holomorphic fiber bundle over V_u (in the obvious sense), having a linear algebraic group as a structure group.*

Proof. Note first that we may clearly assume that all the irreducible components of A_i are mapped surjectively onto X_1 by g. Note next that by the form of the statement of the proposition it suffices to show it after passing to a suitable bimeromorphic model of ($g: X\to X_1$, $h: X_1\to Y$). Then since g is projective, passing to another bimeromorphic model we may assume that there exists a holomorphic vector bundle E on X such that X is a subspace of the associated projective bundle η: $\boldsymbol{P}(E)\to X_1$. In fact, we can write $X\subseteqq P(\mathscr{F})$ for some coherent analytic sheaf $\mathscr{F}$ on X where $P(\mathscr{F})$ is the projective variety associated to $\mathscr{F}$ (cf. [26, V] and [9, 1.9]). Then we take a proper modification σ: $X_1'\to X_1$ such that $\sigma^*\mathscr{F}$ admits a locally free quotient $\mathscr{E}'$ with a torsion kernel. Then the strict transform

X' of X in $X\times_{X_1}X_1'$ is naturally embedded in $P(\mathscr{E}')=\boldsymbol{P}(E)$ where E is the dual of E'. Then taking a suitable nonsingular holomorphic model of $(\sigma^{-1}g\colon X\to X_1', h\sigma\colon X_1'\to Y)$ we get a situation as above.

Let $N+1=\operatorname{rank} E$. Let $\mu\colon P=\operatorname{Isom}_{X_1}(\boldsymbol{P}^N\times X_1, \boldsymbol{P}(E))\to X_1$ be the principal bundle with group PGL $(N+1, \boldsymbol{C})$, associated to η. Then there exists a natural trivialization $\boldsymbol{P}(E)\times_{X_1}P\cong P\times\boldsymbol{P}^N$ of the induced bundle $\boldsymbol{P}(E)\times_{X_1}P\to P$. Let $A_0=X$. Then we have the natural inclusions $A_i\times_{X_1}P\subset P\times\boldsymbol{P}^N$, $0\leqq i\leqq m$. Let $\tau_i\colon P\to D_{\boldsymbol{P}^N}$ be the universal meromorphic map associated to this inclusion, where $D_{\boldsymbol{P}^N}$ is the Douady space of $\boldsymbol{P}^N$ (cf. § 3). On the other hand, μ has the natural compactification $\bar{\mu}\colon \bar{P}=\operatorname{Isom}^*_{X_1}(\boldsymbol{P}^N\times X_1, \boldsymbol{P}(E))\to X_1$ which itself is naturally a holomorphic fiber bundle over X_1. (See [19] for the notation Isom*.) Then it is easy to see that τ_i extends to a meromorphic map $\tilde{\tau}_i\colon \bar{P}\to D_{\boldsymbol{P}^N}$. Let Q be the image of the meromorphic map $\bar{\mu}\times\tilde{\tau}_0\times\tilde{\tau}_1\times\cdots\times\tilde{\tau}_m\colon \bar{P}\to X_1\times D_{\boldsymbol{P}^N}^{m+1}$, where $D_{\boldsymbol{P}^N}^{m+1}=D_{\boldsymbol{P}^N}\times\cdots\times D_{\boldsymbol{P}^N}$ ($(m+1)$-times).

Let $\tilde{\tau}'\colon X_1\to\hat{D}$ be the universal meromorphic map associated to the inclusion $Q\subseteqq X_1\times D_{\boldsymbol{P}^N}^{m+1}$ where $\hat{D}$ denotes the Douady space of $D_{\boldsymbol{P}^N}^{m+1}$ (cf. § 3). $D_{\boldsymbol{P}^N}^{m+1}$ is a disjoint union of projective analytic spaces and hence $\hat{D}$ also is a disjoint union of projective analytic spaces [26]. Then since $a(X_1)=a(Y)$ by our assumption, $\tilde{\tau}'$ must factor through Y. Hence by the definition of the universal meromorphic map (cf. 3.1) we can find Zariski open subsets $U\subseteqq Y$ and $V\subseteqq X_1$ with $h(V)\subseteqq U$ such that if $u\in U$, then for every $v\in V_u$, Q_v is one and the same subspace of $D_{\boldsymbol{P}^N}^{m+1}$.

On the other hand, note that $G:=\mathrm{PGL}\,(N+1, \boldsymbol{C})$ acts naturally on $D_{\boldsymbol{P}^N}$. Let G act on $D_{\boldsymbol{P}^N}^{m+1}$ diagonally. Let $g_0=g$ and $g_i=g|_{A_i}\colon A_i\to X_1$. For any point $p\in P$ we consider $a_p=(X_p, A_{1,p},\cdots, A_{m,p})$ as a point of $D_{\boldsymbol{P}^N}^{m+1}$ where $X_p=(X\times_{X_1}P)_p$ and $A_{i,p}=(A_i\times_{X_1}P)_p$. Then by our construction and the definition of P if we take the above U and V sufficiently small, for any $v\in V$, Q_v is nothing but the closure of the G-orbit of a_p in $D_{\boldsymbol{P}^N}^{m+1}$ for any $p\in P_v$ (which is of course independent of $p\in P_v$). This then implies that for any $u\in U$, $(X_v, A_{1,v},\cdots, A_{m,v})$ are mutually isomorphic (by elements of G) as long as $v\in V_u$. In fact we show that for any such u the map $g_u\colon (X_u, A_{1,u},\cdots, A_{m,u})\to X_{1u}$ is actually locally trivial over V_u.

Restricting V we may assume that g_i are all flat over V. Fix $o\in V_u$ arbitrarily. Take a sufficiently small neighborhood $o\in M\subseteqq V_u$ in such a way that we have a trivialization $\boldsymbol{P}(E)_M\cong M\times\boldsymbol{P}^N$. Then with respect to the induced inclusion $A_{i,M}\subseteqq M\times\boldsymbol{P}^N$ consider $g_{i,M}\colon A_{i,M}\to M$ as a flat family of subspaces of $\boldsymbol{P}^N$ parametrized by M. Let $\tau_{i,M}\colon M\to D_{\boldsymbol{P}^N}$ be the associated universal morphism and $\tau_M=\tau_{0,M}\times\cdots\times\tau_{m,M}\colon M\to D_{\boldsymbol{P}^N}^{m+1}$. Let $\tau_M(o)=b$. Let B be the G-orbit of b in $D_{\boldsymbol{P}^N}^{m+1}$. Then $\tau_M(M)\subseteqq B$ by what we have proved above. Let $\pi\colon G\to B$ be defined by $\pi(g)=gb$. Then π is a holomorphic fiber bundle. Hence if M is sufficiently small, we can get a morphism $\tilde{\tau}_M\colon M\to G$ with $\tilde{\tau}_M(o)=e$, the identity of G, such that $\tau_M=\pi\cdot\tilde{\tau}_M$, i.e., $\tau_M(v)=g(v)b$ for all $v\in M$, where $g(v)=\tilde{\tau}_M(v)$. It then follows that the map $\Phi\colon M\times X_0\to M\times \boldsymbol{P}^N$ defined by $\Phi(v, x)=(v, g(v)x)$ has its image X_M and gives the trivialization $M\times(X_o, A_{1,o}, \cdots, A_{m,o})\cong(X_M, A_{1,M}, \cdots, A_{m,M})$ over M. Thus g_u is locally trivial over V_u. Finally the difference of two trivializations is given by a holomorphic section $M\to G_b:=\pi^{-1}(o)$. Thus the structure group is reduced to the stabilizer G_b of b in G. q.e.d.

Remark 5.1. Recently the author has shown that the set $\mathfrak{M}$ of isomorphism classes of nonuniruled polarized algebraic manifolds has the natural structure of an algebraic space (cf. [17]). Proposition 5.1 follows from this general result immediately if X_{x_1} is not uniruled, e.g. $\kappa(X_{x_1})\geqq 0$ for some $x_1\in X_1$ with X_{x_1} smooth.

5.2. As a preparation for the proof of the next proposition we recall some definitions and notations on relative automorphism groups etc. from [19]. (For the more detail we refer to [19].) Let $f\colon X\to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $U\subseteqq Y$ be a Zariski open subset over which f is smooth. Let $\mathrm{Aut}_U X_U$ be the relative automorphism group over U for the smooth morphism $f_U\colon X_U\to U$. For each $y\in U$ we have the natural identification $(\mathrm{Aut}_U X_U)_y=\mathrm{Aut}\, X_y$. $\mathrm{Aut}_U X_U$ is naturally regarded as a Zariski open subset of the relative Douady space $D_{X_U\times_U X_U/U}$ of $X_U\times_U X_U$ over U which is also Zariski open in $D_{X\times_Y X/Y}$. Let $\mathrm{Aut}^*_Y X$ be the essential closure of $\mathrm{Aut}_U X_U$ in $D_{X\times_Y X/Y}$, i.e., the union of those irreducible components of the closure which are mapped surjectively onto Y. Let $A=(A_1, \cdots, A_m)$ be a sequence of analytic subspaces of X. Restrict U so that A_i are all flat over U. Then we can define a relative group subvariety $\mathrm{Aut}_U(X_U, A_U)$ of $\mathrm{Aut}_U X_U$ by the condition; $\mathrm{Aut}_U(X_U, A_U)_y=\mathrm{Aut}\,(X_y, A_y):=\{g\in \mathrm{Aut}\, X_y;\ gA_{i,y}=A_{i,y}$ for all $i\}$, $y\in U$. (See [19] for the more precise functorial definition.) Suppose now that f is Kähler (cf. [12]), e.g., f is projective. Let $\omega\in\Gamma(Y, R^2f_*\boldsymbol{R})$ be a relative Kähler class, i.e., the restriction $\omega_y\in H^2(X_y, \boldsymbol{R})$ of ω to each fiber X_y is a Kähler class. Then define the relative group subvariety $(\mathrm{Aut}_U X_U)_\omega$ of $\mathrm{Aut}_U X_U$ by $((\mathrm{Aut}_U X_U)_\omega)_y=\{g\in\mathrm{Aut}\, X_y;\ g^*\omega_y=\omega_y\}$, $y\in U$. (See [19] for the functorial definition.) Let $\mathrm{Aut}_U(X_U, A_U)_\omega=(\mathrm{Aut}_U X_U)_\omega\cap\mathrm{Aut}_U(X_U, A_U)$. When Y, and hence U also, is a point, we write simply $\mathrm{Aut}(X, A)_\omega$. Let $\mathrm{Aut}^*_Y(X, A)_\omega$ be the essential closure of $\mathrm{Aut}_U(X_U, A_U)_\omega$ in $\mathrm{Aut}^*_Y X$. This is then a relative meromorphic subgroup of $\mathrm{Aut}^*_Y X$ in the sense of [19], which essentially means that $\mathrm{Aut}^*_Y(X, A)_\omega$ is analytic and compact.

Let G^* be the relative meromorphic subgroup of $\mathrm{Aut}^*_Y X$, say $G^*=\mathrm{Aut}^*_Y(X, A)_\omega$ as above. Then *a relative generic quotient of X by G^** is a compact complex manifold $\bar{X}$ over Y together with a surjective meromorphic Y-map $\varphi: X \to \bar{X}$ such that for general $\bar{x} \in \bar{X}$, the fiber $X_{\bar{x}}$ over $\bar{x}$ is a closure of an orbit of G_y in X_y where $\bar{x} \in \bar{X}_y$ and $G_y = G_Y^* \cap \mathrm{Aut}\, X_y$ (in our case $G_y \cong \mathrm{Aut}(X_y, A_y)_{\omega_y}$). Then a relative generic quotient always exists for the given f and G^* as above by [19, Theorem 1]. We call $\bar{X}$ itself also a relative generic quotient and it is often denoted symbolically by X/G^*. When Y is a point, we simply call $\bar{X}$ a generic quotient of X by G^* or by $G := G^* \cap \mathrm{Aut}\, X$.

Let $f: X \to Y$ and $U \subseteqq Y$ be as above. Suppose that f is a holomorphic fiber bundle over U with typical fiber F and with structure group G. G is called a meromorphic structure group (with respect to f) if G is a meromorphic subgroup of Aut F, i.e., the closure G^* of G in Aut*F (or in $D_{F \times F}$) is analytic and compact. Suppose that G is meromorphic. Let $\bar{F} := F/G$ be the generic quotient of F by G. Then there exists a meromorphic map $\psi: X \to \bar{F}$ canonically associated to f and G, called a *canonical meromorphic map associated to* f and G (cf. [19, Def. 6])

5.3. To state the next proposition in its full generality it is convenient to introduce the following terminology.

Definition. Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $V \subseteqq Y$ and $W \subseteqq X$ be Zariski open subsets with $f(W) \subseteqq V$. Then we say that the triple $(f: X \to Y, W, V)$ has *property* (F) if there exist a projective manifold F, and a linear algebraic subgroup $G \subseteqq \mathrm{Aut}\, F$ such that 1) G acts algebraically and almost homogeneously on F with a Zariski open orbit $F_0 \subseteqq F$, and 2) if we set $A = X - W$ and $B = F - F_0$ then $f: (X, A) \to Y$ is a holomorphic fiber bundle over V (in the obvious sense) with typical fiber (F, B) and with structure group G acting on (F, B) as above.

Proposition 5.2. *Let* $g: X \to X_1$, $h: X_1 \to Y$ *be two fiber spaces of compact complex manifolds in* $\mathscr{C}$. *Suppose that* g *is projective,* $q(g) = 0$, *and* $a(X) = a(Y)$. *Then there exist Zariski open subsets* $U \subseteqq Y$, $V \subseteqq X_1$, $W \subseteqq X$ *with* $g(W) \subseteqq V$, $h(V) \subseteqq U$ *such that* 1) *for any* $u \in U$, X_u, $X_{1,u}$ *are smooth and the triple* $(g_u: X_u \to X_{1,u}, W_u, V_u)$ *has property* (F) *in the sense defined above and* 2) *for any analytic subvariety* $A \subseteqq X$ *with* $g(A) = X_1$ *we have* $A \cap W = \emptyset$.

Remark 5.2. In a special case where Y a point, then the condition $q(g) = 0$ can be deduced from other two conditions; g is projective and $a(X) = 0$.

Proof of Proposition 5.2. α. Let $h_1: Y \to Y'$ be an algebraic reduction of Y. Then from the form of the statement of the proposition we infer readily that it suffices to show the proposition for a suitable bimeromorphic model of $(g: X \to X_1, h_1 h: X_1 \to Y')$, i.e., we may assume from the beginning that h is an algebraic reduction of X_1 so that in particular Y is projective.

β. Let $A=(A_1, \cdots, A_m)$ be any sequence of analytic subspaces of X. (We include the case $m=0$, i.e., $A=\emptyset$.) Then by Proposition 5.1 there exist Zariski open subsets $U \subseteqq Y$, $V \subseteqq X$ with $h(V) \subseteqq U$ such that if $u \in U$, then both X_u and X_{1u} are smooth and $g_u : (X_u, A_u) \to X_{1u}$ is a holomorphic fiber bundle over $V_u \subseteqq X_{1u}$. Since g is projective g is Kähler. Fix a relative Kähler class $\omega \in \Gamma(Y, R^2 g_* \boldsymbol{R})$. Take and fix $v=v(u) \in V_u$, and consider $(F_u, B_u) := (X_v, A_v)$ as a typical fiber of the above bundle. Further $G(u) := \mathrm{Aut}\,(X_v, A_v)_{\omega_v}$ can be taken to be a meromorphic structure group of the bundle (cf. [19, Proposition 6]), the associated relative meromorphic subgroup of $\mathrm{Aut}^*_{X_{1u}} X_u$ being given by $\mathrm{Aut}^*_{X_{1u}}(X_u, A_u)_{\omega_u}$ where $\omega_u \in \Gamma(X_{1u}, R^2 g_{u*} \boldsymbol{R})$ is induced by ω (cf. [19]). Since $q(X_v)=0$ by our assumption, $G(u)$ is a linear algebraic group (cf. [13] [33]). Let $G^* = \mathrm{Aut}^*_{X_1}(X, A)_\omega$ and $G := G^* \cap \mathrm{Aut}_V(X_V, A_V)$.

γ. We now prove the existence of W satisfying 1). More precisely, we show that (after restricting V and U if necessary) there exists a Zariski open subset $W \subseteqq X$ with $g(W) \subseteqq V$ such that if $v \in V$ then G_v acts almost homogeneously on X_v and its unique Zariski open orbit coincides with W_v. Let $\bar{X} := X/G$ be the relative generic quotient of X by G^* over X_1. Let $p : \bar{X} \to X_1$ be the natural map. Then by [19, Proposition 1] for the existence of W as above it suffices to show that p is bimeromorphic. For general $u \in Y$, G^*_u is a relative meromorphic subgroup of $\mathrm{Aut}^*_{X_{1u}} X_u$ over X_{1u}, $\bar{X}_{1u}$ is a relative generic quotient of X_u by G^*_u over X_{1u} and p defines a meromorphic map $p_u : \bar{X}_u \to X_{1u}$ with $\dim p = \dim p_u$ (cf. [19, Proposition 1]). Let $\psi_u : X_u \to \bar{F}_u := F_u/G(u)$ be a canonical meromorphic map associated to the bundle $g_u|_{V_u}$ and the meromorphic structure group $G^*(u)$ (cf. 4.2). Then $\dim p_u = \dim \bar{F}_u$ (cf. [19, 2.2]). Hence it suffices to show that $\dim \bar{F}_u = 0$, or ψ_u is a constant map.

δ. For this purpose we shall construct a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle b} \\ X_1 & \xrightarrow[\ h\ ]{} & Y \end{array}$$

where b is a fiber space of complex varieties and φ is a surjective meromorphic map such that if $u \in U$ φ induces a meromorphic map $\varphi_u : X_u \to Z_u$ which is bimeromorphic to ψ_u. In particular for general, and hence for all, $x_1 \in X_1$, $\varphi(X_{1,x_1})=Z_{h(x_1)}$. In fact, once such a diagram is constructed, we have only to show that $\dim b=0$ and this can be seen as follows. Let $\tilde{\varphi}: X^* \to Z$ be a holomorphic model of φ such that the resulting meromorphic map $X^* \to X_1$ is also holomorphic. Since for each $x_1 \in X_1$, $\tilde{\varphi}|_{X^*_{x_1}}$: $X^*_{x_1} \to Z_{h(x_1)}$ is surjective. $Z_{h(x_1)}$ is Moishezon as well as $X^*_{x_1}$. Further since $0=q(X^*_{x_1}) \geqq q(Z_{h(x_1)})=0$ for general $x_1 \in X_1$, we get that $q(b)=0$. Hence by Proposition 2.5 b is Moishezon. Then Z itself is Moishezon since Y is. Therefore by our assumption $\dim Z=a(Z)=a(Y)=\dim Y$. Thus $\dim b=0$ as was desired.

ε. It remains to construct a diagram as above. For this purpose, taking a flattening of h [28], we may assume that h is flat (though X_1 may then be singular). Then for each $y \in Y$ we can find a neighborhood $y \in N$ and an analytic subvariety $\tilde{N} \subseteqq X_{1,N}$ such that $\mu := h|_{\tilde{N}} : \tilde{N} \to N$ is finite and surjective, and is isomorphic if $y \in U$, and that $\tilde{N} \cap X_{1,U} \subseteqq V$. Take the base change to $\tilde{N}$ by μ. Then $h_{\tilde{N}} : X_{1,\tilde{N}} \to \tilde{N}$ has the canonical section s: $\tilde{N} \to X_{1,\tilde{N}}$ with $s(\tilde{N}) \subseteqq V_{\tilde{N}}$. Then we can apply [19, Proposition 8] to $g_{\tilde{N}}$: $X_{\tilde{N}} \to X_{1,\tilde{N}}$, $h_{\tilde{N}}$ together with the sequences of subspaces $(A_{1,\tilde{N}}, \cdots, A_{m,\tilde{N}})$ and with the relative Kähler class $\omega_{\tilde{N}}$ for $g_{\tilde{N}}$ which is the pull-back of ω by the natural map $X_{1,\tilde{N}} \to X_1$. Therefore we obtain a fiber space $b(\tilde{N})$: $Z(\tilde{N}) \to \tilde{N}$ and a surjective meromorphic $\tilde{N}$-map $\varphi(\tilde{N}): X_{\tilde{N}} \to Z(\tilde{N})$ such that $b(\tilde{N})\varphi(\tilde{N})=f_{\tilde{N}}$, and that for $\tilde{u} \in \mu^{-1}(N \cap U)$ with $u=\mu(\tilde{u})$, $\varphi(\tilde{N})$ induces a meromorphic map $\varphi_{\tilde{u}}: X_u=(X_{\tilde{N}})_{\tilde{u}} \to (Z(\tilde{N}))_{\tilde{u}}$ which is bimeromorphic to ψ_u (if U is restricted smaller). Now we set $M=U$ and $\mathfrak{S}=\{\psi_u\}_{u \in M}$. Then by Proposition 3.2, 1) there exist a fiber space $b: Z \to Y$ and a surjective meromorphic Y-map $\varphi: X \to Z$ (where X is over Y via hg) such that $\varphi_{\tilde{N}}$ is bimeromorphic to $\varphi(\tilde{N})$ for each $\tilde{N}$ as above. In particular we get $hg=b\varphi$. Then restricting U further we may assume that for $u \in U$, φ induces a meromorphic map $\varphi_u: X_u \to Z_u$ which is bimeromorphic to ψ_u. Thus a desired commutative diagram is constructed.

ζ. It remains to show that the above W also satisfies the condition 2) for a suitable choice of $(A_1, \cdots, A_m)$.

Let Λ_0 be the set of those proper analytic subvarieties of X which are not contained in any other proper analytic subvariety of X and which are mapped surjectively onto X_1. Let Λ be the set of finite unions of elements of Λ_0. We show that there exists a unique maximal element in Λ. For this purpose it suffices to show that for any infinite sequence $A_1 \subseteqq A_2 \subseteqq \cdots \subseteqq \cdots$ of subspaces of X with $A_i \in \Lambda$, there exists a proper analytic subset B with $B \supseteqq A_i$ for all i. We set $G(i)^*=\mathrm{Aut}^*_{X_1}(X, A^{(i)})_\omega$ where $A^{(i)}=(A_1, \cdots, A_i)$. Then we have $G(0)^*=\mathrm{Aut}^*_{X_1} X_\omega \supseteqq \cdots \supseteqq G(i)^* \supseteqq G(i+1)^* \supseteqq \cdots$, which must be stationary (cf. 4.2). Hence there exists an index j such that $G(j)^*=G(j+1)^*=\cdots$. Then take $W \subseteqq X$ as in γ for $A=A^{(j)}$. Define B to be the closure of $(X-W) \cap g^{-1}(V)$ in X. Then $B \supseteqq A_i$ for all i.

In fact, for general $x_1 \in X_1$, $W_{x_1} \cap A_{i,x_1} = \emptyset$, for all i, since W_{x_1} is homogeneous with respect $G(i)_x$ and A_{i,x_1} is left invariant by any element of $G(i)_x$. Hence $W \subseteqq X - A_i$ and $B \supseteqq A_i$.

Let $A \in \mathcal{A}$ be the maximal element. Set $G^* = \mathrm{Aut}^*_{X_1}(X, A)$. Let $V \subseteqq X_1$, $W \subseteqq X$ be the Zariski open subsets corresponding to G as in γ (the case where $A_1 = A$ and $A_i = \emptyset$, $i \geqq 2$). Then we claim that $W = (X - A) \cap g^{-1}(V)$. In fact, by the same argument as above we have $X - W \supseteqq A$. On the other hand, by the maximality of A both must coincide over V. Hence W satisfies also the condition 2). q.e.d.

5.4. Propositions 5.1 and 5.2 are also true even if α is Moishezon instead of being projective. However here we shall be content with the following:

Proposition 5.3. *Let $g: X \to X_1$, $h: X_1 \to Y$ be fiber spaces of compact complex manifolds in $\mathscr{C}$. Suppose that g is Moishezon, $q(g) = 0$ and $a(X) = a(Y)$. Then there exist Zariski open subsets $U \subseteqq Y$ and $V \subseteqq X_1$ with $h(V) \subseteqq U$ such that for any $u \in U$, X_u, $X_{1,u}$ are both smooth and $g_u: X_u \to X_{1,u}$ is a holomorphic fiber bundle over V_u with typical fiber an almost homogeneous unirational manifold. Moreover there exists a unique maximal analytic subset M of X each irreducible component of which is mapped surjectively onto X_1 by g.*

We need a lemma.

Lemma 5.4. *Let $g: X \to Y$ and $g': X' \to Y$ be smooth fiber spaces of complex manifolds. Let $\varphi: X \to X'$ be a bimeromorphic Y-morphism such that $\varphi_y: X_y \to X'_y$ is bimeromorphic for any $y \in Y$. Then if g is a holomorphic fiber bundle, g' also is a holomorphic fiber bundle.*

Proof. We consider the following commutative diagram

$$\begin{array}{ccccc} \Theta_Y & \xrightarrow{\ \delta\ } & R^1g_*\Theta_{X/Y} & \xrightarrow{\ \mu\ } & R^1g_*\varphi^*\Theta_{X'/Y} \\ \Big\| & & & \nearrow \nu & \\ \Theta_Y & \xrightarrow{\ \delta'\ } & R^1g'_*\Theta_{X'/Y} & & \end{array}$$

where δ' is the coboundary map in the long exact sequence obtained by applying Rg_* to the short exact sequence

$$0 \longrightarrow \Theta_{X'/Y} \longrightarrow \Theta_{X'} \longrightarrow g'^*\Theta_Y \longrightarrow 0$$

and δ is defined similarly for g; ν is the natural map and μ is induced by the natural sheaf homomorphism $\Theta_{X/Y} \to \varphi^* \Theta_{X'/Y}$. Moreover by the spectral sequence for the composite functor $g_* = g'_* \varphi_*$ we see that ν is isomorphic since $R^1\varphi_*\varphi^*\Theta_{X'/Y} = R^1\varphi_*\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \Theta_{X'/Y} = 0$. Since g is a holomorphic fiber bundle, δ is the zero map and hence δ' also is the zero map. By the definition of δ' this implies that $g'_*\Theta_{X'} \to \Theta_Y$ is surjective, which is equivalent to the local triviality of g' as is well-known. q.e.d.

Proof of Proposition 5.3. Using Chow lemma [28] we can find a projective fiber space $\tilde{g}: \tilde{X} \to X_1$ of compact complex manifolds in $\mathscr{C}$ and a bimeromorphic X_1-morphism $\varphi: \tilde{X} \to X$. Let $U \subseteqq Y$ and $V \subseteqq X$ be Zariski open subsets such that hg is smooth over U, g is smooth over V, $\varphi_v: \tilde{X}_v \to X_v$ is bimeromorphic for any $v \in V$ and that the conclusion of Proposition 5.1 is true for $(\tilde{g}, h)$. Then by Lemma 5.4 for any $u \in U$, $g_u: X_u \to X_{1,u}$ is a holomorphic fiber bundle over V_u. Let $\tilde{W} \subseteqq \tilde{X}$ be as in Proposition 5.2 applied to $(\tilde{g}, h)$ with U and V restricted if necessary. Let $\tilde{M}' = \tilde{X} - \tilde{W}$ and $M' = \varphi(\tilde{M}')$. Let M be the union of those irreducible components of M' which are mapped surjectively onto X by g. Then from the minimality property of $\tilde{W}$ it follows immediately that M has the desired maximality property. Finally let $F \subseteqq X_1$ be the set of indeterminacy for φ^{-1}. Then we have $F \cap X_V \subseteqq M$ if we restrict U and V if necessary. Hence φ_v, $v \in V$, induces an isomorphism $\tilde{X}_v - \tilde{M}_v \cong X_v - M_v$, so that $X_v - M_v$ is homogeneous as well as $\tilde{X}_v - \tilde{M}_v$ (cf. 1) of Proposition 5.2). X_v is thus almost homogeneous (cf. also [13, Remark 2.4.1)]). q.e.d.

We call M obtained in the above proposition *the maximal transversal analytic subset with respect to g*.

§ 6. Quasi-hyperelliptic manifolds

In this section we study some basic properties of quasi-hyperelliptic manifolds to be defined below.

6.1. Definition. Let T be a complex torus and $G \subseteqq \mathrm{Aut}\, T$ be a finite group. Let $Y := T/G$ be the quotient variety. Then Y is called *quasi-hyperelliptic* if codim $B \geqq 2$ in T where $B = \{t \in T; G_t \neq \{e\}\}$, G_t being the stabilizer of t. In this case we call $Y = T/G$ an *admissible representation* of Y. If $B = \emptyset$, Y is called a *hyperelliptic* manifold. A compact complex manifold which is bimeromorphic to a quasi-hyperelliptic manifold is said to be *bimeromorphically quasi-hyperelliptic*.

Remark 6.1. A Kummer manifold X with $a(X) = 0$ is bimeromorphically quasi-hyperelliptic. This follows from the fact that a complex torus T with $a(T) = 0$ contains no divisor.

Let Y be a quasi-hyperelliptic variety and $Y=T/G$ an admissible representation of Y. Let $S=\mathrm{Sing}\, Y$ be the singular locus of Y and $U:=Y-S$. Let E be the affine space which is the universal covering of T.

Lemma 6.1. *There exists a unique group $\tilde{G}$ of affine transformations of E acting properly discontinuously on E such that $Y\cong E/\tilde{G}$ and $E\to Y$ is unramified over U. Moreover the pair $(E, \tilde{G})$ is determined uniquely by Y, being independent of T and G. We have the natural group isomorphism $\pi_1(U)\cong\tilde{G}$.*

Proof. Let $p: T\to Y$ be the natural projection. Let $U_1:=p^{-1}(U)$ and $\mu: \tilde{U}\to U_1$ be the universal covering of U_1. Since $\mathrm{codim}(T-U_1)\geqq 2$, $\pi_1(U_1)\cong\pi_1(T)$. Hence there is a natural inclusion $\tilde{U}\subseteqq E$ which fits into the following commutative diagram

$$\begin{array}{ccccc} \tilde{U} & \longrightarrow & U_1 & \longrightarrow & U \\ \cap\| & & \cap\| & & \cap\| \\ E & \longrightarrow & T & \xrightarrow[p]{} & Y. \end{array}$$

Moreover $U_1\to U$ is unramified; in fact an unramified Galois covering with Galois group G. Therefore $p\mu: \tilde{U}\to U$ gives the universal covering of U. Now consider $\pi_1(U)$ as a group of biholomorphic automorphisms of $\tilde{U}$. Since codim $(E-\tilde{U})\geqq 2$ and $E\cong C^n$, $n=\dim T$, each element $\delta\in\pi_1(U)$ extends uniquely to an automorphism $\tilde{\delta}$ of E. We show that $\tilde{\delta}$ is an affine transformation of E. First note that we have the natural exact sequence

$$e\longrightarrow\pi_1(U_1)\longrightarrow\pi_1(U)\xrightarrow{\lambda} G\longrightarrow e.$$

For $\delta\in\pi_1(U)$, let $\bar{\delta}=\lambda(\delta)$. Then it is well-known that the transformation $\bar{\delta}$, which is an automorphism of T, is induced by an affine automorphism of E which is defined uniquely up to translations by elements of the lattice $\Lambda\subseteqq E$ defining T. Since Λ naturally identified with $\pi_1(U_1)$, from this follows the desired assertion immediately. Let $A(E)$ be the group of affine transformations of E and $\tilde{G}$ (resp. $\tilde{G}_1$) the subgroup of $A(E)$ defined by $\pi_1(U)$ (resp. $\pi_1(U_1)$). Then $\tilde{G}_1\subseteqq\tilde{G}$ and $\tilde{G}/\tilde{G}_1\cong G$. Moreover $E/\tilde{G}\cong(E/\tilde{G}_1)/G=T/G$. This shows the existence of $\tilde{G}$. Also the final assertion follows from our construction.

Uniqueness. Let T_i, $i=1, 2$, be complex tori and $G_i\subseteqq\mathrm{Aut}\, T_i$ finite subgroups such that T_i/G_i are admissible representations for Y. Let T be the normalization of any irreducible component of $T_1\times_Y T_2$. Then we have finite coverings $\nu_i: T\to T_i$ whose branch loci are codimension$\geqq 2$ in

T_i. Thus T is also a complex torus and ν_i are unramified. Passing to a suitable unramified covering of T we assume that $T \to Y$ is Galois. Let G be the Galois group. Then T/G also is an admissible representation of Y. This reduces our problem to the case where there exists a normal subgroup $H \subseteq G_1$ such that $G_1/H \cong G_2$. Then the uniqueness follows almost immediately from the above construction.

Definition. Let Y be a quasi-hyperelliptic variety. Write $Y = E/\tilde{G}$ as in the above lemma. Let G_0 be the normal subgroup of all the translations in $\tilde{G}$. Let $T = E/G_0$ and $G = \tilde{G}/G_0$. Then we call T/G the *canonical representation* of Y.

The abstract characterization of G_0 is also possible. This is essentially due to Uchida-Yoshihara [42].

Lemma 6.2. *G_0 is the unique maximal normal abelian subgroup of G.*

Proof. See Proposition 1 of [42]. In the proof of the proposition the assumption that $\tilde{G}$ acts freely on E is irrelevant as long as $E/\tilde{G}$ is compact.

Lemma 6.3. *Let Y_i, $i = 1, 2$, be quasi-hyperelliptic varieties with admissible representations $Y_i = T_i/G_i$. Then any bimeromorphic map $g: Y_1 \to Y_2$ is necessarily biholomorphic.*

Proof. Suppose that g gives an isomorphism $g': U_1 \to U_2$ of Zariski open subsets $U_i \subseteq Y_i$, $i = 1, 2$. Restricting U_i we may assume that U_i are nonsingular. Let $\tilde{U}_i \to U_i$ be the universal coverings of U_i. Then just as in the proof of Lemma 6.1 there exist natural inclusions $\tilde{U}_i \subseteq E_i$ of $\tilde{U}_i$ into the universal covering spaces E_i of T_i such that any isomorphism $\tilde{g}': \tilde{U}_1 \to \tilde{U}_2$ lifting g' extends to an isomorphism $\tilde{g}''$ of E_1 onto E_2. It then follows that $\tilde{g}''$ induces an isomorphism $g_0': Y_1 \to Y_2$ which extends g'. Then $g = g_0'$. q.e.d.

Let X be a bimeromorphically quasi-hyperelliptic manifold. Let Y be a quasi-hyperelliptic variety bimeromorphic to X. Then the above lemma shows that Y is up to isomorphisms uniquely determined by X. So, if $Y = T/G$ is the canonical representation of Y, then we call T/G the *canonical model of X*.

6.2. We study the structure of the automorphism group of a bimeromorphically quasi-hyperelliptic manifold. First we prove the following general

Proposition 6.4. *Let f: $X \to Y$ be a finite covering of normal compact complex varieties. Let $B \subseteqq Y$ be an analytic subset such that f induces an unramified covering $X - A \to Y - B$, where $A = f^{-1}(B)$. Then there exist a connected closed subgroup $G \subseteqq \mathrm{Aut}_0(X, A)$ and a surjective homomorphism ψ: $G \to \mathrm{Aut}_0(Y, B)$ with finite kernel with respect to which f is $(G, \mathrm{Aut}_0(Y, B))$-equivariant.*

Proof. For $g \in \mathrm{Aut}\, X$ let $\Gamma_g \subseteqq X \times X$ be the graph of g and $\bar{\Gamma}_g$ the image of Γ_g in $Y \times Y$ via $f \times f$. We set $G = \{g \in \mathrm{Aut}\,(X, A); fgf^{-1}(y)$ consists of a single point for every $y \in Y\}$. Then G is a complex Lie subgroup of Aut (X, A). (It is clear that G is a subgroup. So we have only to show that G is an analytic subset of Aut (X, A) and this can be shown by a standard argument which is left to the reader.) If $g \in G$, then one verifies readily that $\bar{\Gamma}_g$ is the graph of a unique element $\bar{g} \in \mathrm{Aut}\,(Y, B)$ and the map $g \to \bar{g}$ defines a homomorphism ψ: $G \to \mathrm{Aut}\,(Y, B)$ of Lie groups. The kernel of ψ is contained in the covering transformation group of f, and therefore is finite. We show that the image of ψ contains $\mathrm{Aut}_0(Y, B)$. In fact, then replacing G by the identity component of G we would obtain the lemma. (Note that Ker ψ is finite.)

Now let $U = X - A$, and $V = Y - B$. Let p: $\tilde{U} \to U$ be the universal covering of U. Then fp: $\tilde{U} \to V$ is the universal covering of V. Let Δ be the covering transformation group of fp. Then every $\bar{g} \in \mathrm{Aut}\ V$ induces an automorphism $\bar{g}_*$ of Δ which is defined up to inner automorphisms of Δ. Fixing such a $\bar{g}_*$ we can find a lift $\tilde{g}$: $\tilde{U} \to \tilde{U}$ of $\bar{g}$ to Aut $\tilde{U}$ such that $\tilde{g}(\delta u) = \bar{g}_*(\delta)\tilde{g}(u)$ for $u \in \tilde{U}$ and $\delta \in \Delta$. Take now $\bar{g}$ from $\mathrm{Aut}_0(Y, B)$, considered naturally as an element of Aut V. Then $\bar{g}$ acts trivially on Δ so that we can take as $\bar{g}_*$ the identity automorphism of Δ so that $\tilde{g}(\delta u) = \delta\tilde{g}(u)$. This implies that $\tilde{g}$ descends to an element g of Aut U. Moreover from $\bar{g} \in \mathrm{Aut}\,(Y, B)$, by considering locally at points of Y and using Riemann extension theorem for holomorphic functions, it follows that $g \in \mathrm{Aut}(X, A)$. Since g induces $\bar{g}$, g belongs to G. Hence $\bar{g} = \psi(g) \in \mathrm{Im}\,\psi$ as was desired. q.e.d.

Proposition 6.5. *Let X be a bimeromorphically quasi-hyperelliptic manifold and $Y = T/G$ the canonical model of X. Then*: 1) $\mathrm{Aut}_0 Y$ (*resp.* $\mathrm{Aut}_0 X$) *is a complex torus. In particular if $q(X) = 0$, then* $\mathrm{Aut}_0 Y = \mathrm{Aut}_0 X = \{e\}$. 2) $\mathrm{BHol}(X, Y) \cong \mathrm{Aut}\ Y$ *if* $\mathrm{BHol}(X, Y) \neq \emptyset$ *where* $\mathrm{BHol}(X, Y)$ *is the set of bimeromorphic morphisms of X onto Y.*

Proof. 1) Let $\pi: T \to Y$ be the natural projection. Let $U = Y - \mathrm{Sing}\, Y$ and $W = \pi^{-1}(U)$. Since $W \to U$ is an unramified covering, by Proposition 6.4 there exist a subtorus $G \subseteqq \mathrm{Aut}_0 T \cong T$ and a surjective homomorphism $G \to \mathrm{Aut}_0\,(Y, \mathrm{Sing}\, Y)$. Thus $\mathrm{Aut}_0 Y = \mathrm{Aut}_0(Y, \mathrm{Sing}\, Y)$ is a complex torus.

Next we consider $\mathrm{Aut}_0 X$. Let L be the linear part of $\mathrm{Aut}_0 X$, i.e., the maximal connected linear algebraic subgroup of $\mathrm{Aut}_0 X$ (cf. [13] [33]). Let $h: X \to Y$ be a fixed bimeromorphic map. Then the formula $\varphi(b) = hbh^{-1}$, $b \in L$, defines a meromorphic map $\varphi: L \to \mathrm{BAut}\, Y = \mathrm{Aut}\, Y$ (Lemma 6.3) where BAut Y denotes the set of bimeromorphic automorphisms of Y. Further it is easy to see that φ is injective on some Zariski open subset on which φ is defined. On the other hand, since each connected component of Aut Y is a complex torus by what we have proved above, $\varphi(L)$ must reduce to a point. Thus L must reduce to the identity and $\mathrm{Aut}_0 X$ is a complex torus (cf. [13] [33]).

2) Let $BH = \mathrm{BHol}\,(X, Y)$. Suppose that $BH \neq \emptyset$, and fix $h_0 \in BH$. Then for any $h \in BH$, we set $\psi(h) = h \cdot h_0^{-1} \in \mathrm{BAut}\, Y$. Since BAut $Y=$ Aut Y by Lemma 6.3 we have $\psi: BH \to \mathrm{Aut}\, Y$. It is then immediate to see that ψ is bijective. This shows 2).

6.3. Let T be a complex torus and $G \subseteq \mathrm{Aut}\, T$ a finite subgroup. Fix the origin $o \in T$ and consider T as a complex Lie group. Then we have the canonical decomposition $\mathrm{Aut}\, T = H(T) \cdot T$ where $H(T) = \mathrm{Aut}\,(T, \{0\})$ (cf. [19]). According to this decomposition any element $g \in G$ can be written uniquely in the form $g(t) = A(g)t + b(g)$ where $A(g) \in H(T)$ and $b(g) \in T$. Let K_g be the kernel of the endomorphism $A(g) - I$ of T where I is the identity. Let $K = \bigcap_{g \in G} K_g$. Let T_0 be the identity component of K which is a subtorus of T. Let $\bar{T} = T/T_0$ and let $\pi: T \to \bar{T}$ be the natural homomorphism. Then the G-action on T induces the natural G-action on $\bar{T}$ so that π is G-equivariant. Thus if we set $Y := T/G$ and $\bar{Y} := \bar{T}/G$, then π induces a morphism $\bar{\pi}: Y \to \bar{Y}$. We check readily that $\bar{\pi}$ is independent of the choice of the origin o and depends only on T and G. Thus $\bar{\pi}$ is associated with (T, G).

Let $\alpha: Y \to \mathrm{Alb}\, Y$ be the Albanese map of Y. Since Y has only quotient singularities, this can be defined (cf. Lemma 7.5 below). Let $\varphi: T \to \mathrm{Alb}\, Y$ be the quotient map $T \to Y$ composed with α.

Lemma 6.6. *T_0 is mapped isogenously onto* Alb Y *by φ. In particular $q(\bar{Y}) = 0$ and* $\dim \bar{Y} = \dim Y - q(Y)$.

Proof. Let $o' = \varphi(o)$, and consider o' as the origin of the complex Lie group $A = \mathrm{Alb}\, Y$. Then h becomes a homomorphism of (T, o) to (A, o'). First note that $\Gamma(A, \Omega_A^1) = \Gamma(Y, \Omega_Y^1) = \Gamma(T, \Omega_T^1)^G$ where $(\ )^G$ denotes the set of G-invariants (cf. [43, Prop. 9.24]). Let $\bar{G}$ be the image of G in $H(T)$ by the natural projection $\mathrm{Aut}\, T \to H(T)$. The action of G on $\Gamma(T, \Omega_T^1)^G$ factors through $\bar{G}$. In particular $\Gamma(T, \Omega_T^1)^G = \Gamma(T, \Omega_T^1)^{\bar{G}}$. Let E be the tangent space of T at o and E^* its dual space. Then we have an isomorphism of transformation spaces $(\Gamma(T, \Omega_T^1), \bar{G}) \cong (E^*, \bar{G})$. Let

E_A be the tangent space of A at o on which G acts trivially. Then h_*, the differential of h at o, induces a $\bar{G}$-equivariant homomorphism $(E, \bar{G}) \to (E_A, \bar{G})$, where the action of $\bar{G}$ on E is the dual action of $\bar{G}$ on E^* and $\bar{G}$ acts on E_A trivially. Let $E_1 = E^{\bar{G}}$. Then h_* induces an isomorphism of E_1 and E_A since $\dim E_1 = \dim (E^*)^G = \dim \Gamma(A, \Omega_A^1) = \dim E_A$. (We have a G-invariant direct sum decomposition $E = E_1 \oplus E_2$ for a subspace $E_2 \subseteq E$ and E_2 has no nontrivial subspace on which G acts trivially. Hence $h_*(E_2) = \{0\}$ and $h_*(E_1) = E_A$.) On the other hand, we have $E_1 = \bigcap_{g \in G} \mathrm{Ker}(A(g)_* - I)$ where $A(g)_*$ is the differential at o of $A(g)$. So E_1 is the tangent space of T_0 at o. It follows that T_0 is mapped isogenously onto A. The first assertion is proved. Thus if $q(\bar{Y}) > 0$, then we can find a holomorphic 1-form on a nonsingular model of $\bar{Y}$ which induces via $\bar{\pi}$ a holomorphic 1-form on Y which is not obtained from a holomorphic 1-form on A. This is a contradiction. Hence $q(\bar{Y}) = 0$. Clearly $\dim \bar{Y} = \dim Y - \dim T_0 = \dim Y - \dim A = \dim Y - q(Y)$. q.e.d.

Remark 6.1. From the above lemma we obtain immediately the following: The Albanese map α: $Y \to \mathrm{Alb}\, Y$ is a holomorphic fiber bundle with finite abelian structure group and with typical fiber a Kummer variety. This result is due to Yoshihara [45] when Y is hyperelliptic.

6.4. Let Y be a quasi-hyperelliptic variety and $Y = T/G$ the canonical representation of Y. Then the map $\bar{\pi}$: $Y \to \bar{Y}$ associated with (T, G) (cf. 6.1) is called the *co-Albanese map of* Y. Clearly $\bar{\pi}$ depends only on Y. By Lemma 6.6 (together with its proof) Y is a complex torus if and only if $\dim \bar{Y} = 0$.

Definition. Let f: $X \to Y$ be a fiber space of compact complex varieties. Suppose that the general fiber of f is quasi-hyperelliptic. Then a relative *co-Albanese map* for f is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {}_{g}\searrow \qquad \nearrow_{h} & \\ & Z & \end{array}$$

where g is a surjective meromorphic map and h is a fiber space, such that for general $y \in Y$, X_y is quasi-hyperelliptic and the induced map g_y: $X_y \to Z_y$ is holomorphic and is the co-Albanese map for the hyperelliptic manifold X_y. We have $q(h) = 0$ and $\dim h = \dim f - q(f)$ by Lemma 6.6.

Proposition 6.7. *Let f: $X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Suppose that the general fiber of f is hyperelliptic. Then* *a relative co-Albanese map for f always exists and is unique up to bimeromorphic equivalences.*

The proof uses a diagram which will be given in a more general context in the next two sections. To avoid repetition we therefore defer the proof of the proposition till the end of Section 8.

§ 7. Kummer reduction and its relativization

In this section we shall prove Propositions 2.2 and 2.3.

7.1. Let X be a compact complex manifold with $a(X)=0$. Then in analogy with the maximal irregularity $q^*(X)$ introduced by Iitaka (cf. [43]) we define the following invariant $q^{**}(X)$ for X; $q^{**}(X):=\sup_{\tilde{X}} q(\tilde{X})$ where $\tilde{X}$ run through all the compact complex varieties which are finite coverings (possibly ramified) of X. Since $a(\tilde{X})=a(X)=0$ for any finite covering $\tilde{X}\to X$ (cf. [43, 3.8]) we have by Proposition 1.2 $q^{**}(X)\leqq \dim X$. Thus we can always find a finite Galois covering $\tilde{X}\to X$ such that $q^{**}(X)=q(\tilde{X})$.

Using this notion we shall now prove the existence of Kummer reduction.

Proposition 7.1. *Let X be a compact complex manifold with $a(X)=0$. Then a Kummer reduction of X exists. Moreover we have $k(X)=q^{**}(X)$ where $k(X)$ is the Kummer dimension of X.*

Proof. Take any finite Galois covering $\tilde{X}'\to X$ with Galois group G such that $q^{**}(X)=q(\tilde{X}')$. Let $r\colon \tilde{X}\to\tilde{X}'$ be an equivariant resolution of $\tilde{X}'$ [27] so that the action of G on $\tilde{X}'$ extends to $\tilde{X}$. Let $\varphi\colon X\to\tilde{X}/G$ be the resulting bimeromorphic map. G acts naturally on the Albanese map $\tilde{\alpha}\colon \tilde{X}\to \mathrm{Alb}\,\tilde{X}$ of $\tilde{X}$. Let $B:=(\mathrm{Alb}\,\tilde{X})/G$. Let $\beta\colon X\to B$ be the composite meromorphic map $X\xrightarrow{\varphi}\tilde{X}/G\to B$ where the last morphism is induced by $\tilde{\alpha}$. We claim that β is a Kummer reduction of X. This would also show the last assertion from our construction. For this purpose we first show that the β constructed above is up to bimeromorphic equivalences independent of the choice of $\tilde{X}'$ as above. So let $\tilde{X}_i'$, $i=1, 2$, be finite Galois coverings of X with $q(\tilde{X}_i')=q^{**}(X)$. Take another Galois covering $\tilde{X}_3'$ of X which dominates both $\tilde{X}_i'$;

$$\begin{array}{ccc} & \tilde{X}_1' & \\ {}^{u_1}\nearrow & & \searrow \\ \tilde{X}_3' & & X. \\ {}_{u_1}\searrow & & \nearrow \\ & \tilde{X}_2' & \end{array}$$

Let G_i be the Galois groups of $\tilde{X}'_i \to X$, and $\tilde{X}_i$ equivariant resolutions of $\tilde{X}'_i$, $i=1, 2, 3$. Then we have to compare the composite meromorphic maps β_i: $X \to \tilde{X}_i/G_i \to B_i := (\mathrm{Alb}\,\tilde{X}_i)/G_i$. We have the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & \tilde{X}_3/G_3 & \longrightarrow & B_3 \\ \| & & \downarrow \bar{u}_i & & \downarrow B(u_i) \\ X & \longrightarrow & \tilde{X}_i/G_i & \longrightarrow & B_i \end{array} \qquad i=1, 2$$

of meromorphic maps where $\bar{u}_i$ and $B(u_i)$ are induced by u_i. Now recall that $\tilde{\alpha}_i$: $\tilde{X}_i \to \mathrm{Alb}\,\tilde{X}_i$ are fiber spaces (Proposition 1.2) so that β_i: $X_i \to B_i$ are meromorphic fiber spaces. Then since $\dim B_3 = \dim B_i$, $B(u_i)$ must be bimeromorphic. Hence β_i and β_3 are bimeromorphic as was desired.

Now we show that the above β is a Kummer reduction of X. Let β': $X \to B'$ be any surjective meromorphic map with B' a Kummer manifold. Let B' be bimeromorphic to T'/G' where T' is a complex torus and G' is a finite group. Passing to another bimeromorphic model of X we may assume that $X \to B' \to T'/G'$ is a morphism. Let $\tilde{X}_1$ be an equivariant resolution of an irreducible component of $X \times_{T'/G'} T'$. Let r': $\tilde{X}_1 \to T'$ be the natural morphism. Then r' is factored by the Albanese map $\tilde{X}_1 \to \mathrm{Alb}\,\tilde{X}_1$. Take a Galois covering $\tilde{X}'$ of X with Galois group H and with $q(\tilde{X}') = q^{**}(X)$ which dominates the above irreducible component of $X \times_{T'/G'} T'$. We have thus the natural meromorphic map $\tilde{X}' \to \tilde{X}_1$, which in turn induces a meromorphic map $\tilde{X}'/H \to \tilde{X}_1/G'$ and then $(\mathrm{Alb}\,\tilde{X}')/H \to (\mathrm{Alb}\,\tilde{X}_1)/G'$. Composing the last map with $(\mathrm{Alb}\,\tilde{X}_1)/G' \to T'/G' \to B'$ we get a meromorphic map γ: $(\mathrm{Alb}\,\tilde{X}')/H \to B'$ such that $\gamma\beta = \beta'$ where β: $X \to (\mathrm{Alb}\,\tilde{X}')/H$. Since β is a Kummer reduction of X by what we have proved above, the assertion is proved.

Remark 7.1. It follows from $k(X) = q^{**}(X)$ that $k(X)$ is invariant under finite coverings.

The advantage of considering Kummer reductions instead of Albanese maps is mainly given by the following:

Proposition 7.2. *Let f: $X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$ with $a(X)=0$. Then $k(X) \geq k(Y) + q(f)$. In particular if $k(X) = k(Y)$, then $q(f) = 0$.*

Proof. Take a normal compact complex variety with a finite covering $\tilde{Y} \to Y$ such that $q(\tilde{Y}) = k(Y)$. Let $\tilde{X} := X \times_Y \tilde{Y}$ and let $\tilde{f}$: $\tilde{X} \to Y$ be the resulting fiber space. Since $k(\tilde{X}) = k(X)$, $k(\tilde{Y}) = k(Y)$ and the general fiber of $\tilde{f}$ is isomorphic to those of f, taking $\tilde{f}$ instead of f we may assume from the beginning that $k(Y)=q(Y)$. (For a singular variety Z we set $k(Z)=k(\tilde{Z})$ for any nonsingular model $\tilde{Z}$ of Z.)

Let U be a Zariski open subset of Y over which f is smooth. Then by Proposition 4.5. there exists a finite covering $\nu: Y_1 \to Y$ which is unramified over U such that if $X_1 := X \times_Y Y_1$, then $q(X_1)=q(Y_1)+q(f)$. Hence $k(X)=q^{**}(X) \geqq q(X_1)=q(Y_1)+q(f)=k(Y)+q(f)$. q.e.d.

We now turn to the relative case. For this we need some preliminaries.

7.2. a) Let X be a compact complex manifold and D an analytic subset of X. Then as a generalization of $q^{**}(X)$ we define a nonnegative integer $q^{**}(X, D)$ as follows; $q^{**}(X, D) := \sup_{\tilde{X}} q(\tilde{X})$ where $\tilde{X}$ run through all the compact complex varieties with a finite covering $\tilde{X} \to X$ which is unramified over $X-D$.

b) Let X be a compact complex manifold with $a(X)=0$. Then by Krasnov (cf. [10]) there exist only a finite number of reduced divisors on X. The union D of all such divisors is called the *maximal divisor* on X. By the purity of branch loci (cf. [9, 4.2]) we have $q^{**}(X)=q^{**}(X, D)$.

Let $f: X \to Y$ be a fiber space of compact complex manifolds. Suppose that $a(X)=a(Y)$. Then there exist only a finite number of reduced and irreducible divisors D_i on X such that $f(D_i)=Y$. (See Fischer-Forster [10].) Let D be the union of all such divisors. Then we call D *the relative maximal divisor with respect to* f.

Lemma 7.3. *Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Suppose that $a(f)=0$ so that in particular $a(X)=a(Y)$. Let $D \subseteqq X$ be the relative maximal divisor with respect to f. Then for 'general' $y \in Y$, $a(X_y)=0$ and D_y is the maximal divisor of X_y.*

Proof. Let $N_1=\{y \in U; a(X_y)=0\}$. Then by [18, Proposition 3] (cf. 2.1), $N_1 \in \mathfrak{Q}(Y)$. Let $U \subseteqq Y$ be a Zariski open subset over which f is smooth. Let $\mathrm{Div}\, X_U/U$ be the space of relative divisors on X over Y and $\mathrm{Div}^- X/Y$ the (analytic) closure of $\mathrm{Div}\, X_U/U$ in $D_{X/Y} \supseteqq D_{X_U/U}$ (cf. [18]). Let $\{D_\beta\}_{\beta \in B}$ be the set of those irreducible components D_β of $(\mathrm{Div}^- X/Y)_{\mathrm{red}}$ (the underlying reduced subspace) such that the natural maps $\varphi_\beta: D_\beta \to Y$ are not surjective. Since φ_β is proper, $\bar{D}_\beta=\varphi_\beta(D_\beta)$ is an analytic subset of Y. Let $N_2=Y-\bigcup_\beta \bar{D}_\beta$ and $N=N_1 \cap N_2$. Then $N \in \mathfrak{Q}(Y)$. Therefore the lemma follows if we show that when $y \in N$, $D_{y,\mathrm{red}}$ is the maximal divisor of X_y. For $y \in N$, let $D(y)$ be the maximal divisor of X_y. Let D_α be an irreducible component of $(\mathrm{Div}^- X/Y)_{\mathrm{red}}$ containing the point $d(y) \in D_{X,y}=D_{X/Y,y}$ corresponding to $D(y)$. Since $y \in N_2$, $\varphi_\alpha: D_\alpha \to Y$ is surjective. Let $Z_\alpha \to D_\alpha$ be the universal family restricted to D_α and $\pi_\alpha: Z_\alpha \to X$ the natural map. Then $E_\alpha = \pi_\alpha(Z_\alpha)$ is easily seen to be a divisor on X which is mapped surjectively onto Y. Hence by the maximality of D, $E_\alpha \subseteqq D$. Therefore $D(y) \subseteqq E_{\alpha,y} \subseteqq D_{y,\mathrm{red}}$. Thus $D_{y,\mathrm{red}} = D(y)$ as was desired. q.e.d.

c) Let $f: X \to Y$ be a fiber space of compact complex manifolds and $D \subseteqq X$ a reduced divisor. Then we say that D is of relative normal crossings at $x \in X$ (with respect to f) if there exist local coordinates $x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}$ of X at x and local coordinates $y_1, \cdots, y_n$ of Y around $f(x)$ such that locally at x f is defined by $f(x_1, \cdots, x_{m+n}) = (x_{m+1}, \cdots, x_{m+n})$ and D is defined by $x_1 \cdots x_k = 0$ for some $1 \leqq k \leqq m$ where $\dim X = m+n$ and $\dim Y = n$.

We call D is *of relative normal crossings* over some open subset $U \subseteqq Y$ if D is of relative normal crossings at each point of X_U. Thus in this case 1) D_y is a divisor with only normal crossings in X_y for $y \in U$, 2) $f: (X, D) \to Y$ is analytically locally trivial at each point of X_U and 3) $X_U - D_U \to U$ is a C^∞-fiber bundle over U.

Lemma 7.4. *Let $f: X \to Y$ be a fiber space of compact complex manifolds with $a(X) = a(Y)$. Then there exists a bimeromorphic model $f^*: X^* \to Y$ of f such that the relative maximal divisor $D \subseteqq X^*$ with respect to f^* is of relative normal crossings over some Zariski open subset $U \subseteqq Y$.*

Proof. Let $D \subseteqq X$ be the relative maximal divisor with respect to f. By Hironaka [27] there exists a proper bimeromorphic morphism $h: X^* \to X$ such that $D^* = f^{-1}(D)$ is a divisor with only normal crossings in X^*. Then D^* is of relative normal crossings over some Zariski open subset $U \subseteqq Y$. (cf. [7, 6.15]). It is clear that D^* is the relative maximal divisor with respect to f^*. q.e.d.

e) Moreover we need the following:

Lemma 7.5. *Let $f: X \to V$ be a fiber space of complex manifolds with $X_y \in \mathscr{C}$ for any $y \in V$. Suppose that X has only quotient singularities and that there exists a resolution $r: \tilde{X} \to X$ such that $fr: \tilde{X} \to V$ is smooth. Then the relative Albanese map $\tilde{\alpha} = \tilde{\alpha}_{\tilde{X}/V}: \tilde{X} \to \mathrm{Alb}\,\tilde{X}/V$ for fr* (cf. [18]) *factors through X. Moreover the resulting V-morphism $\alpha_{X/V}: X \to \mathrm{Alb}\,\tilde{X}/V$ is independent of the chosen resolution r.*

Proof. We show that $\tilde{\alpha} := \tilde{\alpha}_{\tilde{X}/V}$ is constant on each fiber of r. This would imply the lemma since X is normal. Let x be any point of X. Take a neighborhood U of x and a finite covering $p: V \to U$ with V smooth. Let $\tilde{U} = r^{-1}(U)$ and $\tilde{V} = V \times_U \tilde{U}$. Let $\tilde{p}: \tilde{V} \to \tilde{U}$ and $\tilde{r}: \tilde{V} \to V$ be the natural projections. Let $\lambda_1: \tilde{V}_1 \to V$ be a proper bimeromorphic morphism such that $\tilde{V}_1$ is nonsingular and $\lambda = \tilde{r}^{-1}\lambda_1\colon \tilde{V}_1 \to \tilde{V}$ is holomorphic (cf. [27]). Then $\tilde{\alpha}\tilde{p}\lambda\colon \tilde{V}_1 \to \mathrm{Alb}\, \tilde{X}/V$ factors through V (cf. the proof of [15, Proposition]) and hence there exists a V-morphism $\beta\colon V \to \mathrm{Alb}\, \tilde{X}/V$ such that $\tilde{\alpha}\tilde{p} = \beta\tilde{r}$:

$$\begin{array}{ccccc} \tilde{V}_1 & \xrightarrow{\lambda} & \tilde{V} & \xrightarrow{\tilde{r}} & V \\ & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ & & \tilde{U} & \xrightarrow[r]{} & U. \end{array}$$

In view of the finiteness of $\tilde{p}$ and the connectedness of the fibers of r, it follows readily that $\tilde{\alpha}$ is constant on each fiber of $r|_{\tilde{U}}$. Since x was arbitrary, this shows the lemma, the final assertion being clear.

In the situation of the above lemma we shall denote $\mathrm{Alb}\, \tilde{X}/V$ by $\mathrm{Alb}\, X/V$ and call $\alpha_{X/V}\colon X \to \mathrm{Alb}\, X/V$ the *relative Albanese map* for f.

7.3. We now turn to the construction of a relative Kummer reduction. We first give a local construction along the general fiber.

Let $f\colon X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Suppose that $a(f) = 0$. Let $D \subseteq X$ be the relative maximal divisor with respect to f. We assume that there exists a Zariski open subset $U \subseteq Y$ such that $f_U\colon X_U \to U$ are smooth and that D is of relative normal crossings over U. This can always be realized after passing to a suitable bimeromorphic model (Lemma 6.4). Let $W := X - D$. Take any contractible open subset $V \subseteq U$. Then we have the natural isomorphism $b_y\colon \pi_1(W_V) \cong \pi_1(W_y)$ for any $y \in V$. Let G be any subgroup of $\pi_1(W_V)$ of finite index. Corresponding to G we have a finite unramified covering $\nu\colon \tilde{W}_V \to W_V$, which induces by restriction the unramified covering $\nu_y\colon \tilde{W}_y \to W_y$ corresponding to $b_y(G) \subseteq \pi_1(W_y)$.

Let $\mu_V\colon \tilde{X}_V \to X_V$ be the finite covering with $\tilde{X}_V$ normal which completes ν; there exists a natural inclusion $\tilde{W}_V \subseteq \tilde{X}_V$ such that $\mu | \tilde{W}_V = \nu$. Then G acts naturally on $\tilde{X}_V$ and we have $X_V \cong \tilde{X}_V/G$. On the other hand, since $(X, D)|_V \to V$ is locally a product at each point of X_V, the same is true for the induced morphism $\tilde{f}_V = f_V\mu_V\colon \tilde{X}_V \to V$. Therefore we can find by [27] an equivariant resolution $r\colon Z_V \to \tilde{X}_V$ such that the resulting morphism $g_V = \tilde{f}_V r\colon Z_V \to V$ is smooth.

Moreover since D has only normal crossings on X_U, $\tilde{X}_V$ has only quotient singularities (cf. Raynaud [37]). Hence by Lemma 7.5 we have the relative Albanese map $\tilde{\alpha}_V\colon \tilde{X}_V \to \tilde{A}_V := \mathrm{Alb}(\tilde{X}_V/V)$ associated to $\tilde{f}_V$. Moreover we get a natural biholomorphic action of G on $\tilde{A}_V$ making $\tilde{\alpha}_V$ G-equivariant. Hence we get a V-morphism $\varphi_V\colon X_V \cong \tilde{X}_V/G \to \tilde{A}_V/G$. Thus we get the following commutative diagram

$$(5)\qquad \begin{array}{ccc} \tilde{X}_V & \xrightarrow{\tilde{\alpha}_V} & \tilde{A}_V \\ {\scriptstyle \mu_V}\swarrow & \downarrow{\scriptstyle \tilde{f}_V} & \swarrow \\ X_V & \xrightarrow{\varphi_V} & \tilde{A}_V/G \\ {\scriptstyle f_V}\searrow & \downarrow & \swarrow \\ & V. & \end{array}$$

Lemma 7.6. *Set $q(\tilde{f}_V)=q(g_V)$. Then for each V as above we can find a subgroup $G \subseteqq \pi_1(W_V)$ such that $q^{**}(X_y, D_y)=q(\tilde{f}_V)$ for any $y \in V$.*

Proof. Since $\mu_{V,y}\colon \tilde{X}_y \to X_y$ is unramified over W_y and $q(\tilde{f}_V)=q(\tilde{X}_y)$, we get $\sup_G q(f_V) \leqq q^{**}(X_y, D_y)$ for any $y \in V$ where G runs through all the subgroups of $\pi_1(W_V)$ of finite index. On the other hand, for any $y \in V$, take an arbitrary finite covering $\mu'_y\colon \tilde{X}'_y \to X_y$ which is finitely unramified over W_y. Let $G_y \subseteqq \pi_1(W_y)$ be the subgroup corresponding to the covering $\mu_y'^{-1}(W_y) \to W_y$. Then if we make the above construction starting from $G=b_y^{-1}(G_y)$, we see readily that $\mu_{V,y}$ is bimeromorphic to μ'_y. Since μ'_y was arbitrary, it follows that $q^{**}(X_y, D_y) \leqq \sup_G q(\tilde{f}_V)$. Hence the equality must hold here. Since y was arbitrary, if we take y with $a(X_y)=0$ then $q^{**}(X_y, D_y)=q^{**}(X_y) \leqq \dim X_y$. Hence sup is attained for some G as was desired. q.e.d.

Remark 7.2. It follows that $q^{**}(X_y, D_y)$ is independent of $y \in V$ and hence of $y \in U$.

We call the diagram (5) *admissible* if G is chosen as in Lemma 7.6.

Proof of Proposition 2.3. Let $U \subseteqq Y$ be a Zariski open subset over which f is smooth. We shall apply Proposition 3.2. Let $M=\{y \in U; a(X_y)=0\}$. Then $M \in \mathfrak{Q}(Y)$ (cf. 2.1). Let $\psi_y\colon X_y \to B_y$ be the Kummer reduction of X_y (Proposition 7.1). Set $\mathfrak{S}=\{\psi_y\}_{y \in M}$. The existence of the admissible diagram (5) (Lemma 7.6) then shows that for any $y \in U$ there exists a neighborhood $y \in V$ such that f_V is very good with respect to $\mathfrak{S}_V$ in the sense of 3.2. Then by Proposition 3.2, 2) there exist a fiber space $B \to Y$ and a meromorphic Y-map $\tilde{\varphi}\colon X \to B$ such that for any V as above $\tilde{\varphi}_V$ is bimeromorphic to the $\varphi_V\colon X \to \tilde{A}_V/G$ in the admissible diagram (5). Let N be a subset of U with $N \in \mathfrak{Q}(Y)$ such that if $y \in N$, then $a(X_y)=0$, D_y is the maximal divisor of X_y and $\tilde{\varphi}$ defines a meromorphic map $\tilde{\varphi}_y\colon X_y \to B_y$ (cf. Lemma 7.3 and [12, Lemma 5.5]). Then from the proof of Proposition 7.1 we see that $\varphi_{V,y}$ for any $y \in V \cap N$ is bimeromorphic to a Kummer reduction of X_y, and hence so is $\tilde{\varphi}_y$ even for any $y \in N$. q.e.d.

§ 8. Fiber spaces with $a(f)=0$ and $k(f)=\dim f$

Using the local construction obtained in the previous section we shall study more closely the structure of a fiber space with $a(f)=0$ and $k(f)=\dim f$, i.e., the general fiber is a Kummer manifold of algebraic dimension zero.

8.1. First we note the following:

Lemma 8.1. *Let X be a Kummer manifold with $a(X)=0$. Let D be the maximal divisor of X and $V=X-D$. Then the fundamental group $\pi_1(V)$ has a unique maximal abelian normal subgroup.*

Proof. Let $Y:=T/G$ be the canonical model of X and $\varphi: X \to Y$ a bimeromorphic map. Let $U=Y-\text{Sing } Y$. Suppose that φ gives an isomorphism of Zariski open subsets $V_1 \subseteqq V$ and $U_1 \subseteqq U$ so that $\pi_1(V_1) \cong \pi_1(U_1)$. Then since $\text{codim}(V-V_1) \geqq 2$ and $\text{codim}(U-U_1) \geqq 2$, we have $\pi_1(V) \cong \pi_1(V_1) \cong \pi_1(U_1) \cong \pi_1(U)$. Then the lemma follows from Lemma 6.2.

Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$ with $a(f)=0$. Let $D \subseteqq X$ be the relative maximal divisor with respect to f. Let $U \subseteqq Y$ be a Zariski open subset over which f is smooth and D is of relative normal crossings. We assume that $U \neq \emptyset$. Now suppose that $k(f)=\dim f$. Then for any contractible open subset $V \subseteqq U$ we have the canonical choice of G in Lemma 7.6. Namely, in the notation of 7.3 since $\pi_1(W_V) \cong \pi_1(W_y)$ for any $y \in V$, taking y from $V \cap N$ where N is chosen as in 7.3 (just after Remark 7.2) we see that there exists a unique maximal normal abelian subgroup G_0 of $\pi_1(W_V)$ by Lemma 8.1. Then for $y \in V \cap N$ the map $\bar{r}_y: Z_{V,y} \to X_y$ is bimeromorphic to the quotient map $T_y \to T_y/G_y$where T_y/G_y is the canonical model of X_y (cf. the proof of Lemma 8.1). Thus G_0 may serve as G as above. Note that in this case φ_V is bimeromorphic in (5) since $\varphi_{V,y}$ is bimeromorphic for $y \in V \cap N$. In this case we call the diagram (5) *canonically associated to* f_V.

Lemma 8.2. *X_y is bimeromorphically quasi-hyperelliptic for any $y \in U$ and if (5) is canonically associated to f_V, $\tilde{A}_y/G_y$ is the canonical model of X_y where $\tilde{A}_y=(\tilde{A}_V)_y$.*

Proof. Let G_1 be any subgroup of G. Let F_1 be the set of fixed points of G_1. By Lemma 8.3 below F_1 is smooth over V. Then since $\text{codim } F_{1,y} \geqq 2$ in A_y if $y \in V \cap N$, the same is true for all $y \in V$. Since G_1 was arbitrary and X_y is bimeromorphic to $\tilde{A}_y/G_y$, X_y is bimeromorphically quasi-hyperelliptic. Since G_y is the maximal normal abelian subgroup of $\pi_1(W_y)$ for any $y \in V$, this also shows the final assertion (cf. Lemma 6.3).

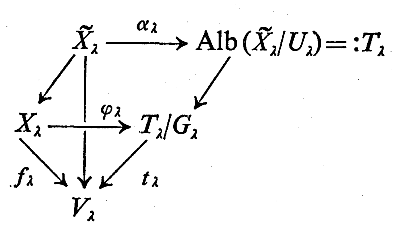
Lemma 8.3. *Let $f: X \to V$ be a smooth fiber space of complex manifolds and $G \subseteq \mathrm{Aut}(X/V)$ a finite subgroup. Let F be the set of fixed points of G on X. Then F is smooth over V.*

Proof. It is well-known that F is nonsingular. Let $x \in F$ and $v = f(x)$. Let T_X (resp. T_F) be the tangent space of X (resp. F) at x and T_V the tangent space of V at v. Let $f_*: T_X \to T_V$ be the differential of f at x. Since f is smooth, f_* is surjective. On the other hand, we get a G-equivariant direct sum decomposition $T_X = T_F \oplus E$ where E has no nontrivial subspace on which G acts trivially. Since f is G-equivariant, E is mapped to zero by f_*. Hence $f_*|_{T_F}: T_F \to T_V$ is surjective, which implies that F is smooth over V.

8.2. Using the description above we now associate to $f: X \to Y$ in 8.1 a variation of real Hodge structure of weight 1 parametrized by U, and hence is a period map defined on U.

Lemma 8.4. *Let $f: X \to Y$ and $U \to Y$ be as in* 8.1. *Then there exists a variation* $(U, \boldsymbol{H}_C, F^p)$ *of real Hodge structure of weight* 1 *parametrized by* U *such that* 1) $\boldsymbol{H}_{C,y} = H^1(Y_y, \boldsymbol{C})$ *where T_y/G_y is the canonical model of X_y and* 2) $F^1 = H^{1,0}(T_y) \subseteq \boldsymbol{H}_{C,y}$.

Proof. In the notation of [38, p. 220] (cf. 3.1): 1) $M = U$, 2) the local system $\boldsymbol{H}_K$, $K = \boldsymbol{Z}, \boldsymbol{R}, \boldsymbol{C}$, is given by $\boldsymbol{H}_{K,y} := H^1(T_y, \boldsymbol{K})$, 3) $k = 1$, 4) a flat nondegenerate bilinear form S on $\boldsymbol{H}_R$ will be defined below, 5) the Hodge subbundle $\boldsymbol{F}^1$ of $\boldsymbol{H}_C$ is given by $\boldsymbol{F}^1_y = H^{1,0}(T_y)$ as above. ($\boldsymbol{F}^0 = \boldsymbol{H}_C$, $\boldsymbol{F}^2 = \{0\}$.) We shall now define S. Fix once and for all a compact Kähler manifold Z with a Kähler form ω and a surjective morphism $g_0: Z \to X$. Set $g = fg_0: Z \to Y$. Fix a Zariski open subset $V \subseteq Y$ such that $V \subseteq U$ and g is smooth over V. Take a locally finite open covering $\{V_\lambda\}$ of V with V_λ contractible. Let

(5)


be the diagram (5) with $f_V = f_\lambda$ which is canonically associated to f_λ where $X_\lambda := X_{V_\lambda}$ and $f_\lambda = f_{V_\lambda}$ etc. and where α_λ and φ_λ are bimeromorphic. Then we have $T_y \cong T_{\lambda,y}$ for $y \in V_\lambda$. Set $Z_\lambda := Z_{X_\lambda} = g_0^{-1}(X_\lambda) = g^{-1}(V_\lambda)$ and $\hat{Z}_\lambda = Z_\lambda \times_{X_\lambda} \tilde{X}_\lambda$. Fix a resolution $r_\lambda: \tilde{Z}_\lambda \to \hat{Z}_\lambda$. Define the composite maps $\tilde{g}_\lambda$,

π_λ, b_λ by $\tilde{g}_\lambda$: $\tilde{Z}_\lambda \xrightarrow{r_\lambda} \hat{Z}_\lambda \to \tilde{X}_\lambda \to V_\lambda$, π_λ: $\tilde{Z}_\lambda \xrightarrow{r_\lambda} \hat{Z}_\lambda \to Z_\lambda$, and b_λ: $\tilde{Z}_\lambda \xrightarrow{r} \hat{Z}_\lambda \to \tilde{X}_\lambda \xrightarrow{\alpha_\lambda} T_\lambda$. Take a Zariski open subset $V'_\lambda \subseteq V_\lambda$ such that $\tilde{g}_\lambda$ is smooth over V_λ and that $\pi_{\lambda,u}$: $\tilde{Z}_{\lambda,u} \to \hat{Z}_{\lambda,u}$ is a resolution of $\hat{Z}_{\lambda,u}$ for each $u \in V'_\lambda$.

Now let $u \in Y$. Suppose that $u \in V'_\lambda$. For each $\varphi_u, \psi_u \in \boldsymbol{H}^1_{\boldsymbol{R},u} = H^1(T_u, \boldsymbol{R})$ we define $S_u(\varphi_u, \psi_u) = \int_{\tilde{Z}_{\lambda,u}} \tilde{\omega}^{m-1}_{\lambda,u} \wedge b^*_{\lambda,u}(\varphi_u \wedge \bar{\psi}_u)$ where $\tilde{\omega}_{\lambda,u} = \pi^*_{\lambda,u}\omega_u$, $(\omega_u = \omega|_{Z_u})$ and $m = \dim Z_{\lambda,u}$ and where φ_u and ψ_u are identified with closed C^∞ 1-forms which represent φ_u and ψ_u. Since $\pi_{\lambda,u}$ and $b_{\lambda,u}$ factor through $\hat{Z}_{\lambda,u} = Z_u \times_{X_u} \tilde{X}_u$, S_u actually is independent of the choice of the resolution r_λ and of the choice of λ with $u \in V'_\lambda$, depending only on u. Since $\tilde{\omega}_{\lambda,u}$ is the pull-back of the Kähler form ω_u by the generically finite surjective morphism $\tilde{Z}_{\lambda,u} \to Z_u$, it is immediate to see that S_u is non-degenerate and the triple $\Phi(u) = (\boldsymbol{H}_{\boldsymbol{R},u}, \boldsymbol{F}^p_u, S_u)$ defines a (real) polarized Hodge structure of weight 1. This is our definition of S. In fact, since β_λ (resp. $\tilde{g}_\lambda$) is smooth over V_λ (resp. V'_λ) where β_λ: $T_\lambda \to T_\lambda/G_\lambda \to V_\lambda$, $\boldsymbol{H}_{\boldsymbol{K}}$ is really a local system on U, $\boldsymbol{F}^1$ is a holomorphic subbundle of $\boldsymbol{H}_{\boldsymbol{C}}$ on U and S is a flat quadratic form on $\boldsymbol{H}_{\boldsymbol{R}}$ defined over $V' := \bigcup_\lambda V'_\lambda$ which is dense in U. Then as in 4.1 a) S extends to a unique flat quadratic form on $\boldsymbol{H}_{\boldsymbol{R}}$ over U. Thus the above data 1)–5) actually gives a variation of Hodge structure of weight 1 parametrized by U. q.e.d.

8.3. Using the period map of Lemma 8.4 we give a condition for bimeromorphic 'quasi-triviality' of a fiber space f with $a(f)=0$ and $k(f) = \dim f$. We need the following:

Lemma 8.5. *Let T be a complex torus with $a(T)=0$. Let V be a complex manifold and $X = T \times V$. Let p: $X \to V$ be the natural projection. Let $G \subseteq \mathrm{Aut}\,(X/V)$ be a finite subgroup. Let $Y = X/G$ and f: $Y \to V$ the induced morphism. Then f is locally trivial.*

Proof. Fixing the origin $o \in T$ we consider T as a complex Lie group. First we show that f is locally a product at each point $y \in Y$. We have the semi-direct decomposition $\mathrm{Aut}\,T = T \cdot H(T)$ where $H(T)$ is the group of automorphisms of T as a complex Lie group (cf. [19]). As we have noted in [19], the $H(T)$-part of $g \in G$ is independent of v. Namely we can write each $g \in G$ in the form

$$g(t) = A(g)t + b(g)(v), \qquad t \in T$$

where $b(g)$ is a T-valued holomorphic function on V and $A(g)$ is independent of v. Fix $y \in Y$ and set $v = f(y)$. Then choose any $x = (t, v) \in \pi^{-1}(y)$ where π: $X \to Y$ is the natural projection. Let G_x be the stabilizer of G at x and F_x the set of fixed points of G_x. Since F_x is smooth over V by

Lemma 8.3, we can take a holomorphic section $s: V\to F$ with $s(v)=x$ locally at v. Then after changing the zero section of p from $\bar{s}(v)=(o, v)$ to $s(v)$, for any $g\in G_x$ g takes the form $t\to A(g)t$ which is independent of v. Thus $(Y, y)\cong(X, x)/G_x$ is locally trivial over (V, v), as was desired.

Let F be the set of those points of X which are fixed by some elements of G. Let $U=X-F$ and $W=Y-\mathrm{Sing}\,Y$. Since $a(T)=0$, codim $F\geqq 2$ in X. Hence Sing $Y=\pi(F)$ and π induces an unramified covering $U\to W$. Now since the family is locally trivial on Y, we get a short exact sequence of $\mathcal{O}_Y$-modules

$$0\longrightarrow\Theta_{Y/V}\longrightarrow\Theta_Y\longrightarrow f^*\Theta_V\longrightarrow 0$$

which splits locally where Θ_Y (resp. $\Theta_{Y/V}$) is the sheaf of germs of holomorphic vector fields on Y (resp. which are tangent to the fibers of f) and Θ_V is defined similarly. We also get a similar exact sequence from the product family $p: X\to V$. From these, we obtain the following diagram of exact sequences

$$\begin{array}{l} f_*\Theta_Y\longrightarrow\Theta_V\xrightarrow{\rho}R^1f_*\Theta_{Y/V}\\ p_*\Theta_X\longrightarrow\Theta_V\xrightarrow{\tilde{\rho}}R^1p_*\Theta_{X/V}.\end{array}$$

Clearly $\tilde{\rho}$ is the zero map. We show that ρ also is the zero map. We consider the following diagram

$$\begin{array}{ccc} & R^1f_*\Theta_{Y/V}\xrightarrow{r}R^1(f|_W)_*\Theta_{W/V}\\ \Theta_V\begin{array}{c}\nearrow^{\rho}\\ \searrow_{\tilde{\rho}}\end{array} & \\ & R^1p_*\Theta_{X/V}\xrightarrow{\tilde{r}}R^1(p|_U)_*\Theta_{U/V}\end{array}$$

where r and $\tilde{r}$ are the restriction maps. Since Y is normal, codim (Sing Y) $\geqq 2$ and depth $\Theta_Y\geqq 2$, and hence, depth $\Theta_{Y/V}\geqq 2$. Hence r is injective. Since π is unramified on U, we have $R^1(p|_U)_*\Theta_{U/V}=R^1(p|_U)_*(\pi_U^*\Theta_{W/V})=R^1(f|_W)_*(\Theta_{W/V}\otimes\pi_*\mathcal{O}_U)$. On the other hand, the natural map $R^1(f|_W)_*\Theta_{W/V}\to R^1(f|_W)_*(\Theta_{W/V}\otimes_{\mathcal{O}_W}\pi_*\mathcal{O}_U)$ is injective since $\mathcal{O}_W$ is naturally a direct summand of $\pi_*\mathcal{O}_U$. Hence we have an injective map $\nu: R^1(f|_W)_*\Theta_{W/V}\to R^1(p|_U)_*\Theta_{U/V}$. Then it is easy to see that $\nu r\rho=\tilde{r}\tilde{\rho}$. Since $\tilde{\rho}$ is the zero map and ν and r are injective, ρ is the zero map. Thus $f_*\Theta_Y\to\Theta_V$ is surjective. Hence there is a holomorphic vector field on Y in a neighborhood of each fiber Y_v which is mapped to a nonvanishing vector field on V. Integrating such a vector field we obtain a desired local isomorphism.

q.e.d.

Proposition 8.6. *Let f: $X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$ with $a(f)=0$ and $k(f)=\dim f$. Suppose that $a(Y)=q(f)=0$. Then there exist a finite covering h: $\tilde{Y} \to Y$ and a Kummer manifold F with $a(F)=q(F)=0$ such that $X \times_Y \tilde{Y}$ is bimeromorphic over $\tilde{Y}$ to the product $\tilde{Y} \times F$.*

Proof. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth. By Lemma 7.4 passing to another bimeromorphic model and restriction to U we may assume that the relative maximal divisor with respect f is of relative normal crossings over U. By Lemma 8.2 X_y is bimeromorphically quasi-hyperelliptic for $y \in U$. Let T_y/G_y be the canonical model of X_y. Then we show that the isomorphism class of T_y is independent of y. Let $(U, \boldsymbol{H}_C, F^p)$ be the variation of real Hodge structure of weight 1 parametrized by U defined in Lemma 8.4. Let Φ: $U \to D/\Gamma$ be the associated period map. Since D is isomorphic to the Siegel upper half space and $a(Y)=0$, by Proposition 4.1 Φ must be a constant map. This implies that the moduli of T_y is constant as was desired. We now consider the diagram $(5)_\lambda$. Let u_λ: $G_\lambda \to \mathrm{Aut}(T_\lambda)_u$ be the restriction map and $G_{\lambda,u} := u_\lambda(G_\lambda)$. Since β_λ: $T_\lambda \to U_\lambda$ is locally trivial by what we have proved above (cf. [11]) t_λ also is locally trivial by Lemma 8.5. Therefore if we set $F_u := (T_\lambda/G_\lambda)_u = T_{\lambda,v}/G_{\lambda,u}$ which is independent of λ, then $F=F_u$ is up to isomorphisms independent of $u \in U$. For each $u \in U$, $\varphi_{\lambda,u}$: $X_u \to F \in \mathrm{BHol}(X_u, F)$ (cf. Proposition 6.5). In particular $\mathrm{BHol}(X_u, F) \neq \emptyset$. Let $X' = F \times Y$ with the natural projection $X' \to Y$. Then for $u \in U$ we have $\mathrm{BHol}(X_u, X'_u) = \mathrm{BHol}(X_u, F) \neq \emptyset$. Moreover by Proposition 6.5, 2) $\mathrm{BHol}(X_u, F) \cong \mathrm{Aut}\, F$. Since $q(F)=q(X_u)=0$ by our assumption, from Proposition 6.5, 1) it follows that Aut F is a discrete group. Hence $\mathrm{BHol}(X_u, X'_u)$ is discrete. Thus by 2) of Proposition 9 of [19] there exists a finite covering $\tilde{Y} \to Y$ such that $X \times_Y \tilde{Y}$ and $X' \times_Y \tilde{Y} = F \times \tilde{Y}$ is bimeromorphic over $\tilde{Y}$. q.e.d.

8.4. *Proof of Proposition* 6.7. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth. For any contractible open subset $V \subseteq U$ we consider the diagram (5) which is canonically associated to f. Since X_y are hyperelliptic for $y \in U$, $\tilde{\alpha}_V$ is isomorphic in our case. Restricting V assume that there is a holomorphic section s: $V \to \tilde{A}_V$ so that $\tilde{A}_V$ is considered as a complex Lie group with $s(V)$ the identity section. We have the natural semidirect product decomposition $\mathrm{Aut}(\tilde{A}_V/V) \cong H(\tilde{A}_V/V) \cdot \Gamma(V, \tilde{A}_V)$ where $H(\tilde{A}_V/V) = \{g \in \mathrm{Aut}(\tilde{A}_V/V);\, g(s(v)) = s(v) \text{ for all } v \in V\}$ (cf. [19]). Let H: $\mathrm{Aut}(\tilde{A}_V/V) \to H(\tilde{A}_V/V)$ be the natural projection. Let $T_{V,0}$ be the connected component of the identity section of the subspace $\bigcap_g (A_V(g) - I)$ where I denotes the identity. Then $T_{V,0}$ is a complex Lie subgroup of T_V over V which is smooth over V. Let π_V: $T_V \to \bar{T}_V := T_V/T_{V,0}$ be the relative (geometric) quotient of T_V with respect to $T_{V,0}$. There exists a natural G-action on $\bar{T}_V$ which makes π_V equivariant. Hence we have a V-morphism $\bar{\pi}_V: X_V \to \bar{Y}_V := \bar{T}_V/G$. From our construction it is clear that $\bar{\pi}_{V,u}: X_{V,u} \to \bar{Y}_{V,u}$ is the co-Albanese map for $X_{V,u}$.

The rest of the proof is essentially the same as that of 2) of Proposition 3.2 except that here we use the relative Barlet space $B_{X/Y}$ (cf. [12, 3.2]) instead of the relative Douady space. In fact, because of the normality of $\bar{Y}$ and of the equidimensionality of the fibers of $\bar{\pi}_V$ we get a V-morphism $j_V: \bar{Y}_V \to B_{X_V/V} = B_{X/Y}|_Y$ induced by the universality of $B_{X_V/V}$ (cf. [2, Theorem 1, p. 38]). Then j_V is actually injective onto some irreducible component, say Z'_V, of $B_{X_V/V}$ and moreover there exist a unique irreducible component Z of $B_{X/Y}$ and a meromorphic Y-map $g': X \to Z'$ which is holomorphic over U such that $Z'|_V = Z'_V$ and $g'_V: X_V \to Z'_V$ is bimeromorphic to $\bar{\pi}_V$ for any V as above (cf. the proof of Proposition 3.2). Let $n: Z \to Z'$ be the normalization of Z'. Then it is easy to see that $g := n^{-1}g': X \to Z$ is the desired co-Albanse map for f.

§ 9. Proof of Theorem 1

9.1. Before the proof we give two important propositions of independent interest. The first one concerns the structure of a (holomorphic) algebraic reduction $f: X \to Y$ whose general fiber is a certain type of complex torus. We begin with making this last point precise.

Let T be a complex torus. Then we say that T is *obtained by a successive extension of abelian varieties* if there exists a sequence of subtori $T_1 \subseteqq T_2 \subseteqq \cdots \subseteqq T_m = T$ such that T_i/T_{i-1}, $1 \leqq i \leqq m$, are all abelian varieties where $T_1/T_0 = T_1$. It is immediate to see that in this case any subtorus or any quotient torus of T has again the same property.

On the other hand, we note the following. Let Z be a subvariety of a complex tours T. Let $\mathrm{Aut}(T, Z) = \{g \in \mathrm{Aut}\, T; g(Z) = Z\}$. Then $\kappa(Z) + \dim \mathrm{Aut}(T, Z) = \dim Z$. This is due to Ueno (cf. [43, 10.9]).

Proposition 9.1. *Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$, which is an algebraic reduction of X. Suppose that* $\dim f$ >0 *and that each smooth fiber is a complex torus which is obtained by a successive extension of abelian varieties. Then there exists no proper analytic subvariety of $Z \subseteqq X$ with $f(Z) = Y$.*

Proof. Let $Z \subseteqq X$ be a subvariety. Supposing that $f(Z) = Y$ we shall derive a contradiction. Let $\tilde{Z} \to Z$ be the normalization of Z and $(\tilde{Z} \to \tilde{Y}, \tilde{Y} \to Y)$ be the Stein factorization of the induced map $\tilde{Z} \to Y$. Then considering instead of f (resp. Z) the base change $\tilde{f}$ of f to $\tilde{Y}$ followed by a resolution (resp. a suitable irreducible component of $\tilde{Z} \times_Y \tilde{Y}$), to derive contradiction, we may assume that the general fiber of $Z \to Y$ is irreducible. Let U be a Zariski open subset over which f is smooth and $f|_Z$ is flat. Let $\mathrm{Aut}^*_{Y,0}(X, Z) \subseteq \mathrm{Aut}^*_Y X$ be the unique irreducible component of $\mathrm{Aut}^*_Y(X, Z)$ which contains the identity section of $\mathrm{Aut}_U(X_U, Z_U) \to U$ (cf. 5.2). Let $y \in U$. Then since X_y is a complex torus, $\mathrm{Aut}^*_Y(X, Z)_y \cong \mathrm{Aut}(X_y, Z_y)$ and further $\mathrm{Aut}^*_{Y,0}(X, Z)_y \cong \mathrm{Aut}_0(X_y, Z_y)$. Hence $\mathrm{Aut}^*_{Y,0}(X, Z)$ is a complex torus. Therefore if $(\varphi: X \to \bar{X}, \bar{f}: \bar{X} \to Y)$ is a relative generic quotient of X by $\mathrm{Aut}^*_{Y,0}(X, Z)$ over Y (cf. 5.2), $\varphi_U: X_U \to \bar{X}_U$ is actually a geometric quotient (cf. [19, Proposition 1]). In particular $\bar{f}_U: \bar{X}_U \to U$ is smooth and any of its fiber is again a complex torus which is obtained by a successive extension of abelian varieties. Since $Z \neq X$, $\dim \bar{f} > 0$. Let $\bar{Z}$ be the image of Z in $\bar{X}$. Then $\bar{Z}_y$ are of general type for all $y \in U$ by the remark preceding the proposition. This implies that $\bar{f}|_{\bar{Z}}: \bar{Z} \to Y$ is Moishezon. Then $\bar{Z}$ is Moishezon as well as Y. Hence we have a subvariety $\bar{Z}' \subseteq \bar{Z}$ such that $\bar{f}|_{\bar{Z}'}: \bar{Z}' \to Y$ is generically finite and surjective. Let $(\bar{g}: \bar{X} \to \bar{X}', \bar{h}: \bar{X}' \to Y)$ be a relative algebraic reduction of $\bar{f}$. Then $\bar{g}$ is holomorphic over some Zariski open subset of Y and the general fiber of $\bar{h}$ is an abelian variety. (An algebraic reduction of a complex torus is given by a quotient by some subtorus.) Thus $\bar{g}(\bar{Z}')$ gives a meromorphic multi-section to $\bar{h}$. Then by [18, Proposition 6] $\bar{h}$ is a Moishezon morphism and so $\bar{X}'$ is Moishezon as well as Y. Since $\bar{h}$ is an algebraic reduction of $\bar{X}$, this implies that $\dim \bar{h} = 0$. This contradicts the fact that each smooth fiber $\bar{X}_y$ of $\bar{f}$ is obtained by a successive extension of abelian varieties and hence $a(\bar{f}) = a(\bar{X}_y) > 0$. q.e.d.

9.2. For the next proposition we need the following:

Lemma 9.2. *Let X be a compact complex manifold on which a linear algebraic group G acts biholomorphically and meromorphically* (cf. [13]). *Suppose that X is almost homogeneous with respect to G so that G has a Zariski open orbit $U \subseteq X$. Let $D = X - U$. Then $q^{**}(X, D) = 0$.*

Proof. Since the identity component of G also acts homogeneously on U, we may assume that G is connected. Let $\pi: \tilde{X} \to X$ be any finite covering which is unramified over U. Then by Proposition 6.4 there exists a connected linear algebraic group $\tilde{G}$ acting biholomorphically and meromorphically on $\tilde{X}$ with open orbit $\tilde{U} = \pi^{-1}(U)$. Hence $\tilde{X}$ is unirational and therefore $q(\tilde{X}) = 0$. Since π can be chosen arbitrarily, $q^{**}(X, D) = 0$.
q.e.d.

Proposition 9.3. *Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $f = f_3 f_2 f_1$ be a decomposition of f into three fiber spaces $f_1: X \to X_1$, $f_2: X_1 \to X_2$, $f_3: X_2 \to Y$ of compact complex manifolds in $\mathscr{C}$.*

Suppose that f_1 *is a relative algebraic reduction of* $f_{12}:=f_2f_1$. *Suppose further that* $a(X)=a(Y)$ *and* $q(f_2)=0$. *Then* $a(f_1)=0$.

Proof. Assuming that $a(f_1)>0$ we shall derive a contradiction. Let $f=f_1'f_1''$, f_1': $X\to X_1'$, f_1'': $X_1'\to X_1$, be a relative algebraic reduction of f_1. Then to derive a contradiction by replacing $f=f_3f_2f_1$ by $f':=f_3f_2f_1''$ we may assume that f_1 is locally Moishezon since also for $f'=f_3f_2f_1''$ the conditions of the proposition are still verified and $a(f_1)>0$. Suppose first that $q(f_1)=0$. Then f_1 is Moishezon by Proposition 2.5. On the other hand, since $q(f_2)=0$ by assumption f_2 also is Moishezon by the same proposition. Then f_{12} is Moishezon, contradicting our assumption that f_1 is a relative algebraic reduction of f_{12} and that $\dim f_1>0$. So we may assume that $q(f_1)>0$. Take Zariski open subsets $U\subseteqq X_2$, $W\subseteqq X_1$ with $f_2(W)\subseteqq U$ such that $f_{12,U}$: $X_U\to U$, $f_{2,U}$: $X_{1,U}\to U$ and $f_{1,W}$: $X_W\to W$ are all smooth. Then by Proposition 4.1 and Proposition 4.5 (restricting U and W if necessary) for each $u\in U$ there exists a finite covering ν_u: $\tilde{X}_{1,u}\to X_{1,u}$ which is unramified over W_u such that if we put $\tilde{X}_u=X_u\times_X\tilde{X}_{1,u}$ and define $\tilde{f}_{1,u}$: $\tilde{X}_u\to\tilde{X}_{1,u}$ by the natural projection, then we get $q(\tilde{X}_u)=q(\tilde{X}_{1,u})+q(\tilde{f}_{1,u})$. On the other hand, since we may assume that f_2 is projective by passing to a suitable bimeromorphic model, and since $a(X_1)=a(Y)$ and $q(f_2)=0$, we can apply Proposition 5.2 to $f_{23}=f_3f_2$: $X_1\to X_2\to Y$. In particular after an eventual restriction of U there exists a Zariski open subset $W_0\subseteqq X_1$ with $f_1(W_0)\subseteqq U$ such that $W_0\subseteqq W$ and $W_{0,u}$ is homogeneous with respect to a linear algebraic subgroup G_u of $\mathrm{Aut}\,X_{1,u}$. Thus, $\tilde{W}_{0,u}:=\nu_u^{-1}(W_{0,u})\to W_{0,u}$ being unramified, by Lemma 9.2 $q(\tilde{X}_{1,u})=0$. Hence we get $q(\tilde{X}_u)=q(\tilde{f}_{1,u})>0$. For $\tilde{w}\in\tilde{W}_u$ let $\tilde{\alpha}_w$: $\tilde{X}_{\tilde{w}}\to\mathrm{Alb}\,\tilde{X}_{\tilde{w}}$ be the Albanese map for $\tilde{X}_{\tilde{w}}=\tilde{f}_{1,u}^{-1}(\tilde{w})$ and let $\tilde{\alpha}_u$: $\tilde{X}_u\to\mathrm{Alb}\,\tilde{X}_u$ be the Albanese map for $\tilde{X}_u$. Then we have the unique affine map β: $\mathrm{Alb}\,\tilde{X}_{\tilde{w}}\to\mathrm{Alb}\,\tilde{X}_u$ with $\tilde{\alpha}_u|\tilde{X}_{\tilde{w}}=\beta\tilde{\alpha}_{\tilde{w}}$. The above equality then implies that β is isogenous (cf. the proof of Proposition 4.6). Then since $\tilde{X}_{\tilde{w}}$ is Moishezon, $\mathrm{Alb}\,\tilde{X}_{\tilde{w}}$, and hence $\mathrm{Alb}\,\tilde{X}_u$ also, is an abelian variety. Now we take $u\in U$ 'general' in such a way that $f_{1,u}$: $X_u\to X_{1,u}$ is an algebraic reduction of X_u, which is possible since f_1 is an algebraic reduction of f_{12}. Then $\tilde{f}_{1,u}$ is again an algebraic reduction of $\tilde{X}_u$ since $a(\tilde{X}_u)=a(X_u)$. This is a contradiction since $\tilde{\alpha}_u$ is not factored by $\tilde{f}_{1,u}$, and $\mathrm{Alb}\,\tilde{X}_u$ is an abelian variety of positive dimension. q.e.d.

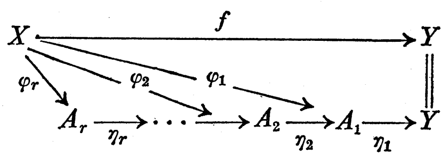
9.3. The universal property of the diagram (3) mentioned in Section 2 is given by the following:

Proposition 9.4. *The diagram* (3) *is characterized by the following universal property. For any fiber space* f^*: $X^*\to Y$ *bimeromorphic to* f *and for any decomposition* $f^*=h^*g^*$ *of* f^* *into two fiber spaces* g^*: $X^*\to X'^*$, h^*: $X'^*\to Y$ *with* $a(g^*)=0$, *there exists a unique meromorphic map*

u: $X'^{*}\rightarrow X'$ such that $g=ug^{*}\varphi$ and $h=h^{*}u$, where φ: $X\rightarrow X^{*}$ is a fixed bimeromorphic map.

Proof. We use the notation of 2.2. We first see that $\dim(X_{x'^{*}})=0$ for general $x'^{*}\in X'^{*}$ where $X_{x'^{*}}$ is the fiber of the meromorphic map $g^{*}\varphi$: $X\rightarrow X'^{*}$. In fact, otherwise, in (2) let i be the smallest index such that $\dim f_{i}(X_{x'^{*}})>0$ for general $x^{*}\in X'^{*}$. Since by our choice of i for any x'^{*} $f_{i}(X_{x'^{*}})$ is contained in a fiber of $X_{i}\rightarrow X_{i-1}$ so that $f_{i}(X_{x'^{*}})$ is Moishezon. On the other hand, since $a(g^{*})=0$, $a(f_{i}(X_{x'^{*}}))=0$ for 'general' $x'^{*}\in X'^{*}$. This is a contradiction. Thus $\dim g(X_{x'^{*}})=0$. It follows then readily that g induces a unique meromorphic map u: $X'^{*}\rightarrow X'$. It is immediate to see that u has the desired commutativity property. q.e.d.

9.4. *Proof of Theorem* 1. We shall construct a diagram of meromorphic fiber spaces which is bimeromorphic to (3) in Section 1 and which satisfies the conclusion of Theorem 1. First we shall construct a commutative diagram

(6)


of compact complex manifolds in $\mathscr{C}$ satisfying the following properties: For any $1\leqq k\leqq r$ $1)_{k}$ the general fiber of the composite morphism $\gamma_{k}=\eta_{1}\cdots\eta_{k}$: $A_{k}\rightarrow Y$ is a complex torus which is obtained by a successive extension of abelian varieties, and $2)_{k}$ φ_{k}: $X\rightarrow A_{k}$ is a fiber space. (Here X may be replaced by a suitable bimeromorphic model of it.) We proceed inductively. So suppose that we have already constructed φ_{i}: $X\rightarrow A_{i}$, η_{i}: $A_{i}\rightarrow A_{i-1}$ for $0\leqq i\leqq k-1$ for some $k>0$ satisfying $1)_{i}$, $2)_{i}$ for $1\leqq i\leqq k-1$, where we set $\varphi_{0}=f$, $A_{0}=A_{-1}=Y$ and $\eta_{0}=\mathrm{id}_{Y}$. Then if $a\text{-}q(\varphi_{k-1})=0$, we set $k-1=r$, and if $a\text{-}q(\varphi_{k-1})>0$, we define ($\varphi_{k}$: $X\rightarrow A_{k}$, η_{k}: $A_{k}\rightarrow A_{k-1}$) to be the relative algebraic Albanese map for φ_{k-1} where we assume that φ_{k} is holomorphic by passing to another bimeromorphic model of X if necessary. We need to show the following:

Claim. $1)_{k}$ *and* $2)_{k}$ *are true for* η_{k} *and* φ_{k} *defined above.*

Admitting the claim for the moment, and hence that the construction of (6) is already done, let (g_{1}: $X\rightarrow X_{1}$, α: $X_{1}\rightarrow A_{r}$) be the relative algebraic reduction for the fiber space φ_{r}: $X\rightarrow A_{r}$. We assume that g_{1} is holomorphic as above. Set $A=A_{r}$ $\varphi=\varphi_{r}$ and $\gamma=\eta_{1}\cdots\eta_{r}$. Then we get the following commutative diagram of fiber spaces

$$
(7)\qquad
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle g_1}\searrow & \overset{\varphi}{\searrow} & \nearrow{\scriptstyle \gamma} \\
X_1 & \xrightarrow[\ \alpha\ ]{} & A.
\end{array}
$$

We have $q(\alpha)=a\text{-}q(\varphi)=0$. Hence α is Moishezon by Proposition 2.5. Now we turn to

Proof of Claim. The claim obviously follows from a) and b) below.

a) We show that $2)_k$ is true assuming that $1)_k$ (and $2)_{k-1}$) are true.

Surjectivity of $\varphi_k\colon X\to A_k$. Let $\bar{X}_k:=\varphi_k(X)$. Suppose that $\bar{X}_k\subsetneqq A_k$. We then show that $\kappa(\bar{X}_k/Y)>0$, which would contradict the fact that $\bar{X}_k\to Y$ is an algebraic reduction of $\bar{X}_k$ (cf. Proposition 1.1). First, by $1)_k$ the general fiber of $A_k\to Y$ is a complex torus. Next, note that for general $y\in Y$, $\bar{X}_{k,y}$ generates A_y. In fact, the natural morphism $\bar{X}_{k,y}\to A_{k-1,y}$ is surjective by $2)_{k-1}$. Moreover, by the definition of φ_k and $\bar{X}_k$ for general (y and) $a\in A_{k-1,y}$, $(\bar{X}_{k,y})_a$ generates $(A_{k,y})_a$. From this it follows readily that $\bar{X}_{k,y}$ generates $A_{k,y}$. Then by a theorem of Ueno ([43, 10.5]), $\kappa(\bar{X}_{k,y})>0$. Hence $\kappa(\bar{X}_k/Y)>0$ as was desired. Thus $\varphi_k(X)=A_k$. In particular $\gamma_k\colon A_k\to Y$ is an algebraic reduction of A_k.

Next, we show that the general fiber of $\varphi_k\colon X\to A_k$ is connected. Let $U_k:=\{a\in A_k;\ \varphi_k$ is smooth along $X_a\}$. Since $A_{k,y}$ is a complex torus obtained by a successive extension of abelian varieties by $1)_k$, $\gamma_k(A_k-U_k)\neq Y$ by Proposition 9.1. In particular for general $y\in Y$, $\varphi_{k,y}\colon X_y\to A_{k,y}$ is smooth. Consider the commutative diagram

$$
\begin{array}{ccc}
X_y & \xrightarrow{\ \varphi_{k,y}\ } & A_{k,y} \\
{\scriptstyle \varphi_{k-1,y}}\searrow & & \swarrow{\scriptstyle \eta_k} \\
 & A_{k-1,y}\,. &
\end{array}
$$

We know that $\varphi_{k-1,y}$ is a fiber space by $2)_{k-1}$ and for 'general' $a\in A_{k-1,y}$, $(\varphi_{k,y})_a\colon (X_y)_a\to(A_{k,y})_a$ is an algebraic Albanese map for $(X_y)_a$. Since $(\varphi_{k,y})_a$ is smooth, it follows that $(\varphi_{k,y})_a$ and hence $\varphi_{k,y}$, φ_k also have connected fibers (cf. 2.5).

b) We show that $1)_k$ holds true. Since $1)_1$ is clearly true by our construction, here we may assume that $k>1$. Let $V_{k-1}:=\{a\in A_{k-1};\ \eta_k$ is smooth along $A_{k,a}:=\eta_k^{-1}(a)\}$. Then as above by $1)_{k-1}$, $2)_{k-1}$ and Proposition 9.1 we see that $\gamma_{k-1}(A_{k-1}-V_{k-1})\neq Y$. Hence for general $y\in Y$, $\eta_{k,y}\colon A_{k,y}\to A_{k-1,y}$ is smooth where every fiber is an abelian variety. Then by

Proposition 4.6 $A_{k,y}$ is hyperelliptic, i.e., the general fiber of $\gamma_k: A_k \to Y$ is hyperelliptic. Let $(\phi_k: A_k \to \bar{A}_k, \bar{\gamma}_k: \bar{A}_k \to Y)$ be the relative co-Albanese map for γ_k (cf. Proposition 6.7). We then claim that for general $a \in A_{k-1,y}$, the induced map $\phi_k^a := \phi_{k,y|A_{k,a}}: A_{k,a} \to \bar{A}_{k,y}$ is surjective. In fact, since $A_{k-1,y}$ is a complex torus by $1)_{k-1}$, there exists a morphism $b_{k,y}$: Alb $A_{k,y} \to A_{k-1,y}$ such that $b_{k,y}\psi_{k,y} = \eta_{k,y}$ where $\psi_{k,y}: A_{k,y} \to$ Alb $A_{k,y}$ is the Albanese map of $A_{k,y}$. Then, since $\phi_{k,y} \times \psi_{k,y}: A_{k,y} \to \bar{A}_{k,y} \times$ Alb $A_{k,y}$ is a finite covering, $\phi_{k,y}: A_{k,y} \to \bar{A}_{k,y}$ being the co-Albanese map for $A_{k,y}$, the surjectivity of ϕ_k^a follows. Hence $\bar{A}_{k,y}$ is Moishezon (in fact, projective). From this, together with the fact that $q(\bar{\gamma}_k) = 0$ (cf. 6.4) it follows that $\bar{\gamma}_k$ is Moshezon by Proposition 2.5. Since $\bar{A}_k \to Y$ is an algebraic reduction of $\bar{A}_k$, this implies that dim $\bar{\gamma}_k = 0$, or equivalently, $A_{k,y}$ is a complex torus for general $y \in Y$. Finally since the general fiber of η_k is an abelian variety it follows from $1)_{k-1}$ that $A_{k,y}$ is obtained by a successive extension of abelian varieties. This completes the proof of Claim and hence the construction of the diagram (7).

We now consider $\alpha: X_1 \to A$. Since α is Moishezon by Proposition 5.3 there exist Zariski open subsets $V \subseteqq A$ and $U \subseteqq Y$ with $\gamma(V) \subseteqq U$ such that for any $y \in U$, the induced morphism $x_y: X_{1,y} \to A_y$ is a holomorphic fiber bundle over V_y with typical fiber an almost homogeneous unirational manifold. In particular α_y is isomorphic to the Albanese map of $X_{1,y}$. On the other hand, by Claim together with Proposition 9.1 we have $\gamma(A - V) \neq Y$. Hence for general $y \in Y$, $V_y = A_y$ and α_y is a holomorphic fiber bundle over the whole A_y. Also, we obtain $a(g_1) = 0$ applying Proposition 9.3 to the decomposition $f = f_3 f_2 f_1 := \gamma\alpha g_1$. Further if dim $\gamma = 0$, then dim $\eta_1 = 0$ so that $r = 0$ in the decomposition (6). Hence we have $f = \alpha g_1$ up to bimeromorphic equivalences. On the other hand, since α is Moishezon, X_1 is Moishezon as well as Y. Hence dim $\alpha = 0$, because α is an algebraic reduction of X_1. Hence $a(f) = 0$ by the definition of α.

Thus by what we have proved above the commutative diagram

(8)
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g_1}\searrow \quad \nearrow{\scriptstyle h_1 := \gamma\alpha} & \\ & X_1 & \end{array}$$

has the properties stated in the theorem. Moreover from the above proof we infer readily that the same is also true for any commutative diagram bimeromorphic to (8). (Use the fact that for any bimeromorphic model $h': X_1' \to Y$ of h with a bimeromorphic Y-map $b: X_1' \to X_1$, αb is holomorphic over some Zariski open subset of Y by Proposition 9.1.)

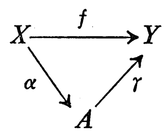
Thus it remains to show that (8) is bimeromorphic to the diagram (3) of the theorem. First, as $a(g_1)=0$, by Proposition 9.4 there exists a unique bimeromorphic map $u: X_1 \to \bar{X}$ such that $g_1 = ug$ and $hu = h_1$.

On the other hand, since any general fiber of $\eta_k: A_k \to A_{k-1}$ is an abelian variety, by the same argument as in the proof of Proposition 9.4 there exists a unique bimeromorphic map $v: \bar{X} \to X_1$ such that $g = vg_1$ and $h_1 v = h$. Since the maps involved are all meromorphic fiber spaces it follows readily that u and v are bimeromorphic as was desired. q.e.d.

9.5. Definition. A compact complex manifold X in $\mathscr{C}$ is called a *compound Moishezon manifold* if in the diagram (3) g is bimeromorphic.

For a compound Moishezon manifold X (7) reduces to

(9)



where for general $y \in Y$, A_y is a complex torus and $\alpha_y: X_y \to A_y$ is a holomorphic fiber bundle with typical fiber almost homogeneous and unirational. In particular $q(f) = \dim \gamma$.

Remark 9.1. Let $\mathscr{CM}$ be the class of compound Moishezon manifolds. Then $\mathscr{CM}$ has the same functorial properties as $\mathscr{C}$ stated in 1.3. Let $X \in \mathscr{CM}$. Then: 1) Any subspace, and any meromorphic image of X is again in $\mathscr{CM}$. 2) If $f: Y \to X$ is a Moishezon morphism then $Y \in \mathscr{CM}$. Further 3) any irreducible component of the Douady space D_X (is compact and) again belongs to $\mathscr{CM}$.

§ 10. Proof of Theorem 2

10.1. Theorem 2 is almost an immediate consequence of Proposition 5.2 and Proposition 10.2 below. First we note the following:

Lemma 10.1. *Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let $f = f_2 f_1$ be a decomposition of f into fiber spaces $f_1: X \to X_1$ and $f_2: X_1 \to Y$. Let $N = \{y \in Y; f$ (resp. f_2) is smooth along X_y (resp. $X_{1,y}$), and $a(f_{1,y}) = 0\}$. Then $N \in \mathfrak{Q}(Y)$ if $N \neq \emptyset$.*

Proof. Let $N_1 = \{x_1 \in X_1; f_1$ is smooth along X_{x_1}, $a(X_{x_1}) = 0\}$. Then $N_1 \in \mathfrak{Q}(X_1)$ by [18] (cf. 2.1). Write $X_1 - N_1 = \bigcup_\mu B_\mu$ where B_μ are analytic subsets in X_1 and the union is at most countable. Let $m = \dim f_2$. Let $A_\mu = \{y \in Y; \dim B_{\mu,y} \geqq m\}$. A_μ is an analytic subsets of A. Let $A = \{y \in$

$Y; f$ is not smooth along X_y, f_2 is not smooth along $X_{1,y}\}$. Then A is analytic and it is easy to see that $N = Y - A \cup (\bigcup_\mu A_\mu)$. Hence $N \in \mathfrak{Q}(Y)$.
q.e.d.

Proposition 10.2. *Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$ with $a(f)=0$. Let $f = f_3 f_2 f_1$ be a decomposition of f into three fiber spaces $f_1: X \to X_1$, $f_2: X_1 \to X_2$, $f_3: X_2 \to Y$ of compact complex manifolds in $\mathscr{C}$. Suppose that $k(f) = k(f_3)$ (note that $a(f_3)=0$) and f_1 is a relative algebraic reduction of $f_{12} := f_2 f_1$. Then $q(f_2) = a(f_1) = k(f_1) = 0$.*

Proof. First we show that $q(f_2) = q(f_1) = a(f_1) = 0$. For 'general' $y \in Y$, X_y, $X_{1,y}$, $X_{2,y}$ are all smooth, $a(X_y) = 0$ and $k(X_y) = k(X_{2,y})$. Since $k(X_y) \geqq k(X_{1,y}) \geqq k(X_{2,y})$, this implies that $k(X_y) = k(X_{1,y})$. Hence by Proposition 7.2 $q(f_{1,y}) = 0$. Thus $q(f_y) = q(f_{1,y}) = 0$. Similarly from $a(X_{1,y}) = 0$ and $k(X_{1,y}) = k(X_{2,y})$ we have $q(f_2) = 0$. Hence by Proposition 9.3 applied to $f = f_3 f_2 f_1$, $g = f_{12}$, $h = f_3$ we get that $a(f_1) = 0$.

Now supposing that $k(f_1) > 0$ we shall derive a contradiction. Let $(f_1': X \to X', f_1'': X' \to X_1)$ be a relative Kummer reduction of f_1. Then in order to get a contradiction by replacing f with a suitable bimeromorphic model of $f_3 f_2 f_1'': X' \to Y$, we may assume that $\dim f_1 = k(f_1)$. Take y sufficiently 'general' in such a way that in addition to the above conditions the following holds true; 1) $a(f_{1,,y}) = 0$ (cf. Lemma 10.1) so that in particular $f_{1,y}: X_y \to X_{1,y}$ is a relative algebraic reduction of $f_{12,y}: X_y \to X_{2,y}$ and 2) $k(f_{1,y}) = \dim f_{1,y}$. Then replacing f by $f_{12,y}$ we may assume from the beginning that Y is a point and then derive a contradiction. So we may omit the subscript y in what follows. Thus we get a commutative diagram of fiber spaces

$$\begin{array}{ccc} X & \xrightarrow{f_{12}} & X_2 \\ & {\scriptstyle f_1}\searrow \quad \nearrow{\scriptstyle f_2} & \\ & X_1 & \end{array}$$

where $a(X) = a(X_1) = 0$, $q(f_1) = a(f_1) = 0$, $\dim f_1 = k(f_1) > 0$ and $k(X) = k(X_2)$. Then applying Proposition 8.6 to f_1 there exist a finite covering $\mu: \tilde{X}_1 \to X_1$ and a Kummer manifold F of dimension $k(f_1)$ such that the induced map $\tilde{f}_1: \tilde{X} := X \times_{X_1} \tilde{X}_1 \to \tilde{X}_1$ is bimeromorphic over $\tilde{X}_1$ to the natural projection $F \times \tilde{X}_1 \to \tilde{X}_1$. Let $(\tilde{f}_2: \tilde{X}_1 \to \tilde{X}_2, \tilde{X}_2 \to X_2)$ be the Stein factorization of $f_2 \mu: \tilde{X}_1 \to X_2$. Then we have $k(\tilde{X}) = k(X) = k(X_2) = k(\tilde{X}_2)$ since the Kummer dimension is invariant under finite coverings (Remark 7.1). In particular if $\varphi: \tilde{X}_2 \to \overline{X}_2$ is a Kummer reduction of $\tilde{X}_2$, then $\varphi \tilde{f}_2$ is a Kummer reduction of $\tilde{X}$. On the other hand, denote the composite meromorphic map $\tilde{X} \to F \times \tilde{X}_1 \to F$ by g where the first arrow is the above bimeromorphic map. Then by the definition of the Kummer reduction there exists a unique meromorphic map $\psi: \bar{X}_2 \to F$ such that $\psi\varphi\tilde{f}_2 = g$. However this is impossible unless dim $F=0$ since the general fiber of $\tilde{f}_2$ is mapped bimeromorphically onto F by g. Hence $k(f_1)=0$. q.e.d.

10.2. *Proof of Theorem* 2. Since $k(f)=k(h)$ by the definition of a relative Kummer reduction, we can apply Proposition 10.2 to the decomposition $g=hbg_1$ of g. Hence $q(b)=a(g_1)=k(g_1)=0$. In particular 2) is proved. Since $q(b)=0$, b is Moishezon by Proposition 2.5. Hence 1) follows from Proposition 5.3. q.e.d.

Remark 10.1. The analogous assertion for the diagram (4)′ mentioned after Theorem 2 also follows in the same way as above noting that $k(g')=0$ implies $k(h')=0$.

§ 11. The case $ca(X)=2$

Let X be a compact complex manifold in $\mathscr{C}$. We set $ca(X):=\dim X - a(X)$ and call it the *co-algebraic dimension* of X. Then the main purpose of this section is to study the structure of X when $ca(X)=2$.

11.1. Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Then we say that f has *property* (A) if for any bimeromorphic model $f': X' \to Y$ of f with a bimeromorphic Y-map $\varphi: X \to X'$ there exists a Zariski open subset $U \subseteq Y$ such that φ gives an isomorphism $X_U \cong X'_U$. The following is an obvious criterion for f to have property (A).

Lemma 11.1. *Let f: $X \to Y$ be as above. Consider the following conditions.* 1) *f has property* (A), 2) *for any bimeromorphic model f': $X' \to Y$ of f there exists no analytic subvariety $F \subseteq X'$ of codimension $\geqq 2$ which is mapped surjectively onto Y, and* 3) *there exists no proper subvariety E of X with $f(E)=Y$. Then* 1) *and* 2) *are equivalent and are implied by* 3).

Proof. It is easy to see that 3) implies 2). We show the equivalence of 1) and 2). Suppose first that a subspace $F \subseteq X'$ as in 2) exists for some f'. Let φ: $\tilde{X} \to X'$ be the blowing up of F followed by a resolution. Then $\tilde{f}=f'\varphi$: $\tilde{X} \to Y$ and f' cannot be biholomorphic over any Zariski open subset $U \subseteq Y$. It follows that 1)→2). Conversely suppose that f does not have property (A). Then we can find f' and φ as above in such a way that if F (resp. F') is the set of indeterminacy of φ (resp. φ^{-1}) then either $f(F)=Y$ or $f'(F')=Y$. Then since F (resp. F') is of codimension $\geqq 2$, 2) is not satisfied. q.e.d.

Definition. Let X be a compact complex manifold. Then we say that X *has property* (A) if for some, and hence for any, holomorphic model f: $X^* \to Y$ of an algebraic reduction of X, f has property (A).

There are special cases where the property (A) is automatically satisfied.

Proposition 11.2. *Let X be a compound Moishezon manifold* (*cf.* 9.5). *Suppose that $q(f) = \dim f$ or $\dim f - 1$. Then X has property* (A).

Proof. When $q(f) = \dim f$, by Proposition 9.1 there is no proper subvariety $E \subseteqq X$ with $f(E) = Y$. So the proposition follows from Lemma 11.1. Suppose that $q(f) = \dim f - 1$. Let $f': X' \to Y$ be any bimeromorphic model of f and $F \subseteqq X'$ any subvariety with $f'(F) = Y$. Let (α: $X' \to A$, $\gamma: A \to Y$) be a decomposition of f' as in (9) where α is a meromorphic fiber space. We have $\dim \alpha = \dim f - q(f) = 1$. Since the general fiber of γ is a complex torus, α is holomorphic over some Zariski open subset of Y. On the other hand, again by Proposition 9.1 we must have $\alpha(F) = A$. Hence $\operatorname{codim} F = 1$. Since f' and F were arbitrary, f has property (A) by Lemma 11.1. q.e.d.

11.2. Let X be a compact complex manifold in $\mathscr{C}$. Let f: $X^* \to Y$ be a holomorphic model of an algebraic reduction of X. Clearly $ca(X) = \dim f$, and $ca(X) = 0$ if and only if f is Moishezon. When $ca(X) = 1$, we have the following well-known:

Proposition 11.3. *Let X be a compact complex manifold with $ca(X) = 1$. Then X has property* (A) *and X_y^* is a nonsingular elliptic curve.*

Before stating our result in case $ca(X) = 2$ we shall introduce the class of quasi-trivial manifolds (cf. Theorem). Let Y_1 and S be compact complex manifolds. Let G be a finite group acting biholomorphically on both Y_1 and S. Let $X_1 := (Y_1 \times S)/G$ be the quotient space. Then we have the natural projection $f_1: X_1 \to Y_1/G$. A fiber space f: $X \to Y$ of compact complex manifolds is called *quasi-trivial* if f is bimeromorphic to f_1 for some Y_1, S and G as above. We call a compact complex manifold X *of quasi-trivial type* if any holomorphic model f: $X^* \to Y$ of an algebraic reduction of X is quasi-trivial. Note that in this case S must be of algebraic dimension zero.

We also recall the following: Let C be a nonsingular elliptic curve. Then there exists a unique indecomposable holomorphic vector bundle over C of rank 2 which admits a trivial line subbundle (cf. Atiyah [1]). We shall denote this vector bundle by $F_2 = F_2(C)$ in what follows.

Theorem 3. *Compact complex manifolds X in $\mathscr{C}$ with $ca(X) = 2$ are* *up to bimeromorphic equivalences classified as follows*:

I. *X has property* (A)

a) $X_y^* \cong$ *complex torus*

α) X_y^* *is an abelian variety*

β) $a(X_y^*) \leqq 1$ *for 'general'* $y \in Y$ *and* f *is a holomorphic fiber bundle over some Zariski open subset of* Y.

b) $X_y^* \cong P^1$ *bundle over an elliptic curve* C_y

α) $X_y^* \cong P(F_2)$, $F_2 = F_2(C_y)$

β) $X_y^* \cong P(1 \oplus L_y)$, L_y: *line bundle on* C_y *with* $\deg L_y = 0$

II. *X is of quasi-trivial type*

a) $X_y^* \sim$ *a complex torus with* $a(X_y^*) = 0$

b) $X_y^* \sim$ *a* $K3$ *surface with* $a(X_y^*) = 0$

Here X_y^* *denotes a general fiber of any holomorphic model* $f: X^* \to Y$ *of algebraic reduction of* X. ($\sim$ *denotes 'is bimeromorphic to'*.)

In particular, hyperelliptic, Enriques, rational surface cannot appear as fibers of algebraic reductions (if $X \in \mathscr{C}$) (cf. Remark 12.5 of Ueno [43]). See also [43a].

11.3. We first study in general the structure of a fiber space whose general fiber is bimeromorphic to a complex torus or a $K3$ surface.

a) We begin with the following:

Proposition 11.4. *Let* $f: X \to Y$ *be a fiber space of compact complex manifolds in* $\mathscr{C}$. *Suppose that* $\dim X = 3$, $\dim Y = 1$ *and the general fiber* X_y *of* f *is a* $K3$ *surface. Then either* $h^{0,2}(X) = 0$ *or* f *is a holomorphic fiber bundle over some Zariski open subset* $U \subseteq Y$. *Moreover the latter is true if* $a(f) \leqq 1$.

Proof. Assuming that $h^{0,2}(X) \neq 0$, we shall show that the latter condition is satisfied. There exists a nonvanishing holomorphic 2-form, say ω, on X since $h^{2,0}(X) = h^{0,2}(X)$. We first show that the restriction ω_y of ω to any smooth fiber X_y is nonzero. In fact, note first that if $\omega_y = 0$ on some X_y, then $\omega_{y'} = 0$ on any smooth fiber $X_{y'}$. In fact, let $r_y: \Gamma(X, \Omega_X^2) \to \Gamma(X_y, \Omega_{X_y}^2)$ be the restriction map. It suffices to show that the kernel of r_y is independent of y. By the Hodge decomposition (1) we may consider r_y naturally as a direct summand of the restriction map $\tilde{r}_y: H^2(X, C) \to H^2(X_y, C)$. Thus we have only to show the corresponding assertion for $\tilde{r}_y$. In this case this is immediate since $\tilde{r}_y$ factors through the space $\Gamma(U, R^2 f_* C)$ of sections of the local system $R^2 f_* C|_U$.

Hence if $\omega_y = 0$ for some $y \in U$, then ω is written in a neighborhood V_y of X_y in the form $\omega = \omega_1 \wedge f^* dt$ where t is a local parameter of Y at y and ω_1 is a holomorphic 1-form on V_y. Since X_y is a $K3$ surface, ω_1

restricted to each fiber must be identically zero. From this it follows that $\omega=0$ on V_y and hence on the whole X. This is a contradiction. Hence $\omega_y\neq 0$ for any smooth fiber X_y, i.e., the closed 2-form ω on X gives by restriction the non-vanishing holomorphic 2-form ω_y on each fiber. In this case by the usual local Torelli argument using Stokes (cf. Bogomolov [3, Theorem 2], Fujita [22, Lemma 4.3]) we see that f is locally analytically trivial along any smooth fiber of f. Finally if $a(f)\leq 1$, then $a(X)=1$ or 2. Then either X is an elliptic threefold or f is an algebraic reduction of X. Hence $h^{0,2}(X)\neq 0$ by Proposition 3 of [21] which states that for any compact complex manifold Z with $h^{0,2}(Z)=0$ we have $a(h)=k(h)=0$ for any holomorphic model $h\colon Z^*\to\bar{Z}$ of an algebraic reduction of Z. Thus the last assertion follows from what we have proved above. q.e.d.

b) Let $f\colon X\to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Let U be a Zariski open subset of Y over which f is smooth. Let $s\colon Y\to X$ be a meromorphic section to f which is holomorphic on U. Let G be a finite group of biholomorphic automorphisms of X_U over U which fix each point of $s(U)$. Let $\bar{X}(U):=X_U/G$ be the quotient of X_U by G. Let $\bar{f}(U)\colon \bar{X}(U)\to U$ be the induced morphism and $q\colon X_U\to\bar{X}(U)$ be the quotient map.

Lemma 11.5. *Suppose that X_u is a complex torus for each $u\in U$. Then there exists a compactification $\bar{f}\colon \bar{X}\to Y$ of $\bar{f}(U)\colon \bar{X}(U)\to U$ such that q extends to a meromorphic map $\bar{q}\colon X\to\bar{X}$.*

Proof. We first show that the action of G extends to a bimeromorphic action on X. Let $g\in G$ be any element. Then by our assumption g defines a holomorphic section g_0 to $\mathrm{Aut}_U(X_U, s(U))\to U$ (cf. 4.2 for the notation). Then since $\mathrm{Aut}_U(X_U, s(U))$ is discrete over U, $g_0(U)$ is a Zariski open subset of a unique irreducible component A of $\mathrm{Aut}_Y^*(X, s(Y))$ (cf. 5.2). Thus g_0 extends to a meromorphic section to $\mathrm{Aut}_Y^*(X, s(Y))\to Y$, which is also equivalent to g extending to a bimeromorphic automorphism $\bar{g}$ of X over Y. Thus our assertion is proved. Now let $\Gamma_{\bar{g}}\subseteq X\times_Y X$ be the graph of $\bar{g}$ and $\Gamma=\bigcup_{g\in G}\Gamma_{\bar{g}}$. Then we obtain the following commutative diagram

$$\begin{array}{ccc} \Gamma & \subseteq & X\times_Y X \\ {\scriptstyle\rho}\downarrow & \swarrow{\scriptstyle p_2} & \\ X & & \end{array}$$

where p_2 is the projection to the second factor. Considering ρ as parametrizing zero cycles on X in the fibers of f we get a meromorphic Y-map τ:

$X \to \mathrm{sym}^k_Y X$ where $\mathrm{sym}^k_Y X$ is the symmetric product of X over Y and k is the order of G (cf. [12]). From the construction it follows readily that $\tau|_{X_U}$ factors through $\bar{X}(U)$, or more precisely, that there exists a unique embedding $\tau'\colon \bar{X}(U) \to \mathrm{sym}^k_Y X$ such that $\tau|_{X_U} = \tau' q$. Hence if $\bar{X}$ is the image of τ and if we identify $\bar{X}_U$ with its image by τ', then $\bar{X}$ together with the natural map $\bar{f}\colon \bar{X} \to Y$ gives the desired compactification of $\bar{f}(U)$. q.e.d.

c) Let $f\colon X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Suppose that there exists a Zariski open subset $U \subseteq Y$ such that X_y is a complex torus for every $y \in U$. Then up to bimeromorphic equivalences over Y, there exists a unique fiber space $f_1\colon X_1 \to Y$ of compact complex manifolds in $\mathscr{C}$ such that 1) $f_U\colon X_U \to U$ and $f_{1,U}\colon X_{1,U} \to U$ are locally isomorphic over U and 2) f_1 admits a meromorphic section $s\colon Y \to X$ which is holomorphic on U. In fact, it suffices to set $X_1 := \mathrm{Aut}^*_{Y,0} X$ (cf. [19, Proposition 7]). We call $f_1\colon X_1 \to Y$ the *basic fiber space associated to f.*

Proposition 11.6. *Let $f\colon X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$ with Y projective. Suppose that $a(f) \leq 1$ and the general fiber X_y of f is bimeromorphic either to a K3 surface or a complex torus. Then any smooth fibers of f are bimeromorphic to each other.*

Proof. Let $U \subseteq Y$ be a Zariski open subset over which f is smooth. It suffices to show that for any $y, y' \in U$, X_y and $X_{y'}$ are bimeromorphic to each other. Take any smooth curve $C \subseteq Y$ passing through y and y' such that if X'_C is the unique irreducible component of X_C which is mapped surjectively onto C then for the induced map $f'_C\colon X'_C \to C$ we have $a(f') = a(f)$ (cf. 2.1). Thus to show the lemma, replacing f by a suitable nonsingular model of f'_C if necessary, we may assume from the beginning that $\dim Y = 1$. Then by [15, Proposition 3] we can pass to another bimeromorphic model to assume that X_y is minimal for every $y \in U$ with U unchanged. Then it suffices to show that X_y, $y \in U$, are isomorphic to each other. First, if X_y is a K3 surface, then this follows from Proposition 11.4. So we assume that X_y is a complex torus.

We first assume that f admits a meromorphic section $s\colon Y \to X$ which is holomorphic on U. Then $f_U\colon X_U \to U$ has the unique structure of a complex Lie group over U with the identity section s (cf. [19]). Then the automorphism $\iota\colon X_U \to X_U$ over U which coincides with $z \to -z$ on each fiber X_y, $y \in U$, extends to a bimeromorphic map $\iota^*\colon X \to X$ over Y (cf. the proof of the previous lemma). Let $\bar{X}(U) = X_U/\langle\iota\rangle$ be the quotient of X_U by $\langle\iota\rangle$, with the natural projection $\bar{f}_U\colon \bar{X}(U) \to U$. Then by Lemma 11.5 there exists a compactification $\bar{f}\colon \bar{X} \to Y$ of $\bar{f}_U$ such that the quotient map $q\colon X_U \to \bar{X}(U)$ extends to a meromorphic map $\bar{q}\colon X \to \bar{X}$. Let $v\colon \tilde{X} \to \bar{X}$ be any resolution of $\bar{X}$ inducing the minimal resolution on the general fiber $\bar{X}_y$. Then the resulting map $\tilde{f}$: $\tilde{X}\to Y$ has as general fibers $K3$ surfaces and we have $a(\tilde{f})\leqq 1$. Hence from the proof of Proposition 11.4 there exists a nonvanishing holomorphic 2-from $\tilde{\omega}$ which restricts to a nonzero holomorphic 2-from on each fiber. Then the pull-back ω of $\tilde{\omega}$ to X via the meromorphic map $v^{-1}\bar{q}$ has the same property. Then as in the proof of Proposition 11.4 this implies that f is locally trivial along any smooth fiber. Hence the proposition is proved in our special case.

Next in the general case we consider the basic fiber space f_0: $X_0\to Y$ associated to f. Then f_0 is a fiber space whose smooth fiber is isomorphic to that of f and which admits a meromorphic section which is holomorphic on U. Hence the proposition is true for f_0, by what we have proved above. Then the same is clearly true for the original f, too. q.e.d.

11.4. We prove some lemmas needed for the proof of Theorem 3.

Lemma 11.7. *Let f: $X\to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$, with* $\dim f=2$. *Suppose that Y is projective and f has property* (A). *Then any smooth fiber X_y of f is relatively minimal.*

Proof. Suppose that X_y is not relatively minimal for some smooth X_y. Let C be an exceptional curve of the first kind on X_y. Let D_α be the irreducible component of $D_{X/Y}$ containing the point c corresponding to the subspace $C\subseteqq X_y$. Then by Kodaira [30] the natural projection u: $D_\alpha\to Y$ is biholomorphic at c. Since D_α is compact u is then generically finite. Let $Z_\alpha\to D_\alpha$ be the universal family restricted to D_α and $\bar{Z}_\alpha\subseteqq X$ the natural image of Z_α in X. Then from the generic finiteness of u it follows that $\bar{Z}_\alpha$ is an irreducible divisor on X and the general fiber of the natural projection $\bar{Z}_\alpha\to Y$ is isomorphic to a disjoint union of $\boldsymbol{P}^1$. Hence $\bar{Z}_\alpha$ is Moishezon as well as Y. Then we can find an irreducible divisor $T\subseteqq\bar{Z}_\alpha$ with $f(T)=Y$. Since codim $T\geqq 2$ in X, f does not have property (A) by Lemma 11.1. This is a contradiction. q.e.d.

Lemma 11.8. *Let f: $X\to Y$ and f_1: $X_1\to Y$ be fiber spaces of compact complex varieties in $\mathscr{C}$ with Y nonsingular. Let φ: $X\to X_1$ be a bimeromorphic map over Y. Let G be a finite group acting biholomorphically on both X and Y in such a way that f is G-equivariant. Suppose that f_1 is a holomorphic fiber bundle the typical fiber S of which is a minimal nonruled analytic surface* (*resp. a complex torus*). *Then there exists a natural biholomorphic action of G on X_1 making f_1 equivariant.*

Proof. Let $\bar{Y}=Y/G$ and let q: $Y\to\bar{Y}$ be the quotient map. Consider

X_1 as spaces over $\bar{Y}$ via qf_1, We define a bimeromorphic action of G on X_1 over $\bar{Y}$ by $g \to g_1 = \varphi g \varphi^{-1}: X_1 \to X_1$. Since qf_1 is locally a product over Y, $X_{1,y}$ is minimal, and $\kappa(X_{1,y}) \geqq 0$, the action of G is actually biholomorphic. (See Viehweg [44, Lemma 2.6].) q.e.d.

Before proceeding we note the following fact. Let S_1 be a nonruled compact analytic surface and S its minimal model. Let BHol (S_1, S) be the set of bimeromorphic morphisms of S_1 onto S. Then since S is (absolutely) minimal, BHol $(S_1, S) \cong \mathrm{Aut}\, S$ by the map $h \to hh_0^{-1}$, for $h \in$ BHol (S_1, S), where $h_0 \in$ BHol (S_1, S) is a fixed element.

Lemma 11.9. *Let $f: X \to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$. Suppose that* $\dim f = 2$ *and* $a(f) = 0$. *Suppose that any smooth fibers of f are bimeromorphic to each other. Suppose further that f has not property* (A) *if X_y is bimeromorphic to a complex torus. Then f is quasi-trivial.*

Proof. In view of Lemma 11.8 it suffices to show that there exist a minimal compact analytic surface S, a finite covering $\nu: \tilde{Y} \to Y$ of compact complex varieties and a bimeromorphic map $X_{\tilde{Y}} \to S \times \tilde{Y}$ over $\tilde{Y}$. (We can then assume ν to be Galois, and then take an equivariant resolution.) Let U be a Zariski open subset of Y over which f is smooth. Let S be the common minimal model of X_y, $y \in U$. For any subset $B \subseteq Y$ we set $S_B = S \times B$. It is then easy to see that taking a suitable open subset $W \subseteq U$ we can obtain a W-morphism $\varphi_W: X_W \to S_W$ (cf. the proof of Lemma 11.7). Then φ_W defines a holomorphic section $\bar{\varphi}_W: W \to \mathrm{BHol}_W (X_W, S_W)$ (cf. [19, § 4] for the notation). Let H be an irreducible component of $\mathrm{BHol}_Y^\sharp (X, S_Y)$ which contains $\bar{\varphi}_W(W)$. Since H is compact, the natural map $H \to Y$ is surjective. Suppose first that S is a $K3$ surface. Then in view of the remark preceding Lemma 11.9, H is generically finite over Y since Aut S is discrete.

Next suppose that S is a complex torus. Then by our assumption together with Lemma 11.1 (possibly after passing to another bimeromorphic model of f) there exists a subvariety $E \subseteq X$ such that $f(E) = Y$ and $\dim E_y = 0$ for general y. Taking the base change to E of f with respect to the natural morphism $f|_E: E \to Y$ and taking resolutions we may assume from the beginning that $f|_E$ is bimeromorphic. Fix the origin $o \in S$ and consider $B := \mathrm{BHol}_Y^\sharp ((X, E), (S_Y, o_Y)) \subseteq \mathrm{BHol}_Y^\sharp (X, S_Y)$ where $o_Y = \{o\} \times Y$ (cf. [19, § 4] for the notation). Then we may take W and φ_W above in such a way that $\bar{\varphi}_W(W) \subseteq \mathrm{BHol}_W ((X_W, E_W), (S_W, o_W))$ (by translating the original $\bar{\varphi}_W$ via a section $W \to S_W$). Let H' be an irreducible component of B which contains $\bar{\varphi}_W(W)$. Then as above H' is proper, generically finite and surjective over Y. Let $\tilde{Y}' = H$ or H' according as S is a $K3$ surface or a complex torus. Then by the property of $\tilde{Y}'$ proved above the assertion follows from [19, Remark 9].

11.5. Further we need some facts on the structure of a $\boldsymbol{P}^1$-bundle over an elliptic curve.

Lemma 11.10. *Let C be a nonsingular elliptic curve and $p: X \to C$ a $\boldsymbol{P}^1$-bundle. Suppose either that $\kappa(K_X^{-1}, X) \leqq 0$ or* $\mathrm{Aut}_0 X$ *is not a complex torus of dimension* 1. *Then $X \cong \boldsymbol{P}(F_2)$ of $\boldsymbol{P}(1 \oplus L)$ where F_2 is as in Theorem* 3 *and L is a line bundle of degree zero on C.*

Proof. Write $X = \boldsymbol{P}(E)$ for some holomorphic vector bundle E of rank 2 on C. Suppose first that E is decomposable; we may assume that E is of the form $E = 1 \oplus L$ with $\deg L \geqq 0$. Let F be the tautological line bundle associated to $E \to C$. Then we have $K_X = F^2 \otimes p^* L$ so that $K_X^{-\nu} = F^{-2\nu} \otimes p^* L$ (cf. [29, Proposition 2.2]). Then we get $p_* \underline{K}_X^{-\nu} = \underline{E}^{*(2\nu)} \otimes_{\mathcal{O}_C} \underline{L}^\nu$ (cf. [29, Theorem 2.1]) and hence $\Gamma(X, \underline{K}_X^{-\nu}) = \Gamma(C, p_* \underline{K}_X^{-\nu}) \cong \Gamma(C, \underline{E}^{*(2\nu)} \otimes_{\mathcal{O}_C} \underline{L}^\nu) \supseteqq \Gamma(C, \underline{L}^\nu)$ where $\underline{E}^{*(2\nu)}$ denotes the 2ν-th symmetric product of $\underline{E}^*$ and the last inclusion is induced by a nonzero section in $\Gamma(C, \underline{1}) \subseteqq \Gamma(C, \underline{E}^{*(2\nu)})$. Here for a holomorphic vector bundle F on C we write $\underline{F} = \mathcal{O}_C(F)$. Therefore $\kappa(L, C) \leqq \kappa(K_X^{-1}, X) \leqq 0$. This implies that $\deg L = 0$. Next we consider the case where E is indecomposable. Tensorizing a suitable line bundle with E we can assume that $\deg E = 0$ or 1. If $\deg E = 0$, then $E \cong F_2 \otimes L$ for some line bundle L with $\deg L = 0$ (Atiyah [1]). So $\boldsymbol{P}(E) \cong \boldsymbol{P}(F_2)$. Next if $\deg E = 1$, then $\mathrm{Aut}_0 X$ is a complex torus of dimension 1 by Maruyama [36, Theorem 3.4]. Thus this case cannot occur.

Remark 11.1. Let X be a holomorphic $\boldsymbol{P}^1$-bundle over a nonsingular elliptic curve C. 1) If $\mathrm{Aut}\, X$ is a complex torus T of dimension 1, then $B := X/T$ is isomorphic to $\boldsymbol{P}^1$ and we have the natural structure of an elliptic surface $X \to B$ on X. 2) $X \cong \boldsymbol{P}(F_2)$ or $\cong \boldsymbol{P}(1 \oplus L)$, $\deg L = 0$, if and only if X is almost homogeneous. In this case there is no curve with negative self-intersection on X. Further $\mathrm{Aut}_0 X$ is an extension of C by $\boldsymbol{C}^*$ (resp. $\boldsymbol{C}$) if $E \cong 1 \oplus L$, $L \not\cong 1$ (resp. F_2). Moreover if $W \subseteqq X$ is the unique Zariski open orbit of $\mathrm{Aut}_0 X$, then $X - W$ is the disjoint union of the two minimal sections B_1 and B_2 (resp. coincides with the unique minimal section B) of the projection $X \to C$. In this case we have $B_i \cdot B_i = 0$ (resp. $B \cdot B = 0$) on X (cf. [36]). 3) If X is homeomorphic to $\boldsymbol{P}(F_2)$ and $\boldsymbol{P}(1 \oplus L)$, then $X \cong \boldsymbol{P}(F_2)$ or $\cong \boldsymbol{P}(1 \oplus L)$ $(L \not\cong 1)$ if and only if $\dim H^0(X, \Theta_X) = 2$ (cf. Suwa [41]).

Proof of Theorem 3. Let $f = gh$ be the decomposition (3) as in Theorem 1 with g a meromorphic fiber space. Since $a(g) = 0$, $\dim g \neq 1$. Hence from the equality $\dim f=\dim g+\dim h$, it follows that there exist two cases to be distinguished; (dim g, $\dim h)=(2, 0)$ or $(0, 2)$. In the former case by the classification theory of surfaces X_y^* is bimeromorphic either to a complex torus or a $K3$ surface, X_y^* being in $\mathscr{C}$. Suppose first that X does not have property (A) provided that X_y^* is bimeromorphic to a complex torus. Then by Proposition 11.6 and Lemma 11.9 it follows that f is quasi-trivial. Hence X belongs to the class II in this case. Suppose next that X_y^* is bimeromorphic to a complex torus and f has property (A). Then by Lemma 11.7 each smooth fiber X_y^* is actually a complex torus. Then from Proposition 11.6 it follows that each smooth fiber is mutually isomorphc and hence f is a holomorphic fiber bundle over some Zariski open subset of Y (cf. Fischer-Grauert [11]). Thus X belongs to class I a) β).

Next we consider the latter case, i.e., the case where $(\dim g, \dim h)=(0, 2)$. In this case f is bimeromorphic to h and then from the inequalities $\dim f\geqq q(f)>0$ we get two cases; $q(f)=2$ or 1. If $q(f)=2$, X_y^* is isomorphic to a complex torus and if $q(f)=1$, X_y^* is isomorphic to a holomorphic $\boldsymbol{P}^1$-bundle over an elliptic curve, by Theorem 1. In both cases X has property (A) by Proposition 11.2. Now suppose first that $q(f)=2$. Then if $a(f)\leqq 1$, by Proposition 11.6 smooth fibers of f are mutually isomorphic and X belongs to the class I a) β). Otherwise X belongs to the class I a) α). Next we consider the latter case so that $q(f)=1$. First, since f is an algebraic reduction of X^*, $\kappa(K_{X_y^*}^{-1}, X_y^*)\leqq 0$ for 'general' $y\in Y$ by Proposition 1.1. Moreover for general $y\in Y$ $\mathrm{Aut}_0 X_y^*$ is not a complex torus of dimension 1. In fact, suppose otherwise. Then it is easy to see that $(\mathrm{Aut}_{Y,0}^* X^*)_y$ is a complex torus of dimension 1 for general $y\in Y$ (cf. the proof of Proposition 9.1). Let $(u: X^*\to C,\ v: C\to Y)$ be the relative generic quotient of X by $\mathrm{Aut}_{Y,0}^* X^*$ (cf. 5.2). Then by Remark 11.1, 1) the general fiber of v is isomorphic to $\boldsymbol{P}^1$. Hence v is Moishezon and so C itself is Moishezon. This is a contradiction since v is an algebraic reduction of C. Thus by Lemma 11.10 X_y^* is of the form $\boldsymbol{P}(E)$ for 'general' $y\in Y$ where $E\cong 1\oplus L$ or F_2 in the notation of that lemma. Then by Remark 11.1, 2) and the upper semicontinuity of $\dim H^0(X_y^*, \Theta_{X_y^*})$ this also is true for general $y\in Y$. Since X is compound Moishezon, passing to a suitable bimeromorphic model X^* we obtain a decomposition (9) of f; $f=\gamma\alpha$, $\alpha: X\to A$, $\gamma: A\to Y$, with α holomorphic. Let M be the unique maximal transversal analytic subspace of X^* with respect to f_1 (cf. 5.4). Since $X_y^*-M_y$ is homogeneous for general $y\in Y$, by Remark 11.1, 2) $M_y\subseteqq X_y^*$ is the union of the minimal sections on X_y^*. Moreover the remark also shows that $M\to A$ is either generically two to one or is bimeromorphic and that X is in the class I, b),β) in the former case and in the class I, b), α) in the latter case. q.e.d.

Remark 11.2. In case I, b), β), for 'general' $y \in Y$, $L^k \neq 1$ for any $k \in Z-\{0\}$.

§ 12. The case $a(X)=0$ and $ck(X) \leqq 2$

Let X be a compact complex manifold in $\mathscr{C}$ with $a(X)=0$. Define the co-Kummer dimension $ck(X)$ of X by $ck(X):=\dim X-k(X)$. Thus $ck(X)=0$ if and only if X is Kummer. In this section we consider the structure of X when $ck(X)=1$ or 2. Let $\beta: X^* \rightarrow B$ be a holomorphic model of a Kummer reduction of X. Then we have $ck(X)=\dim \beta$.

Proposition 12.1. *Let X be as above. Then $ck(X)=1$ if and only if X is a meromorphic $\boldsymbol{P}^1$-fiber space over a Kummer manifold.*

Proof. 'Only if' part follows immediately from Theorem 2. So suppose that X is bimeromorphic to X^* which has a $\boldsymbol{P}^1$-fiber space structure $\gamma: X^* \rightarrow Y$ with Y a Kummer manifold. Then $ck(X) \leqq 1$. Suppose that $ck(X)=0$, i.e., X is Kummer. To derive a contradiction from this, by passing to a suitable finite covering of Y and to a suitable bimeromorphic model of X, we may assume that Y is a complex torus (cf. Remark 7.2). Then γ is the Albanese map of X. Hence if $h: X \rightarrow C$ is the co-Albanese map of X (cf. 6.4), then $\dim C=1$. Hence $a(X) \geqq a(C)=1$, which is a contradiction. q.e.d.

Proposition 12.2. *Suppose that $ck(X)=2$. Then there exists a Zariski open subset $U \subseteqq B$ such that β_U: $X_U^* \rightarrow U$ is a holomorphic fiber bundle such that if F is the typical fiber of β_U and if G is a structure group of β_U, then either of the following two cases occurs*: 1) *F is a rational surface on which G acts almost homogeneously, or* 2) *F is a $K3$ surface of algebraic dimension zero and G is finite. In the latter case β is quasi-trivial.*

Proof. Let $(g_1: X' \rightarrow X_1, b: X_1 \rightarrow B)$ be the decomposition (4) (cf. Theorem 2) applied to the constant map $X \rightarrow Y=\{\text{point}\}$ where X' is a bimeromorphic model of X. In particular b is Moishezon and $a(X_1)=a(g_1)=k(g_1)=0$. Since $\dim g_1 \neq 1$ $(a(g_1)=0)$, it follows that $(\dim g_1, \dim b)=(0,2)$ or $(2,0)$. In the former case β is bimeromorphic to b and hence by Proposition 5.3 β satisfies the conditions of 1). Next, consider the case of $(2,0)$. Then β is bimeromorphic to g_1. Since $a(g_1)=k(g_1)=0$, $X_{1,b}^*$, and hence X_b^*, is bimeromorphic to a $K3$ surface for general $b \in B$. Let $U \subseteqq B$ be a Zariski open subset such that β is smooth over U. Restricting U we may assume that we have the period map $\Phi: U \rightarrow D/\Gamma$ as in 4.1. By Proposition 4.1 Φ is a constant map. Hence the bimeromorphic moduli of X_y, $y \in U$, is constant, i.e., X_y are mutually bimeromorphic. Hence by

Lemma 11.9 β is quasi-trivial and the case 2) occurs. q.e.d.

§ 13. Proof of Theorem

In this section we shall prove Theorem in Section 1.

13.1. We begin with an easy criterion for a meromorphic map to be holomorphic. In general a surjective meromorphic map $f: X \to Y$ of compact complex varieties is said to be *almost holomorphic* if there exist Zariski open subsets $W \subseteqq X$ and $U \subseteqq Y$ such that $f|_W$ is holomorphic, $f(W) \subseteqq U$ and $f|_W: W \to U$ is proper.

Lemma 13.1. *Let $f: X \to Y$ be a meromorphic fiber space of compact complex manifolds. Suppose that* dim $Y=1$. *Then f is holomorphic if one of the following conditions is satisfied*; 1) *there is no subvariety E on X with $f(E)=Y$ and* 2) *f is almost holomorphic.*

Proof. Suppose that 1) is true. Let $f^*: X^* \to Y$ be any holomorphic model of f with a bimeromorphic morphism $\varphi: X^* \to X$ over Y. Let E be the exceptional divisor for φ. Then by our assumption $f^*(E) \neq Y$. This implies that $f = f^* \varphi^{-1}|_{X - \varphi f^{*-1} f^*(E)}$ is holomorphic and proper over the Zariski open subset $U := Y - f^*(E)$. Thus it suffices to show that 2) implies the holomorphy of f. Let F be the set of indeterminacy for f. Then F corresponds by f to a finite set of points $Y - U$. From this it follows readily that f is actually holomorphic. q.e.d.

Using this we prove the following:

Proposition 13.2. *Let X be a compact complex manifold with* dim $X=3$ *and $a(X)=1$, i.e., $ca(X)=2$. Suppose that X is in the class* I *in Theorem* 3. *Then an algebraic reduction $f: X \to Y$ is necessarily holomorphic.*

Proof. Let $f^*: X^* \to Y$ be any holomorphic model of an algebraic reduction of X. Suppose first that X^* is in the class I, a) and there exists a proper subvariety $F \subseteqq X$ with $f^*(F) = Y$. Since f^* has property (A) by Lemma 11.1, F is a divisor. Then $a(f) \geqq 1$, which contradicts Proposition 9.1. Hence there is no F as above. Then the assertion follows from Lemma 13.1. Next we assume that X is in the class I, b). Taking X^* suitably we may assume that there exist 1) a bimeromorphic morphism φ: $X^* \to X$ and 2) a decomposition $f^* := \gamma\alpha$, $\alpha: X^* \to A$, $\gamma: A \to Y$, of $f^*: X^* \to Y$ as (9). Let $E \subseteqq X^*$ be the exceptional set for φ. If $f^*(E) \subsetneqq Y$, then $f := f^* \varphi^{-1}: X \to Y$ is almost holomorphic as in the proof of Lemma 13.1 and hence is holomorphic by that lemma. So supposing that $f^*(E) = Y$ we shall derive a contradiction. If $\alpha(E) \subsetneqq A$, $\alpha(E)$ is a divisor on A with

$\gamma(\alpha(E))=Y$. This is impossible since γ is an algebraic reduction of A and $\dim \gamma=1$. So $\alpha(E)=A$.

Let E be any irreducible component of E with $\alpha(E)=A$. Then $a(E_i) =a(A)=1$. Assume first that $\dim \varphi(E_{iy})=0$. Then E_{iy} is exceptional in X_y in the sense of Grauert [23], which is impossible because X_y contains no curve with negative self-intersection (cf. Remark 13.1, 2)). Thus $\dim \varphi(E_{iy})=1$. Let $F_i=\varphi(E_i)$. Then $\dim F_i=1$. Then $\varphi(E_{iy})=F_i$ for any $y \in Y$, and hence for any $x \in F_i$, $\varphi^{-1}(x)$ is a divisor on E_i which is mapped surjectively onto Y. This again is impossible since $a(E_i)=1$.

q.e.d.

13.2. a) Before the proof of Theorem we remark on curves on a $K3$ surface S of algebraic dimension zero. On S there exists only a finite number of irreducible curves, say $C_1, \cdots, C_m$, and we have $C_i \cong \boldsymbol{P}^1$ and $C_i \cdot C_i=-2$[31]. On the other hand, by Riemann-Roch we see that there is no curve D on S with $D \cdot D \geqq 0$. From this it follows readily that the intersection matrix $(C_i \cdot C_j)$, $1 \leqq i, j \leqq m$, is negative definite, and we can then obtain from S a unique normal analytic surface S' by contracting these curves to normal points of S'. We call S' the minimal normal $K3$ surface.

In general let S be a compact analytic surface with $a(S)=0$. Then we call a normal compact analytic surface S' the *normal minimal model of* S if S' is a complex torus or the minimal normal $K3$ surface bimeromorphic to S according as S is bimeromorphic to a complex torus or a $K3$ surface. Among the surfaces bimeromorphic to S, S' is uniquely characterized by the property that it contains no curve.

b) *Proof of Theorem.* If $a(X)=3$, then X is by definition Moishezon. If $a(X)=2$, X is an elliptic threefold as is well-known (cf. Proposition 11.3). So suppose that $a(X)=1$, i.e., $ca(X)=2$. Then we can apply Theorem 2 to X. If X is in the class I, by Proposition 13.2 for any bimeromorphic model X' of X an algebraic reduction of X' is necessarily holomorphic. Thus Theorem follows from Theorem 3 in this case.

Next, suppose that $a(X)=0$. Then $k(X)=3$, 2 or 0. If $k(X)=3$, then X is Kummer. If $k(X)=2$, X is a meromorphic $\boldsymbol{P}^1$-fiber space over a Kummer manifold S of dimension 2 by Proposition 12.1. Since S is simple (i.e., there is no analytic family of curves on S which covers the whole S), the natural map $\beta: X \to S$ (Kummer reduction) is almost holomorphic [20]. Let S' be the normal minimal model of S. Let $\beta': X \to S'$ be the resulting almost holomorphic meromorphic map. Then since for any Zariski open subset $U' \subseteqq S'$, $S'-U'$ is a finite set, β' is actually holomorphic as one sees immediately. Finally we assume that $a(X)=k(X)=0$. In this case we consider a semisimple reduction $h: X \to T$ of X, which is a meromorphic fiber space with T semisimple, having the universal property among the surjective meromorphic maps of X onto semisimple varieties (cf. [20]). First, by [20] $\dim T>0$. Further since $a(X)=0$, $\dim T\neq 1$. Suppose that $\dim T=2$. Then by Proposition 9.2 h is a meromorphic $\boldsymbol{P}^1$-fiber space. Replacing T by its normal minimal model as above, we can take h to be holomorphic. Thus X is in the class II. Finally assume that $\dim T=3$. Then X is semisimple. But since $a(X)=0$ X must be simple. q.e.d.

Remark 13.1. Let X be as in Theorem. 1) If $a(X)=1$, and for some $y\in Y$ X_y is a complex torus with $a(X_y)\leqq 1$, then there exist a torus bundle $\tilde{X}\to\tilde{Y}$ over a compact Riemann surface $\tilde{Y}$ and a finite group G acting on $\tilde{X}$ and $\tilde{Y}$ fiber-preservingly such that X is bimeromorphic to $\tilde{X}/G$. 2) There is no known example of X which is simple with $a(X)=k(X)=0$. It is highly interesting to know if such an X actually exists or not.

1) of the above remark follows from the following:

Proposition 13.3. *Let $f\colon X\to Y$ be a fiber space of compact complex manifolds in $\mathscr{C}$ with* $\dim Y=1$. *Suppose that there exists a Zariski open subset $U\subseteq Y$ such that $f_U\colon X_U\to U$ is a holomorphic fiber bundle with typical fiber a complex torus T. Then there exists a torus bundle $\tilde{f}\colon \tilde{X}\to\tilde{Y}$ over a compact Riemann surface $\tilde{Y}$ and a finite group G acting fiber-preservingly on $\tilde{X}$ and $\tilde{Y}$ such that f is bimeromorphic to the induced morphism $\tilde{X}/G\to\tilde{Y}/G$.*

We need a local version of this proposition.

Lemma 13.4. *Let $f\colon X\to D$ be a fiber space of complex manifolds with $f\in\mathscr{C}/D$* (*cf.* [14]), *where D is a* 1-*dimensional disc $D=\{|t|<\varepsilon\}$, $\varepsilon>0$. Let $D'=D-\{0\}$. Assume that f is a holomorphic fiber bundle over D' with typical fiber a complex torus T. Then* (*after a possible restriction of D*) *there exists a finite covering $\mu\colon\tilde{D}\to D$, unramified over D', such that the induced map $f_{\tilde{D}}\colon X_{\tilde{D}}\to\tilde{D}$ is bimeromorphic to the projection $T\times\tilde{D}\to\tilde{D}$. Here the bimeromorphic map can be taken to be isomorphic over $\tilde{D}'=\tilde{D}-\{0\}$.*

Proof. Passing to a suitable finite covering of D we may assume that f admits a holomorphic section $s\colon D\to X$. Fix the origin $o\in T$. Let $X':=T\times D$ considered naturally as a complex space over D. Set $I^*=\mathrm{Isom}_D^*((X,s(D)),(X',o_D))$ where $o_D=\{o\}\times D$ (cf. [19] for the notation). The fiber over $d\in D'$ of I_d^* is then given by $\mathrm{Isom}((X_d,s(d)),(T,o))\cong\mathrm{Aut}((T,o))$. Hence I^* is discrete over D'. Then the lemma follows from Remark 9 of [19].

Proof of Proposition 13.3. Let $y_1,\cdots,y_r\in Y$ correspond to the singular fibers of f. Then by Lemma 13.4 there exist for each y_i a disc neighborhood $D_i \ni y_i$ and a finite covering $\mu_i: \tilde{D}_i \to D_i$ such that $f_{\tilde{D}_i}: X_{\tilde{D}_i} \to \tilde{D}_i$ is bimeromorphic to $T \times \tilde{D}_i \to \tilde{D}_i$. Let m_i be the covering degree of μ_i. Then take a finite Galois covering $\pi: \tilde{Y} \to Y$ with Galois group G such that for each $\tilde{y}_i \in \tilde{Y}$ over y_i the ramification index at $\tilde{y}_i$ is divisible by m_i. Let $\tilde{f} = f_{\tilde{Y}}: \tilde{X} = X_{\tilde{Y}} \to \tilde{Y}$ be the induced map. Then for each $\tilde{y}_\alpha \in \tilde{Y}$ corresponding to a singular fiber there exist a neighborhood $\tilde{U}_\alpha$ of $\tilde{y}_\alpha$ and a bimeromorphic map $\varphi_\alpha: \tilde{X}_{\tilde{U}_\alpha} \to \tilde{U}_\alpha \times T$ over $\tilde{U}_\alpha$ which is isomorphic over $\tilde{U}_\alpha - \{\tilde{y}_\alpha\}$. Then set $\tilde{X}' = \tilde{X} - \bigcup_\alpha \tilde{f}^{-1}(\tilde{y}_\alpha)$ and define $\tilde{X}_0 := \tilde{X}' \cup (\bigcup_\alpha (\tilde{U}_\alpha \times T))$ where $\tilde{X}'$ and $\tilde{U}_\alpha \times T$ are identified via $\varphi_\alpha|_{\tilde{X}' \cap \tilde{X}_{\tilde{U}_\alpha}}$. (Here we take $\{\tilde{U}_\alpha\}$ in such a way that $\tilde{U}_\alpha \cap \tilde{U}_\beta = \emptyset$ if $\alpha \neq \beta$.) Then we have a natural morphism $\tilde{f}_0: \tilde{X}_0 \to \tilde{Y}$ such that $\tilde{f}$ and $\tilde{f}_0$ are bimeromorphic and $\tilde{f}_0$ is a holomorphic fiber bundle with typical fiber T. Now G acts naturally on $\tilde{X}$ so that $\tilde{X}/G \cong X$. The proposition then follows from Lemma 11.8.

§ 14. A bimeromorphic classification of non-algebraic uniruled manifolds of dimension 3

14.1. Recall that a compact complex manifold X is said to be *uniruled* if there exists a covering family of rational curves on X

$$(10)\qquad \begin{array}{ccc} X \times T \supseteq Z & \xrightarrow{\ \pi\ } & X \\ \searrow & \downarrow \rho & \\ & T & \end{array}$$

i.e., π is surjective and the general fiber of ρ is an irreducible rational curve. We may assume that ρ is a universal family restricted to a subspace T of the Douady space D_X of X.

The following proposition gives a rough classification of non-algebraic uniruled threefolds in $\mathscr{C}$.

Proposition 14.1. *Let X be a compact complex manifold with* dim $X = 3$ *in $\mathscr{C}$. Suppose that X is uniruled and is not Moishezon. Then X is in either of the following two classes;* i) *a fiber space over a compact Riemann surface Y whose general fiber is isomorphic to a $\boldsymbol{P}^1$-bundle over an elliptic curve of the form $\boldsymbol{P}(1 \oplus L)$ or $\boldsymbol{P}(F_2)$ where L is a line bundle with* $\deg L = 0$, *or* ii) *a $\boldsymbol{P}^1$-fiber space over a normal compact analytic surface S with* $a(S) = 0$. *If X is in* i), *the relative Albanese map $\psi: X \to S := \mathrm{Alb}^* X/Y$ gives the structure of a meromorphic $\boldsymbol{P}^1$-fiber space on X.*

Remark 14.1. 1) The relation with classification table in Theorem 3 is as follows.

class / $a(X)$	i)	ii)
2	$X_y \cong \boldsymbol{P}(1 \oplus L)$, ($L^k \cong 1$ for some $k \geqq 1$)	
1	$X_y \cong \boldsymbol{P}(1 \oplus L)$ ($L^k \not\cong 1$ for any $k \geqq 1$, y 'general') $\cong \boldsymbol{P}(F_2)$	II, β) (quasi-trivial type) $X \sim (\boldsymbol{P}^1 \times S)/G$
0		II, $\boldsymbol{P}^1$-fiber space over S

2) A covering family of rational curves on X is unique; the one given by the natural meromorphic $\boldsymbol{P}^1$-fiber space structure over S. This follows readily from the non-ruledness of S.

3) For any compact complex manifold X Mabuchi [35] introduced an invariant $\beta(X) := \max\{\dim \bar{X}$: there exists a surjective meromorphic map $X \to \bar{X}$ with $\kappa(\bar{X}) \geqq 0\}$. From the above proposition, for a uniruled non-Moishezon manifold X we always have $\beta(X) = 2$.

Proof. α) Suppose first that $\pi^{-1}\pi(Z_t) = Z_t$ for general $t \in T$ in (10). Then π is bimeromorphic, $\dim T = 2$, and $\rho\pi^{-1}$: $X \to T$ gives a structure of a meromorphic $\boldsymbol{P}^1$-fiber space over T. If $a(T) = 0$, then replacing T by its normal minimal model S we get a $\boldsymbol{P}^1$-fiber space $X \to S$ as in the proof of Theorem. Thus X is in the class ii). On the other hand note that $a(T) \neq 2$; otherwise X would be Moishezon.

β) Next, suppose that $\pi^{-1}\pi(Z_t) \neq Z_t$. Then we infer readily that there exist irreducible components $Z_{t,i}$ of $\pi^{-1}\pi(Z_t)$ other than Z_t for general $t \in T$. Then since X is nonsingular $\dim Z_{t,i} \geqq 1$, and hence also $\dim \pi(Z_{t,i}) \geqq 1$ because $T \subseteqq D_X$. Then if we set $S_t := \pi\rho^{-1}\rho\pi^{-1}\pi(Z_t)$, $\dim S_t \geqq 2$. On the other hand, since π and ρ are Moishezon morphisms (cf. [14, Proposition 4]), S_t also is Moishezon. Hence $S_t \neq X$ for any $t \in T$ by our assumption. Thus $\dim S_t = 2$ for any $t \in T$. Thus $\{S_t\}_{t \in T}$ defines a covering family of divisors on X. Let τ: $T \to \mathrm{Div}\, X$ be the universal meromorphic map. Let $Y = \tau(T)$. Then $\dim Y = 1$, the general fiber of τ being of the form $\rho\pi^{-1}\pi(Z_t)$, $t \in T$, which is a divisor on T since $\dim \rho\pi^{-1}\pi(Z_t) = \dim \pi^{-1}\pi(Z_t) = 1 + \dim Z - \dim X = \dim Z - 2 = \dim T - 1$. Then restricting the universal family onto Y and taking a suitable irreducible component we obtain a covering family

$$\begin{array}{ccc} Y\times X \supseteq Z' & \xrightarrow{\pi'} & X \\ \searrow & \downarrow \rho & \\ & Y & \end{array}$$

with Z'_y an irreducible divisor for general $y \in Y$.

Since Z'_y admits a covering family of rational curves by our construction, Z'_y is ruled for $y \in Y$. It is not rational, since otherwise Z', and hence X also, is Moishezon (cf. Proposition 2.5), contradicting our assumption. Let $r: \tilde{Z} \to Z'$ be a resolution and $\tilde{\rho} := \rho' r: \tilde{Z} \to Y$. Let $\alpha: \tilde{Z} \to \mathrm{Alb}^* \tilde{Z}/Y$ be the relative Albanese map for $\tilde{\rho}$ (which we may assume to be holomorphic). Let S be the image of α and $\eta: S \to Y$ the natural morphism. Then $\dim S = 2$ since $\tilde{Z}_y$ is not rational. If S is Moishezon, X would be Moishezon, again leading to a contradiction. So $a(S)=1$. In particular S is an elliptic surface over Y. (Thus we have actually $S = \mathrm{Alb}^* \tilde{Z}/Y$.) We claim that $\tilde{\pi} = r\pi': \tilde{Z} \to X$ is bimeromorphic. In fact, let $B_y = \tilde{\pi}^{-1}\tilde{\pi}(\tilde{Z}_y)$. Suppose that $\tilde{\rho}(B_y) = Y$. Let $B_{y,i}$ be any irreducible component of B_y other than $\tilde{Z}_y$. Then $B_{y,i}$ is a divisor on $\tilde{Z}$, $\tilde{\rho}(B_{y,i}) = Y$ and $B_{y,i}$ is Moishezon, both $\tilde{\pi}$ and Z_y being Moishezon (cf. [14, Proposition 4]). Thus $\alpha(B_{y,i}) \neq S$. Hence $\alpha(B_{y,i})$ is a divisor with $\eta\alpha(B_{y,i}) = Y$. However this also is impossible since η is an algebraic reduction of S by Proposition 1.1. Thus $B_y = \tilde{Z}_y$. This implies that π is bimeromorphic as was desired.

γ) By α) and β) we have shown that if X is not in the class ii), then X is a meromorphic $\boldsymbol{P}^1$-fiber space over an elliptic surface S with $a(S)=1$. Let $\eta: S \to Y$ be the algebraic reduction giving the structure of an elliptic surface. Let $f: X \to Y$ be the composite meromorphic map and $f^*: X^* \to Y$ a holomorphic model of f. We consider the general fiber X^*_y of f^* in case i). When $a(X)=1$, $X^*_y \cong \boldsymbol{P}(1 \oplus L)$ or $\boldsymbol{P}(F_2)$ by Theorem 3.

So assume that $a(X)=2$ so that X is an elliptic threefold. In this case we infer readily that X^*_y has the structure of an elliptic surface over $\boldsymbol{P}^1$ also. Hence by [36], [41] $X^*_y \cong \boldsymbol{P}(1 \oplus L)$ with $L^k \cong 1$ for some $k \geqq 1$ or $\cong \boldsymbol{P}(F_2 \otimes L_1)$ where L_1 is a line bundle of degree 1 on S_y. We shall see that the latter case does not occur. In fact, if $X^*_y \cong \boldsymbol{P}(F_2 \otimes L_1)$ for general y, there exists a unique irreducible divisor E on X such that E_y coincides with the unique minimal section of $X^*_y \to S_y$ for general $y \in Y$. Then we have $a(E)=1$ and $\deg [E]|_{E_y} = 1$ for general y where $[E]$ denotes the line bundle defined by E on X^*. This is impossible by Proposition 1.1.

Finally we have to show that f is actually holomorphic so that we can take $f = f^*$ in the above argument. This follows from Proposition 13.2 when $a(X)=1$, and the same proof also applies to the case $a(X)=2$ since we have proved that $X^*_y \cong \boldsymbol{P}(1 \oplus L)$. q.e.d.

14.2. Finally we shall state without proof a bimeromorphic classification of non-Moishezon uniruled threefolds in the class i) of Proposition 14.1. We first introduce some notations. Let X be a minimal elliptic surface with $a(S)=1$. Let $\pi: S \rightarrow Y$ be the algebraic reduction giving the structure of an elliptic surface. Let $\pi_0: B \rightarrow Y$ be the associated basic elliptic surface (cf. 10.3, c)), which is algebraic. Consider B as an elliptic curve over $K:=\boldsymbol{C}(Y) \cong \boldsymbol{C}(S)$. Then we denote by $E(K)$ the abelian group of K-rational points of B. On the other hand, put the unique structure of an algebraic curve on Y. Then the coherent analytic sheaf $R^1\pi_*\mathcal{O}_S$ has the unique structure of a coherent algebraic sheaf on Y. Let $\eta \in Y$ be the (scheme-theoretic) generic point of Y. Let $E(S):=(R^1\pi^*\mathcal{O}_S)_\eta$, which is a finite dimensional vector space over K.

Proposition 14.2. *The set of bimeromorphic equivalence classes of compact non-Moishezon uniruled threefolds in $\mathscr{C}$ which is in the class* i) *in Proposition* 14.1 *is in natural bijective correspondence with the set of pairs* (S, e) *consisting of a minimal elliptic surface S with $a(S)=1$ and an element* $e \in E(K)$ *(resp.* $e \in \boldsymbol{P}(E(S)) := (E(S)-\{0\}/K^*)$ *if* $X_y^* \cong \boldsymbol{P}(1 \oplus L)$ *(resp.* $X_y^* \cong \boldsymbol{P}(F_2)$*)).*

Remark. If $X_y^* \cong \boldsymbol{P}(1 \oplus L)$, then $a(X)=2$ if and only if $e \in E(K)_{\mathrm{tor}}$, the torsion subgroup of $E(K)$.

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