Advanced Studies in Pure Mathematics 1, 1983
Algebraic Varieties and Analytic Varieties
pp. 219–230

Introduction to the Theory of Compact Complex Spaces in the Class $\mathscr{C}$

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Introduction

The main purpose of the present notes is to give an introduction to Fujiki's paper in this volume. A compact complex space X belongs to $\mathscr{C}$, if its reduced analytic space X_{red} is an image of a meromorphic map from a compact Kähler manifold. Fujiki studied Douady spaces of compact complex spaces in $\mathscr{C}$ and obtained important results. Some of them were obtained by Campana independently. Fujiki applied his results to the classification theory of compact complex manifolds and obtained interesting results. (see also Campana [4]) In the present notes we shall discuss his methods. Of course, his methods cannot be necessarily applied to compact complex manifolds which are not in $\mathscr{C}$. This means that the structure of such manifolds may be quite different from that of manifolds in $\mathscr{C}$. In this paper we shall give several such examples.

In § 1, we shall give the definition of a compact complex spaces in $\mathscr{C}$ and exhibit several examples. In § 2, we shall discuss the Douady spaces of compact complex spaces and give important results on them. In § 3, we shall discuss applications of Fujiki's results on the Douady space to the classification theory. Details can be found in Fujiki's paper in this volume.

Notation and convention

All complex manifolds and complex spaces satisfy the second countability axiom.

A holomorphic map $f: V \to W$ of a complex space V onto a complex space W is called a *fibre space*, if f is proper and has connected fibres.

A fibre space $f: V \to W$ is said to be bimeromorphic to a fibre space $g: X \to Y$, if there exist bimeromorphic maps $h_1: V \to X$ and $h_2: W \to Y$ with $g \circ h_1 = h_2 \circ f$.

$\kappa(V)$ means the Kodaira dimension of a compact complex manifold

Received October 9, 1981.

V. For a line bundle L on V, $\kappa(L, V)$ means the L-dimension of L. (For the definition see, for example, Ueno [19] or Iitaka' paper in this volume.)

§ 1. Compact complex space in the class $\mathscr{C}$

Let us begin with the following definition.

Definition 1.1. *Let X be a complex space and χ a real C^∞ type* (1, 1)-*form on X. The form χ is called* hermitian *if there exist an open covering $\mathfrak{A} = \{U_\alpha\}_{\alpha \in A}$ of X, an embedding j: $U_\alpha \to O_\alpha$ of U_α into a domain O_α of $\mathbf{C}^{n_\alpha}$ and a C^∞ positive type* (1, 1)-*form χ_α on O_α such that $j^*\chi_\alpha = \chi \mid U_\alpha$. Further a hermitian form χ is called a* Kähler form *if in the above definition all χ_α can be taken to be d-closed. A complex space with a Kähler form is called a* Kähler space.

From the above definition it follows that a non-singular Kähler space X is a Kähler manifold in the usual sense. Moreover, a complex subspace Y of a Kähler space X is again a Kähler space. But an image of a meromorphic mapping of a Kähler space is not necessarily Kähler. Thus we need the following definition.

Definition 1.2. *A compact complex space X is following in the class $\mathscr{C}$ if there exists a surjective meromorphic mapping g: $\tilde{X} \to X_{\text{red}}$ from a compact Kähler space $\tilde{X}$ to X_{red}.*

By the resolution of singularities of complex spaces due to Hironaka [11], in the above definition we may assume that $\tilde{X}$ is a compact Kähler manifold. Moreover, we may assume that g: $\tilde{X} \to X_{\text{red}}$ is holomorphic. Note that in the above definition it may happen that $\dim \tilde{X} > \dim X$.

From the definition we infer readily the following.

1–3) For a compact complex space X in $\mathscr{C}$, any irreducible closed complex subspace Y of X is in $\mathscr{C}$.

1–4) If X is in $\mathscr{C}$ and g: $X \to Y$ is a surjective meromorphic map of complex spaces, then Y belongs to $\mathscr{C}$.

1–5) If a surjective morphism g: $X \to Y$ is bimeromorphically equivalent to a Kähler morphism (such a morphism is called a *$\mathscr{C}$-morphism*) and Y is in $\mathscr{C}$, then X is in $\mathscr{C}$. Here a holomorphic map f: $M \to N$ is called a Kähler morphism if there exist an open covering $\{U_\alpha\}_{\alpha \in A}$ of M and C^∞-functions P_α on U_α such that for each point $y \in N$, P_α is strictly plurisubharmonic on $U_\alpha \cap M_y$ and $P_\alpha - P_\beta$ is pluriharmonic on $U_\alpha \cap U_\beta \cap M_y$. (For example, a projective morphism is a Kähler morphism.)

Proposition 1.3. *Let M be a compact complex manifold in $\mathscr{C}$. Then* *the Hodge decomposition theorem holds for the cohomology group $H^*(M, \boldsymbol{C})$. Moreover we can define a primitive cohomology $H^*(M, \boldsymbol{C})_{\mathrm{prym}}$ and it carries a polarized Hodge structure defined over $\boldsymbol{R}$.*

Proof. There exist a compact Kähler manifold $\tilde{M}$ and a surjective morphism $g: \tilde{M} \to M$. Then we have a commutative diagram

$$\begin{array}{ccc} E_1^{p,q}(M) = H^q(M, \Omega_M^p) & \Rightarrow & H^{p+q}(M, \boldsymbol{C}) \\ \downarrow g^* & & \downarrow g^* \\ E_1^{p,q}(\tilde{M}) = H^q(\tilde{M}, \Omega_{\tilde{M}}^p) & \Rightarrow & H^{p+q}(\tilde{M}, \boldsymbol{C}) \end{array}$$

of the Hodge spectral sequences. As $\tilde{M}$ is Kähler, the bottom spectral sequence degenerates at E_1 terms. Now we shall show that both arrows g^* are injective. Assume that $g^*\tau = \bar{\partial}\eta$ for $\bar{\partial}$-closed (p, q)-form τ on M where η is $(p, q-1)$-form on $\tilde{M}$. Let ω be a Kähler form on $\tilde{M}$ and put $k = \dim \tilde{M} - \dim M$. There exist Zariski open subsets $\tilde{M}'$ of $\tilde{M}$ and M' of M such that $g|_{\tilde{M}'}: \tilde{M}' \to M'$ is smooth. Then for each point $x \in M'$, $\int_{M_x} \omega^k = C(x)$ is positive and independent of x. If τ is not zero in $H^q(M, \Omega_M^p)$, then by the Serre duality, there is a $\bar{\partial}$-closed $(m-p, m-q)$-form ζ with $\int_M \tau \wedge \zeta \neq 0$ where $m = \dim M$. Then we have

$$\int_{\tilde{M}} \omega^k \wedge g^*(\tau \wedge \zeta) = \int_{M'} C(x)\tau \wedge \zeta = C \int_M \tau \wedge \zeta \neq 0$$

On the other hand, since we have $g^*\tau = \bar{\partial}\eta$, by the Stokes theorem we have

$$\int_{\tilde{M}} \omega^k \wedge g^*(\tau \wedge \zeta) = 0.$$

This is a contradiction. Hence $g^*: H^q(M, \Omega_M^p) \to H^q(\tilde{M}, \Omega_{\tilde{M}}^p)$ is injective. Similarly we can prove that $g^*: H^{p+q}(M, \boldsymbol{C}) \to H^{p+q}(M, \boldsymbol{C})$ is injective. Hence the Hodge spectral sequence $E_1^{p,q}(M) \Rightarrow H^{p+q}(M, \boldsymbol{C})$ degenerates at E_1-terms. Moreover, since g^* commutes with the complex conjugation, $H^{p+q}(M, \boldsymbol{C})$ carries a Hodge structure. Define $H^{p+q}(M, \boldsymbol{C})_{\mathrm{prym}}$ to be $g^{*-1}H^{p+q}(\tilde{M}, \boldsymbol{C})_{\mathrm{prym}}$. Then $H^{p+q}(M, \boldsymbol{C})_{\mathrm{prym}}$ carries a polarized Hodge structure induced from that of $H^{p+q}(\tilde{M}, \boldsymbol{C})_{\mathrm{prym}}$. Q.E.D.

Remark. In the above proposition, a primitive cohomology does depend upon a choice of a surjective morphism $g: \tilde{M} \to M$.

Corollary 1.4. *Let M be a compact complex manifold in $\mathscr{C}$. Then we have the following.*

1) *Every holomorphic form on M is d-closed.*

2) *The Albanese map α: $M \to A(M)$ gives an isomorphism α^*: $H^1(A(M), \boldsymbol{C}) \simeq H^1(X, \boldsymbol{C})$.*

3) *The ν-th Betti number $b_\nu(M)$ of M is even, if ν is odd.*

Remark 1.5. For a compact complex manifold S of dimension 2 (analytic surface), the Hodge spectral sequence always degenerates at E_1-terms, even if S is not in $\mathscr{C}$. Moreover, every holomorphic form is d-closed. But the Hodge decomposition theorem does not necessarily hold for $H^1(S, \boldsymbol{C})$. On the other hand $H^2(S, \boldsymbol{C})$ carries always a Hodge structure. These fact follows easily from Kodaira [14].

Now we give some examples.

Example 1). Let V be a complete algebraic variety defined over $\boldsymbol{C}$. Then by Chow's lemma, V is in $\mathscr{C}$.

Example 2). A reduced compact complex space X is called *Moishezon* if its meromorphic function field $\boldsymbol{C}(X)$ has transcendence degree $n = \dim X$. The Moishezon space X is bimeromorphically equivalent to a projective variety. Hence X is in $\mathscr{C}$. Moishezon [17] showed that a Moishezon *manifold* M is projective if and only if M is Kähler.

Example 3). *Hopf manifold.* Put $W = \boldsymbol{C}^n - \{(0, \cdots, 0)\}$. We let g be an analytic automorphism of W defined by

$$g\colon (z_1, \cdots, z_n) \longrightarrow (\alpha z_1, \cdots, \alpha z_n), \qquad |\alpha| < 1.$$

The infinite cyclic group G generated by g acts on W property discontinuously and freely. The quotient manifold $H_\alpha = W/G$ is called a Hopf manifold. The manifold H_α is diffeomorphic to $S^1 \times S^{2n-1}$, hence $b_1(H_\alpha) = 1$. Thus H_α is *not* in $\mathscr{C}$.

Example 4). Let C be a non-singular curve of genus g and L a very ample line bundle on C. Put $V = L \oplus L$. Take general sections $s_1, s_2, s_3, s_4 \in H^0(C, V)$ such that for each point $x \in C$, $s_1(x), s_2(x), s_3(x), s_4(x)$ generate a lattice in the fibre $V_x \simeq \boldsymbol{C}^2$. Then $\Lambda = \boldsymbol{Z}s_1 + \boldsymbol{Z}s_2 + \boldsymbol{Z}s_3 + \boldsymbol{Z}s_4$ operates on V as translations in every fibre of V. The action is properly discontinuous and free. There is a natural morphism $\pi\colon M = V/\Lambda \to C$. The morphism π is smooth and each fibre of π is a complex 2-torus. It is easy to calculate the canonical bundle K_M of M and we obtain

$$K_M = \pi^*(L^{-2} \otimes K_C).$$

Hence we have

$$\omega_{M/C}=\pi^*\mathcal{O}(L^{-2}).$$

From the following theorem we infer that M is *not* in $\mathscr{C}$.

Theorem 1.6. *Let φ: $M\to N$ be a proper and smooth holomorphic map with connected fibres. Assume M is in $\mathscr{C}$. Then the curvature form of $\varphi_*\omega_{M/N}$ with respect to a natural inner product on $\varphi_*\omega_{M/N}$ is positive semi-definite.*

Proof. If M is Kähler, this is proved by Griffiths [10]. By virtue of Proposition 1.3, it is easy to see that his proof works also in our situation. Q.E.D.

§ 2. Douady space

The *Douady space* D_X of a complex space X is defined by the following universal property.

(2.1) There exists a closed analytic subspace $W\subset X\times D_X$ which is proper and flat over D_X such that for any complex space T and a closed analytic subspace $Z\subset X\times T$ which is proper and flat over T, there exists a unique morphism f: $T\to D_X$ such that $Z=(\mathrm{id}_X\times f)^*(W)$.

Equivalently, the Douady space D_X of a complex space X is a complex space which represents a functor D defined by

$$D(T)=\{\text{closed complex subspaces } Z\subset X\times T, \text{ proper and flat over } T\}.$$

Existence of the Douady space was proved by Douady [5], by using technique of Banach analytic manifolds. Recently, Bingener [2] obtained a different proof.

If X is a projective scheme, D_X is nothing but the Hilbert scheme of X. We can also define the *relative Douady space* $D_{X/Y}$ of a complex space f: $X\to Y$ over Y in a natural way. Existence of the relative Douady space was proved by Pourcin [18].

Now we state an important result due to Fujiki.

Theorem 2.2. *If X is in $\mathscr{C}$, each irreducible component D_α of the Douady space D_X is compact. If f: $X\to Y$ is a $\mathscr{C}$-morphism* (cf. (1–5)), *then, after restricting to any relatively compact subdomain of Y, each irreducible component D_α of the relative Douady space $D_{X/Y}$ is proper over Y.*

A proof can be found in Fujiki [6]. To prove the theorem, Fujiki first proved the corresponding theorem for the *Barlet space* B_X of a complex space X. The Barlet space was introduced by Barlet [1] which is a counterpart of the Chow scheme of a projective scheme. Compactness of each irreducible component of the Barlet space was proved independently by Liebermann [16].

Now let us consider Example 4) in § 1. Take $C=\boldsymbol{P}^1$ and consider the irreducible component D_α of the Douady space D_M of M which contains a point corresponding to the zero section of $\pi: M\to\boldsymbol{P}^1$. (Note that the zero section of V induces the zero section of π.) Let $g_\alpha: X_\alpha\to D_\alpha$ be the universal family over D_α. For the point O corresponding to the zero section of $\pi: M\to\boldsymbol{P}^1$ the fibre $g_\alpha^{-1}(O)$ is a non-singular rational curve. Since g_α is proper and flat, it follows that reduced structure of each irreducible component of a fibre g_α is $\boldsymbol{P}^1$. On the other hand, every fibre of π contains no $\boldsymbol{P}^1$. Hence each irreducible component of a fibre of g_α intersect transversally with a fibre of π. As we have $(g_\alpha^{-1}(O)\cdot l_\pi)_M=1$ for a fibre l_π of π, every fibre of g_α is irreducible and reduced. This means that there is one to one correspondence between the point set of D_α and the set of holomorphic sections of V. Hence as sets we have an isomorphism $D_\alpha\simeq H^0(\boldsymbol{P}^1, L\oplus L)/(\oplus_{i=1}^4 \boldsymbol{Z}s_i)$. This implies that D_α is non-compact. Thus Theorem 2.2 does not necessarily hold for an analytic space which is not in $\mathscr{C}$.

Fujiki [8] also proved the following theorem.

Theorem 2.3. *The Douady space D_X of a complex space X satisfies the second countability axiom. Hence there are countably many irreducible components of D_X.*

Note that we always assume that a complex space X satisfies the second countability axiom.

Fujiki [9] and Campana [3] proved the following theorem, independently.

Theorem 2.4. *If X is a complex space in $\mathscr{C}$, then each irreducible component of D_X is in $\mathscr{C}$.*

Thus the category whose objects consist of complex spaces in $\mathscr{C}$ is closed under several operations.

Theorem 2.2 or the corresponding theorem on Barlet spaces has several applications. In this section we consider an application to automorphism groups of Kähler manifolds due to Fujiki [7] and Lieberman [16]. Let X be a compact complex space and Aut (X) the analytic automorphism group of X. For each element $g\in$ Aut (X) we let $\rho(g)$ be the point of the Douady space $D_{X\times X}$ of $X\times X$ which corresponds to the graph $\Gamma_g\subset X\times X$ of g. Thus we have a map ρ: Aut $(X)\to D_{X\times X}$. Douady [5] proved the following theorem as an application of his existence theorem of Douady spaces.

Theorem 2.5. 1) *Im* ρ *is open in* $D_{X\times X}$. *Hence* Aut (X) *inherits a natural complex structure.*

2) *With respect to this complex structure* Aut (X) *is a complex Lie group.*

The second part of the theorem was originally proved by Kaup [12] and Kerner [13].

It is not difficult to show that Im ρ is Zariski open in $D_{X\times X}$, that is, the closure of Im ρ consists of union of irreducible components of $D_{X\times X}$ which contain a point corresponding to the graph of an analytic automorphism. Hence, if X is in $\mathscr{C}$, the identity component $\mathrm{Aut}_0(X)$ of Aut (X) has a natural compactification. This is a key point to prove the structure theorem of $\mathrm{Aut}_0(X)$. For the detail we refer the reader to Fujiki [7] and Lieberman [16].

Another application is the following. Assume that X is in $\mathscr{C}$. Put $G=\mathrm{Aut}_0(X)$. We let G^* be the natural compactification of G given above. Then the graph $\Gamma\subset G\times X\times X$ of the action of G on X has also a natural compactification $\Gamma^*\subset G^*\times X\times X$. For a closed analytic subset A of X the stabilizer G_A of A is defined by $G_A=\{g\in G\,|\,g(A)=A\}$. It is easy to show that the closure $\bar{G}_A$ of G_A in G^* is given by $p_1(\Gamma^*\cap G^*\times A\times A)$, where p_1 is the projection from $G^*\times X\times X$ to the first factor. As G^* is compact, G_A is compact. This means that G_A *has only finitely many irreducible components.*

The following example is due to Fujiki.

Example. Consider Example 3) in § 1. Suppose $n=2$. It is easy to show that $G=\mathrm{Aut}_0(H_\alpha)$ is given by $GL(2, \boldsymbol{C})\Big/\left\langle\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right\rangle$ where $GL(2,\boldsymbol{C})$ acts linearly on $W=\boldsymbol{C}^2-\{(0,0)\}$ and induces automorphisms of H_α. The stabilizers at points $u=[1, 0]$, $v=[0, 1]$ are given by

$$G_u=\left\{\begin{pmatrix}\alpha^n & b\\ 0 & d\end{pmatrix}\right\}\Big/\left\langle\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right\rangle,\qquad G_v=\left\{\begin{pmatrix}a & 0\\ c & \alpha^n\end{pmatrix}\right\}\Big/\left\langle\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right\rangle.$$

Hence we have $G_u\cap G_v\simeq\left\{\begin{pmatrix}1 & 0\\ 0 & \alpha^n\end{pmatrix}\right\}$. Thus $G_u\cap G_v$ is an infinite discrete group. This implies that G_A, $A=\{u, v\}$ is also an infinite discrete group. Thus, the irreducible component of $D_{H_\alpha\times H_\alpha}$ containing $\rho(G)$ is *not* compact.

Fujiki [7] also showed the following theorem.

Theorem 2.6. *Let X be Kähler manifold, and* ω *a Kähler form on X. The Kähler form* ω *gives a cohomology class* $\bar{\omega}\in H^2(X, \boldsymbol{R})$. *Put* $\mathrm{Aut}_{\bar{\omega}}(X)=\{g\in \mathrm{Aut}(X)\,|\,g^*\bar{\omega}=\bar{\omega}\}$. *Then* $\mathrm{Aut}_{\bar{\omega}}(X)$ *is a complex Lie group with only finitely many connected components.*

§ 3. Algebraic reduction

Let $C(M)$ be the meromorphic function field of M. The algebraic dimension $a(M)$ of M is defined by $a(M)=\text{tr.}\deg_C C(M)$. It is well-known that $C(M)$ is an algebraic function field, that is, there exist a projective variety $V\subset P^N$ and an isomorphism $C(M)\simeq C(V)$. We may assume that V is non-singular. Note that $\zeta_1/\zeta_0|_V, \cdots, \zeta_N/\zeta_0|_V$ are generators of $C(V)$ where $(\zeta_0:\zeta_1:\cdots:\zeta_N)$ are homogeneous coordinates of P^N. We let ψ_i, $i=1, 2, \cdots, N$ be elements of $C(M)$ corresponding to $\zeta_i/\zeta_0|_V$, $i=1, 2, \cdots, N$. Then we have a meromorphic mapping

$$\begin{array}{rcl}\varphi: M & \longrightarrow & P^N \\ \cup & & \cup \\ z & \longmapsto & (1:\psi_1(z):\psi_2(z):\cdots:\psi_N(z)).\end{array}$$

Note that $\varphi(M)=V$. Taking a suitable bimeromorphically equivalent model $\hat{M}$ of M we have a morphism

$$\hat{\varphi}: \hat{M} \longrightarrow V.$$

The morphism $\hat{\varphi}$ is call the *algebraic reduction* of M. This is uniquely determined up to bimeromorphic equivalence, and $\hat{\varphi}$ induces an isomorphism between function fields of M and V.

For example, let us consider the Hopf manifold H_α defined in Example 3. In this case we have a morphism

$$\begin{array}{rcl}\varphi: H_\alpha & \longrightarrow & P^{n-1} \\ \cup & & \cup \\ [z_1, z_2, \cdots, z_n] & \longmapsto & (z_1:z_2:\cdots:z_n).\end{array}$$

The morphism φ induces an isomorphism $\varphi^*: C(P^{n-1})\rightrightarrows C(H_\alpha)$. Hence, $\varphi: H_\alpha\to P^{n-1}$ is an algebraic reduction. It is easy to show that each fibre of φ is isomorphic to an elliptic curve $E=C^*/\langle\alpha\rangle$.

We define the algebraic codimension $ca(M)$ of M as $ca(M)=\dim M-a(M)$. Then general fibres of the algebraic reduction of M are of dimension $ca(M)$.

In general we have the following theorem. For the proof see Ueno [19].

Theorem 3.1. *Let $\hat{\varphi}: \hat{M}\to V$ be an algebraic reduction of M. For each line bundle L on $\hat{M}$ we have*

$$\kappa(\hat{M}_x, L_x)\leq 0$$

for general point $x\in V$, where $\hat{M}_x=\hat{\varphi}^{-1}(x)$, $L_x=L|_{M_x}$.

Corollary 3.2. *If $ca(M)>0$, then for a general fibre M_x of the algebraic reduction $\hat{\varphi}$: $\hat{M}\to V$ we have $\kappa(M_x)\leq 0$. Moreover we have*

1) *if $ca(M)=1$, every smooth fibre of $\hat{\varphi}$ is an elliptic curve:*

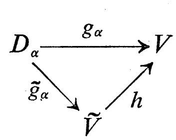
2) *if $ca(M)=2$, then every smooth fibre of $\hat{\varphi}$ is bimeromorphically equivalent to one of the following surfaces:* 1) *complex torus,* 2) *hyperelliptic surface,* 3) *$K3$ surface,* 4) *Enriques surface,* 5) *ruled surface of genus* 1, 6) *rational surface,* 7) *elliptic surface with trivial canonical bundle,* 8) *surface of type* VII_0.

Proof. Apply Theorem 3.1 to the case $L=K_M$ and $L=K_M^{-1}$. We only need to prove that if $ca(M)=2$, then ruled surfaces of genus $g\geq 2$ do not appear as general fibres. This was proved by Kuhlman [15]. Q.E.D.

Now assume that M is a complex manifold in $\mathscr{C}$. In this case, if $ca(M)=2$, Fujiki has shown that only very few types of surfaces appear as fibres of algebraic reductions. For example, Enriques and hyperelliptic surfaces never appear as general fibres. We shall discuss the main idea to prove this fact due to Fujiki.

Note that if S is an Enriques surface or a hyperelliptic surface, there is surjective morphism π: $S\to \boldsymbol{P}^1$ whose general fibres are elliptic curves. We call such a fibre space an elliptic fibration. If S is a hyperelliptic surfaces, then an elliptic fibration over $\boldsymbol{P}^1$ is uniquely determined. But if S is an Enriques surface, in general there are several different elliptic fibrations over $\boldsymbol{P}^1$.

Now let us consider the algebraic reduction $\hat{\varphi}$: $\hat{M}\to V$ of a complex manifold M in $\mathscr{C}$ with $ca(M)=2$. Assume that general fibres of $\hat{\varphi}$ are Enriques surfaces or hyperelliptic surfaces. Take dense open set U of V such that for each point $u\in U$ the fibre $\hat{M}_u$ is non-singular. Since $\hat{M}_u$ is also an Enriques surface or a hyperelliptic surface, it contains an elliptic curve l_u which is a general fibre of an elliptic fibration of $\hat{M}_u$ over $\boldsymbol{P}^1$. For each point $u\in U$ we fix such an elliptic curve l_u. Since we can consider the relative Douady space $D_{\hat{M}/V}$ as a subspace of $D_{\hat{M}}$, by virtue of Theorem 2.3, $D_{\hat{M}/V}$ has countably many irreducible components. Hence there exist an irreducible component D_α of $D_{\hat{M}/V}$ and a dense subset U' of U such that D_α contains a point corresponding to the elliptic curve l_u for all $u\in U'$. We let π_α: $Z_\alpha\to D_\alpha$ be the universal family over D_α and g_α: $D_\alpha\to V$ the canonical morphism. By Theorem 2.2, g_α is proper. Hence by our choice of D_α, g_α is surjective. We let



be the Stein factorization of g_α. Put $f=\tilde{g}_\alpha\circ\pi_\alpha$. For each point $u\in U'$ we find a point $\tilde{u}\in h^{-1}(u)$ such that one of the fibres of $\pi_{\alpha,\tilde{u}}=\pi_\alpha|_{f^{-1}(\tilde{u})}: Z_{\alpha,\tilde{u}}=f^{-1}(\tilde{u})\to D_{\alpha,\tilde{u}}=\tilde{g}_\alpha^{-1}(\tilde{u})$ is l_u. Since l_u is a general fibre of an elliptic fibration of $\hat{M}_u$ over P^1, $\pi_{\alpha,\tilde{u}}: Z_{\alpha,\tilde{u}}\to D_{\alpha,\tilde{u}}$ is isomorphic to this elliptic fibration. Hence $D_{\alpha,\tilde{u}}$ is P^1 and $Z_{\alpha,\tilde{u}}$ is isomorphic to $\hat{M}_u$. On the other hand, there is a natural mapping $j: Z_\alpha\to\hat{M}$ and there is a factorization

$$\begin{array}{ccc} Z_\alpha & \xrightarrow{\ j\ } & \hat{M} \\ {\scriptstyle\tilde{j}}\downarrow & \nearrow & \\ \hat{M}\times_V\tilde{V} & & \end{array}$$

As U' is dense in V, it follows that $\tilde{j}: Z_\alpha\to\tilde{M}=\hat{M}\times_V\tilde{V}$ is bimeromorphic. Hence we have $a(\tilde{M})=a(Z_\alpha)\geq a(D_\alpha)$. Moreover, $\tilde{M}\to\hat{M}$ is generically finite, hence we have $a(\tilde{M})=a(\hat{M})$. On the other hand, since $\tilde{g}_\alpha: D_\alpha\to\tilde{V}$ is a P^1-fibre space, we have $a(D_\alpha)=a(\tilde{V})+1$. Therefore $a(\hat{M})=a(\tilde{M})\geqslant a(D_\alpha)=a(\tilde{V})+1=a(V)+1$. Since $\phi: \hat{M}\to V$ is the algebraic reduction, this is a contradiction.

By a similar argument we can prove that if M is in $\mathscr{C}$, rational surfaces do not appear as general fibres of an algebraic reduction. In all cases, the important fact is that D_α is proper over V. This is not necessarily true for a complex manifold which is not in $\mathscr{C}$. Hence the above results may not be true for such a manifold. We give such an example.

Example. Let S be a Kummer surface associated with a Jacobian of a non-singular curve of genus 2. In Ueno [22] it has been shown there is a finitely generated subgroup $G\subset\mathrm{Aut}(S)$ such that if the first Chern class of a line bundle L on S is fixed under the action of G, then $\kappa(L,S)\leq 0$. Let $\{g_1, g_2, \cdots, g_n\}$ be a system of generators of G. Take a non-singular curve C of genus g with a system of generators $\{\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n\}$ of the fundamental group $\pi_1(C)$ with a relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}=e$. Then a mapping

$$\alpha_i\longrightarrow g_i$$
$$\beta_i\longrightarrow g_i^{-1}$$

gives a group homomorphism $\rho: \pi_1(C)\to G$ and we can define an analytic fibre bundle $f: X\to C$ associated with the group homomorphism ρ whose fibre is S. By Corollary 2.6 it is easy to show that X is non-Kähler.

Now we prove that $a(X)=1$. This follows from the following lemma.

Lemma 3.3. *Let $f: V\to W$ be an analytic fibre bundle over a projective manifold W whose fibre is a compact complex manifold F with structure* *group $G \subset \mathrm{Aut}(F)$. Suppose that the structure group can not be reduced to a smaller subgroup. Moreover, assume $H^1(F. \mathcal{O}_F)=0$. Then we have $a(V) > a(W)$, if and only if there exists a line bundle L on F such that the first Chern class $c_1(L) \in H^2(F, \boldsymbol{Z})$ of L is invariant under the action of G on $H^2(F, \boldsymbol{Z})$ and $\kappa(L, F) \geqslant 1$.*

Proof. Take a meromorphic function g on V which does not come from W. Let D be the zero divisor of g and L the line bundle determined by D. We let $\tilde{W}$ be the universal covering of W and we fix a trivialization $\psi: \tilde{V} = V \times_W \tilde{W} \rightrightarrows F \times \tilde{W}$. Then, for each point $\tilde{w} \in \tilde{W}$, the pull back $\boldsymbol{L}$ of L to $\tilde{V}$ induces a line bundle $\boldsymbol{L}_{\tilde{w}}$ on F through the trivialization ψ. The line bundles $\boldsymbol{L}_{\tilde{w}}$'s have the same Chern class. Our assumption that $H^1(F, \mathcal{O}_F) = 0$ implies that $\boldsymbol{L}_{\tilde{w}}$'s are isomorphic to each other. Therefore we denote this line bundle by L. Then by our assumption, $c_1(L)$ is invariant under the action of G. Moreover L is a line bundle corresponding to a zero divisor of a meromorphic function $g|_{V_w}$, where $V_w = f^{-1}(w)$. Hence we have $\kappa(L, F) \geqslant 1$.

Conversely, if we have such a line bundle L on F, then we can define a line bundle $\boldsymbol{L}$ on V. By our assumption there exists a positive integer m such that $f_*\mathcal{O}_V(\boldsymbol{L}^m)$ is locally free of rank$\geqslant 2$. Take a very ample line bundle H on W such that $f_*\mathcal{O}_V(\boldsymbol{L}^m) \otimes \mathcal{O}_W(H)$ is generated by its global sections. Then there are two linearly independent sections. $h_1, h_2 \in H^0(V, \mathcal{O}_V(\boldsymbol{L}^m \otimes f^*H))$ such that h_2/h_1 is a meromorphic function which is not constant on a general fibre of f. Hence we have

$$a(V) > a(W). \qquad \text{Q.E.D.}$$

Now assume that our threefold X is in $\mathscr{C}$. Then, since the Kummer surface S has an elliptic fibration over $\boldsymbol{P}^1$, we can apply the same argument as above and we conclude $a(X) \geqslant 2$. This is a contradiction. Hence X is not in $\mathscr{C}$.

Compact complex threefolds which are not in $\mathscr{C}$ have several strange properties. We refer the reader to Ueno [20], [21].

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