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The Zeroes of Characteristic Function χ_f for the Exponents of a Hypersurface Isolated Singular Point

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§ 0. Introduction

In this note, exponents mean a certain set of numbers $\alpha_1, \cdots, \alpha_\mu$ associated with an isolated critical point of a holomorphic function f defined in an open neighbourhood of 0 in C^{n+1}. For instance, if f is a simple germ of a function of the type of Dynkin diagram A_l, D_l, E_6, E_7 or E_8, then the exponents of f are given by $m_j/h+n/2, j=1, \cdots, l=$(rank of the Dynkin diagram) where m_j, $j=1, \cdots, \mu$ are the Coxeter exponents and h is the Coxeter number of the diagram.

Such exponents are introduced in [6] (or see [7] for the summary) for the study of the period mapping, associated with f. In fact, roughly speaking, they are given as exponents of Fuchs type differential equation ($=$the Gauß-Manin connection of f) so that they become the exponents of the Fourier expansion of the period mapping associated to f. The existence and well definedness of exponents introduced above depend on the existence of another object, the primitive form $\zeta^{(0)}$, introduced there. Unfortunately the existence of such primitive forms is shown only for simple singularities and simple elliptic singularities for the moment, so that, rigorously speaking, the above definition of the exponents is valid only for that type of singularities.

Thus in this note, in § 1, we give a tentative way of defining exponents and discuss their relation to the characteristic pairs of the mixed Hodge structure by J. Steenbrink [13] (see also M. Saito [10]). A duality property of exponents will be explained in § 1 using local dualities (see (1.3)).

Then the main purpose of this note is to give a provisional report on some computer experiments concerning such exponents. More precisely, our interest in this note concentrates mainly on two problems, namely, one of the "distribution" of exponents in § 2 and the other—the zeroes of characteristic function χ_f introduced in § 3.

The exponents $\alpha_1, \cdots, \alpha_\mu$, which lie in the interval $(0, n+1)$ with

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center at $(n+1)/2$, seem to be distributed rather densely near the center $(n+1)/2$. We shall give a continuous model of such distribution in § 2. As a consequence of such discussion, one conjectures that the geometric genus of the singular point of $f^{-1}(0)$ is less than $\mu/(n+1)!$.

To study the distribution of the exponents, in § 3 characteristic function χ_f for f is introduced as

$$\chi_f=\sum_{i=1}^{\mu} T^{\alpha_i}=\sum_{i=1}^{\mu} \exp(2\pi\sqrt{-1}\,\alpha_i\tau)$$

where $\alpha_1, \cdots, \alpha_\mu$ are the exponents of f. By choosing a variable $X=T^{1/d_0}$ $=\exp(2\pi\sqrt{-1}\,\tau/d_0)$ for some $d_0 \in N$, χ_f becomes a polynomial in X.

If f is a quasi-homogeneous function, the polynomial χ_f is a cyclotomic polynomial. For a search for the zeroes of χ_f for further examples we needed the help of computer.

Then it seems to be a remarkable fact that a computer experiment shows that *though in general the polynomial χ_f in X is not a cyclotomic polynomial, it has rather many roots of absolute values equal to* 1 *which are not roots of* 1.

In the execution of computations by the computer DEC System-2020 and the XY-plotter TETRONIX 4662 in RIMS of Kyoto University, many efforts has been done by T. Mitsui.

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§ 1. The definition of exponents

We introduce exponents of a holomorphic function f of $n+1$-variables at an isolated critical point 0 in this paragraph which will be used in the later paragraphs. They are defined as eigenvalues of a certain endomorphism N on the module $\Omega_f:=\Omega^{n+1}_{C^{n+1},0}/df\wedge\Omega^{n}_{C^{n+1},0}$ (cf. (1.3)).

The definition is tentative so that we give several conjectures on some basic properties of exponents, which in all the known examples turn out to be true.

(1.1) Below, in (1.1), (1.2), we give a brief summary of local Gauß-Manin connection and residue pairings for f from [1], [6].

Let f be the germ of a holomorphic function defined in a neighbourhood of 0 in C^{n+1}. Put

$$\mathscr{H}^{(0)}_f:=\Omega^{n+1}_{C^{n+1},0}/df\wedge d\Omega^{n-1}_{C^{n+1},0}$$
$$\mathscr{H}^{(-1)}_f:=\Omega^{n}_{C^{n+1},0}/df\wedge\Omega^{n-1}_{C^{n+1},0}+d\Omega^{n-1}_{C^{n+1},0}$$

where $\Omega^{p}_{C^{n+1},0}$ is the module of germs of holomorphic p-forms at 0 in C^{n+1}.

The modules $\mathscr{H}_f^{(0)}$, $\mathscr{H}_f^{(-1)}$, which were first studied by E. Brieskorn [1] and are denoted $\mathscr{H}''$, $\mathscr{H}'$ respectively, are naturally $\mathcal{O}_{C,0}=C\{f\}=C\{t\}$ free modules of rank μ ($=$Milnor's number of f at $0=\dim_C \mathcal{O}_{C^{n+1},0}/(\partial f/\partial z_0, \cdots, \partial f/\partial z_n)$.) (see [1] [12])

The exterior differentiation d: $\Omega^n_{C^{n+1}}\to\Omega^{n+1}_{C^{n+1}}$ induces a differential operator, the Gauß-Manin connection,

$$\nabla: \mathscr{H}_f^{(-1)}\longrightarrow\mathscr{H}_f^{(0)}. \tag{1.1.1}$$

The wedge product with the form $df\wedge$: $\Omega^n_{C^{n+1}}\to\Omega^{n+1}_{C^{n+1}}$ induces a short exact sequence of $\mathcal{O}_{C,0}$-modules

$$0\longrightarrow\mathscr{H}_f^{(-1)}\hookrightarrow\mathscr{H}_f^{(0)}\xrightarrow{r^{(0)}}\Omega_f\longrightarrow 0 \tag{1.1.2}$$

where Ω_f: $=\Omega^{n+1}_{C^{n+1}}/df\wedge\Omega^n_{C^{n+1}}$ is a torsion $C\{t\}$ module and has rank μ over C.

(1.2) The following theorem is proved in [6], [5], [9].

Theorem. *There exists an infinite sequence of C-bilinear forms*

$$K^{(k)}: \mathscr{H}_f^{(0)}\times\mathscr{H}_f^{(0)}\longrightarrow C, \qquad k=0, 1, 2, \cdots$$

such that

i) *$K^{(k)}$ is symmetric (skew-symmetric) if k is even (odd) respectively.*
ii) $K^{(k)}(\nabla\omega, \omega')=K^{(k+1)}(\omega, \omega')$ *for* $\omega\in\mathscr{H}_f^{(-1)}$, $\omega'\in\mathscr{H}_f^{(0)}$.
iii) $K^{(k)}(t\omega, \omega')-K^{(k)}(\omega, t\omega')=(n+k)K^{(k)}(\omega, \omega')$ *for* $\omega, \omega'\in\mathscr{H}_f^{(0)}$.
iv) *$K^{(0)}$ is given by the residue pairing,*

$$K^{(0)}(\varphi(x)dx, \psi(x)dx)=\mathrm{Res}\left[\frac{\varphi\psi dx}{\partial f/\partial x_0\cdots\partial f/\partial x_0}\right].$$

Hence factoring through $r^{(0)}$ of (1.1.2), $K^{(0)}$ induces a non-degenerate symmetric bilinear form,

$$J: \Omega_f\times\Omega_f\longrightarrow C. \tag{1.2.1}$$

(1.3) Let us consider a splitting v of the exact sequence (1.1.2), i.e. v: $\Omega_f\to\mathscr{H}_f^{(0)}$ is a C-linear map such that

$$r^{(0)}\cdot v=id_{\Omega_f}. \tag{1.3.1}$$

Associated with v let us define a C-endomorphism

(1.3.2) N: $\Omega_f\to\Omega_f$, by the composition,

$$e \in \Omega_f \longrightarrow \frac{tv(e)-v(te)}{df} \in \mathscr{H}_f^{(-1)} \xrightarrow{\nabla} \mathscr{H}_f^{(0)} \xrightarrow{r^{(0)}} \Omega_f.$$

(1.4) **Proposition.** Suppose that a splitting v satisfies

(1.4.1) $$K(v(e), v(e'))=0 \qquad \text{for } e, e' \in \Omega_f.$$

Then

(1.4.2) $$N+N^*=(n+1)id_{\Omega_f}$$

where N^* is the adjoint endomorphism of Ω_f w.r.t. the bilinear form J of (1.2.1).

Corollary. *The set of eigenvalues* $\{\alpha_1, \cdots, \alpha_\mu\}$ *of* N *has the duality property*:

(1.4.3) $$\{\alpha_1, \cdots, \alpha_\mu\}=\{n+1-\alpha_1, \cdots, n+1-\alpha_\mu\}.$$

(1.5) *Note.* The duality property of N and $\alpha_1, \cdots, \alpha_\mu$ above has several consequences. First, one gets an algebraic representation of the Poincaré duality of the Milnor's fiber using N. The duality is also used to extend the period map associated with f to the boundaries (cf. [6])

(1.6) Let $f(x)$ and $g(y)$ be holomorphic functions on $\boldsymbol{C}^{n+1}$ and $\boldsymbol{C}^{m+1}$ respectively with isolated critical points at the origins. Then the joint $f(x)+g(y)$ defined on $\boldsymbol{C}^{n+m+2}$ has isolated critical point at 0 such that

(1.6.1) $$\Omega_{f+g} \cong \Omega_f \underset{\boldsymbol{C}}{\otimes} \Omega_g.$$

Let v_f and v_g be splittings for f and g, and N_f and N_g be the associated endomorphism of Ω_f and Ω_g respectively. Then $v_f \wedge v_g : \Omega_f \otimes \Omega_g \to \mathscr{H}_f^{(0)} \otimes \mathscr{H}_g^{(0)} \xrightarrow{\wedge} \mathscr{H}_{f+g}^{(0)}$, defines a splitting for $f+g$ and the associated endomorphism of Ω_{f+g} is given by

(1.6.2) $$N_{f+g}=N_f \otimes id_{\Omega_g}+id_{\Omega_f} \otimes N_g.$$

Hence if $\{\alpha_1, \cdots, \alpha_\mu\}$ and $\{\beta_1, \cdots, \beta_\nu\}$ be eigenvalues of N_f and N_g respectively, then the eigenvalues of N_{f+g} is given by

(1.6.3) $$\{\alpha_i+\beta_j : 1 \leqq i \leqq \mu, 1 \leqq j \leqq \nu\}.$$

(1.7) In [6], we constructed splittings v with the help of primitive forms $\zeta^{(0)}$. Then the set of eigenvalues $\{\alpha_1, \cdots, \alpha_\mu\}$ of N does not depend on $\zeta^{(0)}$ and has nice properties listed in the conjecture below. Unfortunately the existence of primitive form is not known for a general f. So we state the conjecture in the following, weaker formulation.

Conjecture. 1) *For any function f with an isolated critical point there exists a section v of* (1.3), *such that*

i) $K^{(k)}(v(e), v(e'))=0$ *for* $k\geqq 1$, $e, e' \in \Omega_f$

ii) $K^{(k)}(\nabla(tv(e)-v(te)), v(e'))=0$ *for* $k\geqq 2$, $e, e' \in \Omega_f$.

2) *The eigenvalues* $\alpha_1, \cdots, \alpha_\mu$ *of* N, *which we shall call the exponents of* f, *do not depend on the choice of a section* v *satisfying the conditions of* 1) *and have the following properties.*

i) *rationality,* $\alpha_i \in \boldsymbol{Q}$ *for all* i. *Moreover the set* $\exp(2\pi\sqrt{-1}\,\alpha_i)$ *gives the eigenvalues of Milnor's monodromy associated to* f.

ii) *positivity* $\alpha_i>0$ *for all* i.

iii) *duality* $\{\alpha_1, \cdots, \alpha_\mu\}=\{n+1-\alpha_1, \cdots, n+1-\alpha_\mu\}$

iv) *simplicity the multiplicity of the smallest* (*or biggest*) α_i *is equal to* 1.

v) *stability The set* $\{\alpha_1, \cdots, \alpha_\mu\}$ *is not changed by a* μ*-constant deformation of* f.

(1.8) **Example.** For a quasi-homogeneous function of degree 1 with respect to the weights $r_0, \cdots, r_n$ of coordinates, the conjecture is true. Explicitly the endomorphism N is given by

$$N\colon \varphi(x)dx \in \Omega_f \longmapsto (X\varphi)dx + r\varphi(x)dx \in \Omega_f$$

where $X=\sum_{i=0}^n r_i x_i(\partial/\partial x_i)$ and $r=\sum_{i=0}^n r_i$ is the smallest exponent.

(1.9) In [13] (5.3) J. Steenbrink introduced in a different way the concept of exponents as a part of characteristic pairs using the mixed Hodge structure on the vanishing cohomology.

Here we recall an equivalent definition due to M. Saito [10] (3.2)

Let $H^n(X_\infty)_\lambda$ be the generalized eigensubspace of the n-th cohomology group of the Milnor's fiber, with the eigenvalue λ with respect to the Milnor's monodromy which carries a mixed Hodge structure. Let h_λ^{pq} be the Hodge number $\dim_{\boldsymbol{C}} Gr_F^W Gr_{p+q}^W H^n(X_\infty)_\lambda$.

Definition. We define μ rational numbers $\{\alpha_1, \cdots, \alpha_\mu\}$ and call them the exponents of f:

$$\lambda \in \boldsymbol{C},\quad p \in \boldsymbol{Z},\quad \lambda\neq 1 \Rightarrow \#\{j\colon \exp 2\pi\sqrt{-1}\,\alpha_j=\bar{\lambda},\ [\alpha_j]=n-p\}=\sum_q h_\lambda^{pq}$$

$$\lambda=1 \Rightarrow \#\{j\colon \alpha_j=n-p+1\}=\sum_q h_1^{pq}$$

where [ ] is the Gauß' symbol.

M. Saito proved the following theorem ([10] § 1, Theorem 1)

Theorem. *The geometric genus p_g of the singular point of $f^{-1}(0)$ equals the number of exponents not greater than* 1.

(1.10) In the computed examples including quasi-homogeneous functions and cusp singularities, the set of exponents in the sense (1.7) and that in the sense of (1.9) coincide. One conjectures that they coincide for any f.

In practice the exponents of the examples in § 3 are calculated by the method of (1.9).

(1.11) For the future use let us introduce some notation.

Definition. $r(f)$: = the smallest exponent of f

$s(f)$: = the biggest exponent of f — the smallest exponent of f

In view of the duality of exponents (1.7) 2) iii) (cf. [8] (3.2))

$$2r(f)+s(f)=n+1.$$

The theorem (1.9) implies that $r(f)>1$ if and only if $f^{-1}(0)$ is rational. Here one makes

Conjecture. i) $s(f)<1$ *if and only if f is simple* (*i.e. one of* A_k, D_k, E_6, E_7, E_8).

ii) $s(f)=1$ *if and only if f is simple elliptic or cusp* (*i.e. one of* $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $T_{p,q,r}$ *with* $1/p+1/q+1/r<1$). *For quasi-homogeneous functions f, the conjecture is true* (*see* [8] (0.8)).

§ 2. The distribution of the exponents of a function f

Let $\alpha_1, \cdots, \alpha_\mu$ be the exponents of a function f of $n+1$-variables with an isolated critical point. They lie in the interval $(0, n+1)$. So, one can define a discrete probability density: $1/\mu \sum_{i=1}^{\mu} \delta(s-\alpha_i)ds$, where $\delta(s)$ is Dirac's delta function and s is a variable in the interval $(0, n+1)$. A priori there is no description of the distribution except that it is symmetric at $s=(n+1)/2$.

In this paragraph, we propose a continuous distribution $N_{n+1}(s)ds$ on the interval (see (2.3)), and ask whether this distribution is the "limit" of the distribution of the exponents as f "moves".

This paragraph also gives a motivation for the investigations in the next paragraph.

(2.1) **Definition.** *A distribution of exponents of a function f is*

(2.1.1) $$\frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds$$

where $\delta(s)$ is Dirac's delta function and $s \in (0, n+1)$.

(2.2) To get an intuitive understanding of a distribution let us compute it in a simple example:

$$f=z_0^{m_0+1}+z_1^{m_1+1}+\cdots+z_n^{m_n+1}.$$

Since f is a joint of monomials $z_i^{m_i+1}$, $i=0, \cdots, n$, by applying (1.6.3), exponents of f are given by

$$\frac{j_0}{m_0+1}+\frac{j_1}{m_1+1}+\cdots+\frac{j_n}{m_n+1} \quad \text{for } 1\leqq j_i \leqq m_i, \quad i=0, \cdots, n.$$

Then the distribution of exponents of f is given by the following n-dimensional integral representation:

(2.2.1) $$\int_{x_0+\cdots+x_n=s}\prod_{i=0}^{n}\left(\frac{1}{m_i}\sum_{j=1}^{m_i}\delta\left(x_i-\frac{j}{m_i+1}\right)dx_i\right)$$

(2.3) Motivated by this formula (2.2.1) let us introduce a one dimentional continuous distribution $N_{n+1}(s)ds$ by integrating

(2.3.1) $$N_{n+1}(s)ds=\int_{x_0+\cdots+x_n=s}\varphi(x_0)\varphi(x_1)\cdots\varphi(x_n)dx_0\cdots dx_n$$

where

$$\varphi(x)=\begin{cases}0 & \text{if } x\notin[0,1]\\ 1 & \text{if } x\in[0,1]\end{cases}.$$

It is obvious by definition that $N_{n+1}(s)$ is an $n+1$-st convolution $\varphi*\cdots*\varphi(s)$ of φ so that it is an $n-1$-times smoothly differentiable function satisfying the recursion relation.

i) $N_0(s)=\varphi(s)$

ii) $(d/ds)N_{n+1}(s)=N_n(s)-N_n(s-1)$

(2.4) In the next paragraph in (3.7) 2) and (3.9) 2), we show the following,

Assertion. i) *The distribution of exponents for a quasi-homogeneous function of degree* 1 *w.r.t. the weights* $(r_0, \cdots, r_n)$ (*which depends only on the weights*), *converges to the distribution* $N_{n+1}(s)ds$ *as* $r_0, \cdots, r_n$ *tend to zero.*

ii) *The distribution of exponents for $f(x, y)$ which defines a plane curve with two Puiseau characteristic pairs (n_1, l_1), (n_2, l_2), (which depends only on n_1, l_1, n_2, l_2), converges to $N_2(s)ds$ as n_2 tends to ∞.*

Here "converge" means that the Fourier transformation of the distribution converges uniformly on compact sets.

(2.5) One would like to generalize the above assertion in two directions.

i) Instead of quasi-homogeneous function or two variable function, the assertion must be valid for any function f in a suitable formulation.

ii) $N_{n+1}(s)ds$ is not only a limit of the distribution of exponents, but it must be a "bound" of the distributions of exponents for all f in a suitable sense.

Below we state the question more explicitly

(2.6) i) In [7] (6.1), the following inequality was conjectured

$$\text{(2.6.1)} \qquad \frac{1}{\mu}\int_0^r \sum_{i=1}^{\mu} \delta(s-\alpha_i)ds < \int_0^r N_{n+1}(s)ds \qquad \text{for } 0<r<\frac{n+1}{2}.$$

ii) The conjecture is false by the following simple example $f=z^3$ where $\mu=2$, $\alpha_i=i/3$, $i=1, 2$, and

$$\frac{1}{2}\int_0^r \left(\delta\left(s-\frac{1}{3}\right)+\delta\left(s-\frac{2}{3}\right)\right)ds=\frac{1}{2}\not<\int_0^r ds=r \qquad \text{for } \frac{1}{3}\leqq r\leqq\frac{1}{2}.$$

iii) Even though the inequality (2.6.1) does not hold for all r, $0<r<(n+1)/2$, the inequality holds for a specific value of r as we see in the following example.

(2.7) **Example.** *Let $n=1$ so that f defines a local plane curve with an isolated singular point. Then for $r=1/2$, the inequality* (2.6.1) *holds. i.e.*

$$\#\left\{\alpha\text{: exponent of } f \text{ s.t } \alpha\leqq\frac{1}{2}\right\}<\mu\int_0^{1/2} N_1(s)ds=\frac{1}{8}\mu.$$

Proof. In [14], M. Tomari has shown the following theorem.

Theorem. *Let $f(x, y, z)$ be a holomorphic function in 3-variables, which defines a normal singular point of a surface with multiplicity 2. Then the geometric genus p_g of the singular point is less than $\mu/8$, where μ is the Milnor number of f.*

Let us apply this assertion to $f=z^2+g(x, y)$. In view of (1.6.3) and (1.9) theorem, the geometric genus of $z^2+g(x, y)$ is equal to the number of exponents of g less than or equal to 1/2 and the Milnor number of z^2+g is equal to that of g.

(2.8) i) Let us call r with $0<r<(n+1)/2$ dominating, if

$$\mu^{-1}\#\{\alpha: \alpha \text{ is an exponent of } f \text{ s.t. } \alpha\leqq r\}<\int_0^r N_{n+1}(s)ds \tag{2.8.1}$$

for all functions f of $n+1$ variables.

Let us call r weakly dominating, if

$$\mu^{-1}\#\{\alpha: \alpha \text{ is an exponent of } f \text{ s.t. } \alpha<r\}\leqq\int_0^r N_{n+1}(s)ds \tag{2.8.2}$$

for all functions f of $n+1$ variables.

ii) **Problem.** Determine the set of all dominating values and weakly dominating values respectively for each n.

For $n=0$, the set of dominating values is void and the set of weakly dominating values is $=\{1/m: m=2, 3, 4\cdots\}$.

iii) Is 1/2 a dominating value for all $n\geqq 1$? i.e. Is the $\#$ of exponents of f with $\alpha\leqq 1/2$ less than $\mu/(n+1)!\,2^{n+1}$ for any function of $n+1$ variables $n\geqq 1$?

iv) Is 1 a dominating value for all $n\geqq 2$? i.e. Is the $\#$ of exponents of f with $\alpha<1$ less than $\mu/(n+1)!$ for any function f of $n+1$ variables, $n\geqq 2$?

In particular for $n=2$, the conjecture is "The geometric genus of a surface singular point of $f^{-1}(0)<\mu/6$." This was conjectured also by A. Durfee in [2] and partially solved by Randeli.

(2.9) To compute the values of $\int_0^r N_n(s)ds$ for intergers $r\in \boldsymbol{N}$, it is convenient to use the following combinatorial method similar to the Pascal's triangle.

Define integers $N(n, r)$ indexed by two integers $n\in \boldsymbol{N}$ and $0\leqq r\leqq n$,

$$N(n, r) := n!\int_{r-1}^r N_n(s)ds. \tag{2.9.1}$$

By putting formally,

$$N(n, 0)=N(n, n+1)=0 \quad \text{and} \quad N(1, 1)=1 \tag{2.9.2}$$

one obtains the following recursion formula.

$$N(n+1, r+1)=(n-r+1)N(n, r)+(r+1)N(n, r+1). \tag{2.9.3}$$

The recursion form gives an algorithm to compute $N(n, r)$ by induction on n, like the Pascal's triangle.

$n=1$						1						
$n=2$					1		1					
$n=3$				1		4		1				
$n=4$			1		11		11		1			
$n=5$		1		26		66		26		1		
$n=6$	1		57		302		302		57		1	

In a private letter to the author H. Hitotumatu has computed independently both the recursion formula (2.9.3) and the following explicit formula

$$N(n, r)=\sum_{k=0}^{r-1}(-1)^k\binom{n+1}{k}(r-k)^n. \tag{2.9.4}$$

(2.10) *Note.* Inspired by the characteristic function introduced in the next paragraph § 3, let us introduce a sequence of polynomials,

$$L_n(T)=\sum_{r=1}^{n} N(n, r)T^{r-1} \qquad n=1, 2, \cdots. \tag{2.10.1}$$

Formally putting $L_0=1$, from (2.9.3) we get a recursion formula

$$L_{n+1}(T)=(1+nT)L_n(T)+T(1-T)\frac{\partial}{\partial T}L_n(T). \tag{2.10.2}$$

From this recursion formula (2.10.2) one proves the following assertions

Assertion 1). i) *The polynomials $L_n(T)$ have only simple real roots lying in* $(-\infty, 0)$.

ii) *In each connected component of* $(-\infty, 0)-\{\text{roots of } L_n(T)\}$, *there is exactly one root of* $L_{n+1}(T)$.

Assertion 2). *Put*

$$H(T, t)=\frac{(T-1)}{Te^{(1-T)t}-1}e^{(1-T)t} \tag{2.10.3}$$

which satisfies the following differential equation

$$H=(1-tT)\frac{\partial H}{\partial t}-T(1-T)\frac{\partial H}{\partial T}. \tag{2.10.4}$$

Then using the recursion (2.10.2), one computes

$$(2.10.5)\qquad \left(\frac{\partial}{\partial t}\right)^n H(T,t)=\left(\frac{T-1}{Te^{(1-T)t}-1}\right)^{n+1} e^{(1-T)t}L_n(Te^{(1-T)t}).$$

Hence

$$(2.11.6)\qquad \left(\frac{\partial}{\partial t}\right)^n H(T,t)\Big|_{t=0}=L_n(T),\qquad n=0,1,2,\cdots$$

so that $H(T, t)$ is a generating function for the sequence of polynomials $L_n(T)$.

The author is indebted to I. Naruki for the computation of (2.10.3).

Assertion 3). *The $[(n-1)/2]$ roots of $L_n(T)$ which are less than -1, (which may be regarded as real units in a certain algebraic number field) are multiplicatively independent for $n\leqq 8$.*

Hopefully the last assertion might be true for any n. Compare this assertion with (3.8) ii) of § 3.

§ 3. The zeroes of the characteristic function $\chi_f(T)$

In this paragraph we introduce the characteristic function $\chi_f(T)$ as a generating function of the exponents of f, or as the Fourier transform of the distribution of exponents (see (3.1)). Our main interest lies in the zero locus of the function χ_f and we give a report on some experimental results on the zeroes of χ_f calculated by a computer.

It is remarkable that in all the examples, the equation $\chi_f(T)=0$ has rather many roots on the unit circle $|T|=1$. Because of the limited number of examples studied, the author does not know whether this is a general phenomenon for any f or not.

(3.1) **Definition.** The characteristic function χ_f for a germ f of an analytic function with an isolated critical point is,

$$(3.1.1)\qquad \chi_f(T):=\frac{1}{\mu}\sum_{i=1}^{\mu} T^{\alpha_i}$$

where $\alpha_1,\cdots,\alpha_\mu$ are the exponents of f.

Note. Using a new variable τ with $T=\exp(2\pi\sqrt{-1}\,\tau)$, one gets the Fourier transform representation

$$(3.1.2)\qquad \chi_f:=\frac{1}{\mu}\int \exp(2\pi\sqrt{-1}\,t\tau)\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds$$

and hence, using the N of (1.3.2), we obtain,

$$\chi_f := \frac{1}{2\mu\pi\sqrt{-1}} \text{ tr. } \oint \exp(2\pi\sqrt{-1}\, t\tau)(s1_{\Omega_f} - N)^{-1} ds. \tag{3.1.3}$$

(3.2) *The product formula.* Let $f(x)+g(y)$ be the joint of two functions f and g. Then by using (1.6.3), one gets, easily,

$$\chi_{f+g}(T) = \chi_f(T)\chi_g(T). \tag{3.2.1}$$

(3.3) *The zeroes of* χ_f. Let us now study the roots of $\chi_f=0$ in the following steps.

i) Choose integers $d_0, d_1, \cdots, d_\mu$ and represent the exponents $\alpha_i = d_i/d_0$, $i=1, \cdots, \mu$.

ii) Introduce a new variable X such that

$$X = T^{1/d_0} = \exp(2\pi\sqrt{-1}\,\tau/d).$$

Using X, write the characteristic function as $\chi_f = X^{d_0 r(f)} P(X)$, where $P(X) := \sum_{i=1}^{\mu} X^{d_0(d_i - r(f))}$ is a polynomial in X of degree $d = d_0 s(f)$.

Note. The duality of the exponents (1.4) implies the relation $X^d P(X^{-1}) = P(X)$. Furthermore if the conjecture (1.9) is true, $P(X)$ is a monic polynomial with the constant term 1. Hence the roots of $P(X)=0$ are units in some algebraic number field.

iii) Let $\beta_1, \cdots, \beta_d$ be the roots of $P(X)=0$. Then we get a "virtual decomposition"

$$\chi_f = T^{r(f)} \prod_{j=1}^{s(f)d_0} (T^{1/d_0} - \beta_j). \tag{3.3.1}$$

Even though the decomposition (3.3.1) depends on the choice of d_0, all such decompositions are related in an obvious manner.

In particular, the set $\{\beta_j^{d_0} : j=1, \cdots, d=s(f)d_0\}$ does not depend on the choice of d_0. Let us call this set the zero locus of χ_f. All the elements β of the zero locus are counted with multiplicities $= \#\{j \in \{1, \cdots, s(f)d_0\} : \beta = \beta_j^{d_0}\}/d_0$. It is obvious by definition that the sum of the multiplicities of the elements of the zero locus of χ_f is equal to $s(f)$.

(3.4) *The numerical invariant* $s(f, 0)$. As we shall see in the examples below, the zero locus of χ_f has rather many elements on the unit circle $|T| = 1$. Motivated by this fact let us introduce the following invariants;

$$\begin{aligned} s(f, r) :&= \textstyle\sum \text{ multiplicity of } \beta \in \text{ zero locus of } \chi_f \\ &\qquad \text{such that } |\log|\beta|| \leqq r \\ &= d_0^{-1} \#\{\beta : \text{ root of } P(X)=0 \text{ s.t. } |\log|\beta|| \leqq d_0^{-1} r\} \\ &\qquad\qquad \text{for } 0 \leqq r \leqq \infty. \end{aligned} \tag{3.4.1}$$

By definition $s(f, r)$ is an increasing function of r such that $s(f, \infty)=s(f)$. By definition, $s(f, 0)/s(f)$ is equal to the $\sharp$ of roots of $P(X)$ on the unit circle $|X|=1$ divided by $\deg P(X)$.

(3.5) *An algorithm to compute* $s(f, 0)$. We shall compute $s(f, 0)$ in the following manner. The computer programming is due to T. Mitsui.

i) Take the polynomial $P(X)$ of (3.3) ii).

ii) Write $P(X)$ in the form $(X+1)^l Q(X)$, where $Q(X)$ and $X+1$ are co-prime. Notice that the degree $d-l$ of the polynomial $Q(X)$ is even.

iii) Introduce a new polynomial $R(Z)$ with integer coefficients by $R(X+X^{-1})=X^{-(d-l)/2}Q(X)$. Notice that $R(Z)=0$ doesn't have a root equal to 2 or -2.

iv) Let m be the number of real roots of $R(Z)=0$ which lie in the interval $(-2, 2)$, counted with multiplicities.

The Sturm algorithm has been employed to calculate the number m. From the computational point of view, the algorithm is so algebraic that any software for symbolic and algebraic manipulation is suitable. We have carried out the computations by REDUCE 2 on DEC System 2020.

v) Since the roots of $R(z)$ lying on the interval $(-2, 2)$ correspond to the roots of $Q(X)=0$ lying on the unit circle by 1 : 2, we get the formula,

$$s(f, 0)=d_0^{-1}(l+2m) \tag{3.5.2}$$

(3.6) *Graphical search of the roots of* $P(X)=0$. In [4] T. Mitsui has developed a graphical method for the search of zeroes of a polynomial $P(X)$ in the complex X-plane by the use of graphical devices. In the following examples (3.7)—(3.11) we use this method to study the zeroes of the $P(X)$ of (3.3) ii).

In the figures 1–8, the following data are displayed.

i) the figures are drawn in the half plane $\operatorname{Im} X \geqq 0$.

ii) the small half-circle in a figure expresses the unit half circle $|X|=1$, $\operatorname{Im} X \geqq 0$.

iii) the big half circle in a figure expresses the half circle of radius $|X|=2$.

iv) the zero locus of $\operatorname{Re} P(X)$ is displayed by the solid line.

v) the zero locus of $\operatorname{Im} P(X)$ is displayed by the dotted line.

vi) Since the polynomial $P(X)$ has real coefficients, the figure can be extended to the lower half plane $\operatorname{Im} X \leqq 0$ by the reflexion along the real axis which is shown by the dotted straight line in the figures.

Thus the zeroes of $P(X)$ are at the intersections of the solid lines with the dotted lines. If they cross on the small circle, the intersection shows a root of $P(X)=0$ on the unit circle $|X|=1$.

vii) In the explanation of a figure, $\sharp$ means the number of the roots of the polynomial $P(X)$ which lie on $|X|=1$.

(3.7) **Example 1** (Quasi-homogeneous singularity).

1) *the characteristic function.* Let f be a quasi-homogeneous function of degree 1 with the weights $(r_0, \cdots, r_n)$. By computing the Poincaré-polynomial of the graded module Ω_f, we get,

$$\chi_f = \frac{1}{\mu} \frac{\prod_{i=0}^n (T - T^{r_i})}{\prod_{i=0}^n (T^{r_i} - 1)}. \tag{3.7.1}$$

2) *the limit of the characteristic function.* Let us vary the weights $(r_0, \cdots, r_n)$ in the expression (3.7.1) so that $r_i \to 0$ for $i=0, \cdots, n$. Then noting that $\mu = \prod_{i=0}^n (r_i^{-1} - 1)$, one computes easily the limit as,

$$\lim_{r_0, \cdots, r_n \to 0} \chi_f = \left(\exp(\sqrt{-1}\,\pi\tau) \frac{\sin(\pi\tau)}{\pi\tau} \right)^{n+1}. \tag{3.7.2}$$

On the other hand, note that the right hand side of (3.7.2) is nothing but the Fourier transform $\int \exp(2\pi\sqrt{-1}\,\tau s) N_{n+1}(s) ds$ of the distribution in (2.3).

Thus in the probabilitistic sense, the distribution of the exponents converges to the distribution;

$$\lim_{r_0, \cdots, r_n \to 0} \left(\sum_{i=1}^{\mu} \delta(s - \alpha_i) ds \right) = N_{n+1}(s) ds. \tag{3.7.3}$$

3) *The zero locus of χ_f and $s(f, 0)$.* It is obvious from the expression (3.7.1), that by a choice of $X = T^{1/d_0}$, where d_0 is a common denominator of $r_0, \cdots, r_n$, $\chi_f = T^{r(f)} P(X)$, where *$P(X)$ is a cyclotomic polynomial in X.*

In particular, all the roots of $\chi_f = 0$ are roots of 1 and therefore $s(f, 0) = s(f)$.

One conjectures that the zeroes of χ_f are roots of 1 if and only if f is a semi quasi-homogeneous singularity.

4) **Examples.** For simple germs in three variables, one computes,

$$\chi_{A_m} = T^{1+1/(m+1)} \prod_{j=1}^{m-1} \left(T^{1/(m+1)} - \exp\left(\frac{2\pi\sqrt{-1}}{m} j \right) \right) \Big/ m$$

$$\chi_{D_m} = T^{1+1/2(m-1)} \prod_{j=1}^{m-1} \left(T^{1/2(m-1)} - \exp\left(\frac{2\pi\sqrt{-1}\, j}{m} \right) \right)$$
$$\times \prod_{j=1}^{m-2} \left(T^{1/2(m-1)} - \exp \frac{2\pi\sqrt{-1}\, j}{2(m-2)} \right) \left(T^{1/2(m-1)} + 1 \right)^{-1} \Big/ m$$

$$\chi_{E_6}=T^{1+1/30}(T^{1/3}+1)\left(T^{1/4}-\exp\left(\frac{2\pi\sqrt{-1}}{3}\right)\right)$$
$$\times\left(T^{1/4}-\exp\left(\frac{4\pi\sqrt{-1}}{3}\right)\right)\Big/6$$
$$\chi_{E_7}=T^{1+1/18}\prod_{j=1}^{6}\left(T^{1/9}-\exp\left(\frac{2\pi\sqrt{-1}j}{7}\right)\right)$$
$$\times\left(T^{1/9}+\exp\left(\frac{2\pi\sqrt{-1}}{3}\right)\right)\left(T^{1/9}+\exp\left(\frac{4\pi\sqrt{-1}}{3}\right)\right)\Big/7$$
$$\chi_{E_8}=T^{1+1/30}(T^{1/3}+1)(T^{1/5}-\sqrt{-1})(T^{1/5}+\sqrt{-1})(T^{1/5}+1)/8$$

Note. As we see from the above examples (and from the formula (3.7.1)), the zero locus of χ_f is related to the μ-th power roots of 1. Is there any good explanation of that?

(3.8) **Example 2** (Cusp singularities $T_{p,q,r}$).

1) *equation* $f=x^p+y^q+z^r+xyz$ for $1/p+1/q+1/r<1$.
exponents 1, $1+j/p$ for $j=1,\cdots,p-1$, $1+j/q$ for $j=1,\cdots,q-1$, $1+j/r$ for $j=1,\cdots,r-1$, 2. $\mu=p+q+r-1$.

(3.8.1) $$\chi_{T_{p,q,r}}=T\left(1+\frac{T-T^{1/p}}{T^{1/p}-1}+\frac{T-T^{1/q}}{T^{1/q}-1}+\frac{T-T^{1/r}}{T^{1/r}-1}+T\right)\Big/\mu$$
$$r(T_{p,q,r})=1,\qquad s(T_{p,q,r})=1.$$

The program (3.5) gives the following results.

$s(T_{2,3,7},0)=11/21$	(see Figure 1)
$s(T_{2,3,8},0)=12/24$,	$s(T_{2,3,9})=15/27$
$s(T_{2,3,10},0)=18/30$,	$s(T_{2,3,11})=38/66$
$s(T_{2,4,5},0)=8/20$	(see Figure 2)
$s(T_{3,3,4},0)=4/12$	(see Figure 3)
$s(T_{p,p,p},0)=(p-2)/p$	(see Figure 4)

2) Let us investigate more closely the last case above.

Assertion. i) *The characteristic function of* T_{ppp} *is decomposed in the following form*

(3.8.2) $$\chi_{T_{ppp}}(T)=T(T^{1/p}+\alpha_p)(T^{1/p}+1/\alpha_p)\prod_{i=1}^{[p/2]-1}(T^{1/p}-\varepsilon_i)(T^{1/p}-\bar{\varepsilon}_i)$$

for even p

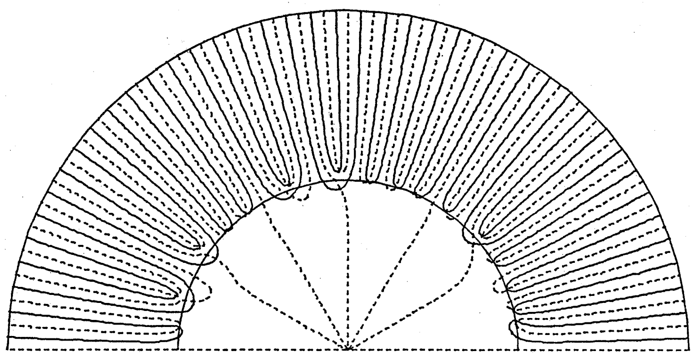

Figure 1*. example 2. 1)
$f=T_{7,3,2}$ $d_0=42$, $\deg P=42$, $\sharp=22$

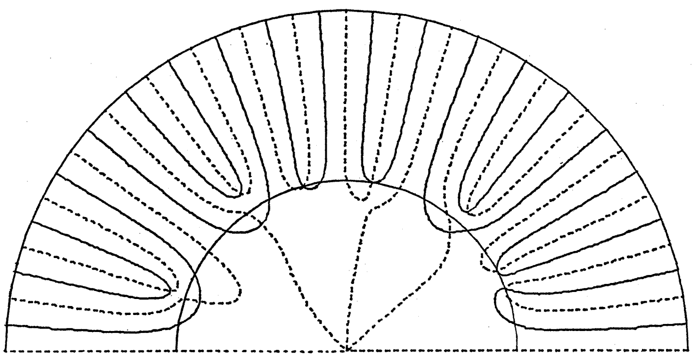

Figure 2. example 2. 1)
$f=T_{5,4,2}$ $d_0=20$, $\deg P=20$, $\sharp=8$

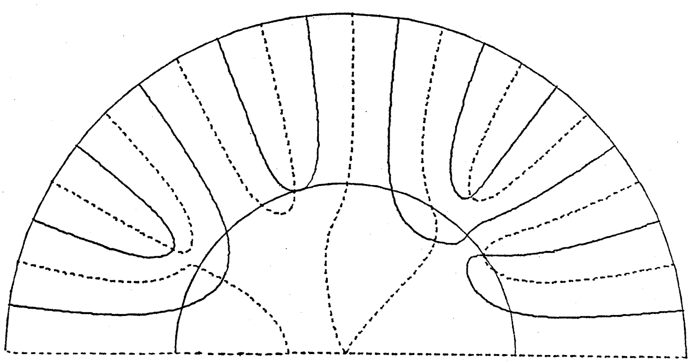

Figure 3. example 2. 1)
$f=T_{4,3,3}$ $d_0=12$, $\deg P=12$, $\sharp=4$

* The explanation of the figures is found in (3.6).

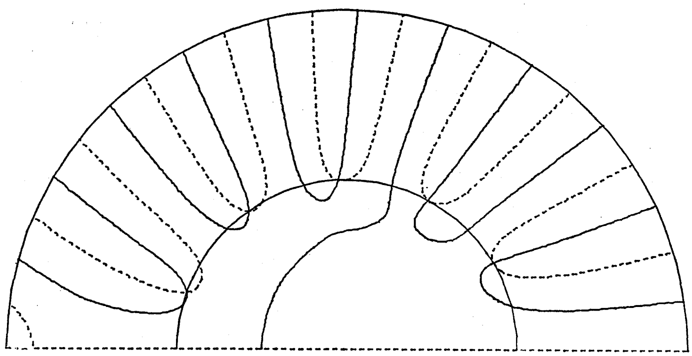

Figure 4. example 2. 2)
$f=T_{12,12,12}$ $d_0=12$, $\deg P=12$, $\sharp=10$

$$\chi_{T_{ppp}}(T)=T(T+1)(T^{1/p}+\alpha_p)(T^{1/p}+1/\alpha_p)\prod_{i=1}^{[p/2]-1}(T^{1/p}-\varepsilon_i)$$
$$\times(T^{1/p}-\bar{\varepsilon}_i) \qquad \textit{for odd } p$$

where ε_i *are complex numbers with* $|\varepsilon_i|=1$ *for* $i=1,\cdots,[p/2]-1$ *and* α_p *is a real algebraic number with*

(3.8.3) $$0<|\alpha_p+\tfrac{1}{2}|<3/2^p.$$

Assertion. ii) *In the above expression, numbers* $\alpha_p, \varepsilon_1, \cdots, \varepsilon_{[p/2]-1}$ *are multiplicatively independent in* $\boldsymbol{C}^*/\pm 1$.

Note iii) As a consequence of (3.8.3) we get

(3.8.4) $$\inf\{r: s(f,r)=s(f)\}=p\log\alpha_p\approx p\log 2\longrightarrow\infty \quad (\text{as } p\longrightarrow\infty).$$

Note. iv) Note that the fact ii) is in contrast with the case when f is quasi-homogeneous of example 1, where all the roots of $\chi_f=0$ are roots of 1 so that they are torsions in $\boldsymbol{C}^*$.

A sketch of the proof of i) *and* ii)

Choose p as d_0 in (3.3) i) and set $\chi_{T_{ppp}}=X^pP_p(X)$ for $T^{1/p}=X$ and $P_p(X)=1+X^p+3\sum_{i=1}^{p-1}X^i$.

First one checks easily the following recursion formula.

(3.8.5) $$P_{p+2}(X)=(X^2+1)P_p(X)-X^2P_{p-2}(X).$$

Put

$$Q_q(Z)=X^{-q}P_{2q}(X) \qquad q=2,3,4,\cdots$$

$$R_q(Z)=(X+1)^{-1}X^{-q}P_{2q+1}(X) \qquad q=2, 3, 4, \cdots$$
$$\text{where} \quad Z=X+X^{-1}.$$

Then from (3.8.5) one obtains recursion formulae,

$$Q_{q+1}(Z)=ZQ_q(Z)-Q_{q-1}(Z)$$
$$R_{q+1}(Z)=zR_q(Z)-R_{q-1}(Z).$$

Using the above recursion formulae one proves the following assertion by induction on q.

Assertion. v) *The equation* $Q_q(Z)=0$ (resp. $R_q(Z)=0$) *has q real simple roots.* $q-1$ *of them lie in the interval* $(-2, 2)$ *and one root lies in the interval* $(-\infty, 2)$. *Furthermore there is one root of* $Q_{q+1}(Z)=0$ (resp. $R_{q+1}(Z)=0$) *in each of the connected components of* $(-\infty, 2)-\{$roots of $Q_q(Z)=0\}$ (resp. $(-\infty, 2)-\{$roots of $R_q(Z)=0\}$).

From the *assertion* v) above, the *assertion* i) follows immediately. The approximation of the root α_p is done by a direct calculation.

Now let us prove the next assertion.

Assertion. vi) *The polynomials* $P_{2q}(X)$, $(X+1)^{-1}P_{2q+1}(X)$ *are irreducible over* $\boldsymbol{Z}$.

Proof. Let us show that if $P_p(X)$ is reducible, $P_p(X)$ should be divisible by a cyclotomic polynomial.

Remember that except two negative real roots $-\alpha_p$ and $-\alpha_p^{-1}$, all the roots of $P_p(X)$ have absolute values equal to 1. Let $Q(X)$ be a factor of $P_p(X)$ over $\boldsymbol{Z}$. If $X+\alpha_p$ divides $Q(X)$, then $X+\alpha_p^{-1}$ divides $Q(X)$, otherwise $Q(0)$ is not equal to ± 1. Hence, either $Q(X)$ or $P_p(X)/Q(X)$ has only roots of absolute value 1, so one of the two is a cyclotomic polynomial. Thus if $P_p(X)$ is reducible, it should have a root of the form $\exp(2\pi\sqrt{-1}/m)$ for some integer $m\geqq 0$.

Using the expression $P_p(X)=(X-1)^{-1}(X^{p+1}-1+2X(X^{p-1}-1))$ one has the representation.

$$(3.8.6) \qquad P_p(e^{i\theta})=\frac{e^{i(p/2)\theta}}{\sin\theta/2}\left(\sin\left(\frac{p+1}{2}\theta\right)+2\sin\left(\frac{p-1}{2}\theta\right)\right).$$

Let m be any integer $\geqq 2$. Divide p by m so that $p=rm+t$ for some t with $0\leqq t<m$. Then

$$P_p(e^{2\pi\sqrt{-1}/m})=\frac{e^{p\pi\sqrt{-1}/m}}{\sin\pi/m}(-1)^r\left(\sin\left(\frac{t+1}{m}\pi\right)+2\sin\left(\frac{t-1}{m}\pi\right)\right).$$

Because of the condition $0\leqq t<m$, one can see that the right hand cannot

be zero except for the case $m=2$, $t=1$. This proves *assertion* vi).

Let us now show the *assertion* ii). Suppose that there exist integers $m, m_1, \cdots, m_{[p/2]-1}$ such that

$$\alpha^m \prod_{j=1}^{[p/2]-1} \varepsilon_j^{m_j} = \pm 1. \tag{3.8.7}$$

Then taking the absolute value on both sides, one gets $\alpha^m=1$. Thus $m=0$. Now because of the assertion vi) α and ε_j's are conjugate by the Galois group action. For instance let σ be an element of the Galois group s.t. $\sigma(\varepsilon_1)=\alpha$. The element σ induces a permutation of the roots of $P_p(X)$. One checks easily that $\sigma(\varepsilon_2), \cdots, \sigma(\varepsilon_{[p/2]-1})$ are neither equal to α nor to α^{-1}. Then applying σ to the relation (3.8.7), one gets a new relation,

$$\alpha^{m_1} \prod_j \varepsilon_j^{n_j} = \pm 1$$

for suitable n_j. Then again by the same argument above $m_1=0$.

Repeating this argument, one proves $m_j=0$ for $j=1, \cdots, [p/2]-1$. Together, these prove the *assertion* ii).

(3.9) **Example 3** (irreducible plane curves).

In [11] M. Saito determined the characteristic function for an irreducible plane curve as follows.

1) Let a germ of an irreducible plane curve is given by,

$$y=c_1x^{l_1/n_1}+\cdots+c_2x^{l_1/n_1+l_2/n_1n_2}+\cdots+c_gx^{l_1/n_1+\cdots+l_g/n_1\cdots n_g}+\cdots$$

such that $c_i\neq 0$, $(l_i, n_i)=1$, $i=1, \cdots, g$, $l_1>n_1$.

Put by induction on g, $w_1=l_1$, $w_g=w_{g-1}n_gw_{g-1}+l_g$ and $\Phi_1(l_1, n_1; T)=(T^{1/l_1}-T)/(1-T^{1/l_1})(T^{1/n_1}-T)/(1-T^{1/n_1})$

$$\begin{aligned}
&\Phi_g(l_1, n_1, \cdots, l_g, n_g; T)\\
&\quad=(1-T)/(1-T^{1/n_g})\cdot\Phi_{g-1}^{<1}(l_1, n_1, \cdots l_{g-1}, n_{g-1}; T^{1/n_g})\\
&\quad\quad+T(1-T)/(1-T^{1/n_g})\cdot\Phi_{g-1}^{>1}(l_1, n_1, \cdots, l_{g-1}, n_{g-1}; T^{1/n_g})\\
&\quad\quad-1(T^{1/w_g}-T)/(1-T^{1/w_g})\cdot(T^{1/n_g}-T)/(1-T^{1/n_g})
\end{aligned}$$

where

$$\Phi^{>1}(T):=\sum_{i/N>1} a_iT^{i/T}, \qquad \Phi^{<1}(T):=\sum_{i/N<1} a_iT^{i/N}$$

for

$$\Phi(T)=\sum a_iT^{i/T}\in \boldsymbol{C}[T^{1/N}].$$

Then the characteristic function is given by

$$\chi_{l_1,n_1,\cdots,l_g,n_g}(T)=\Phi_g(l_1, n_1, \cdots, l_g, n_g; T)/\mu_g$$

where

$$\mu_g=(n_g-1)(w_g-1)+n_g\mu_{g-1}, \qquad \mu_0=0.$$

2) One sees directly,

$$\lim_{n_g\to\infty}\chi_{e_1,n_1,\cdots,e_g,n_g}(T)=\left(\frac{T-1}{\log T}\right)^2.$$

3) Let us examine the zero locus for a simple case.

$$\chi_{3,2,3,2}=T^{5/12}(1+T^{1/2}+T^{2/3}+T^{7/6})$$
$$+T^{17/30}(1+T^{1/15}+T^{2/15}+\cdots+T^{13/15})$$
$$r=5/12 \qquad s=14/12$$

Let us choose $d_0=60$ and put $\chi_{3,2,3,2}=X^{25}P(X)$ such that $\deg P=70$.

The program (3.7) shows that $P(X)$ has 48 roots of $|X|=1$, and therefore

$$s(f_{3,2,3,2}, 0)=48/60 \qquad \text{(See figure 5).}$$

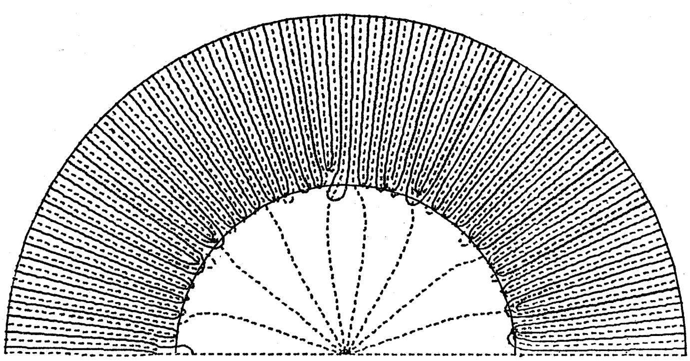

Figure 5. example 3.
f: Puiseau pairs (3, 2) (3, 2) d_0=60, deg P=70, #=48

(3.10) **Example 4** (An example by B. Malgrange [3], see also [13]). Let $f=x^8+y^8+z^8+x^2y^2z^2$. Then $\mu=215$ and

$$\chi_f(T)=T^{1/2}(1+3T^{1/8}+6T^{1/4}+9T^{3/8}+13T^{1/2}+18T^{5/8}+21T^{3/4}+24T^{7/8}$$
$$+25T+24T^{9/8}+21T^{5/4}+18T^{11/8}+13T^{3/2}+9T^{13/8}$$
$$+3T^{15/8}+T^2)/215.$$

$$r(f)=1/2, \qquad s(f)=2.$$

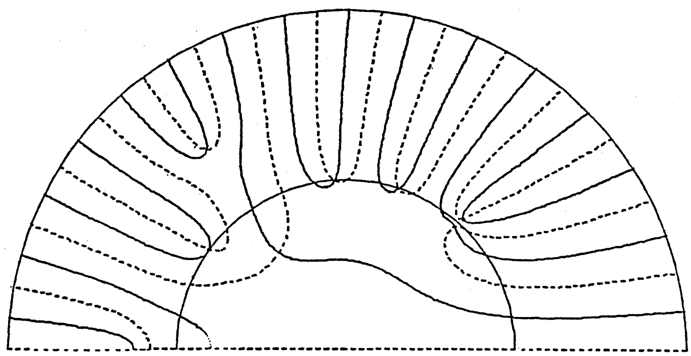

Figure 6. example 4.
$f=x^8+y^8+z^8+x^2y^2z^2$ $d_0=8$, $\deg P=16$, $\sharp=6$

Choose $d_0=8$ and put $\chi_f=X^4P(X)$ s.t $\deg P=16$.

The program of (3.5) shows that $P(X)$ has 6 roots on $|X|=1$ and therefore one gets

$$s(f, 0)=3/4. \qquad \text{(See Figure 6)}$$

(3.11) **Example 5** (Plane curves with two Newton-boundaries).

1) Let $f=x^p+y^q+x^2y^3$ with $2/p+3/q<1$. Then $\mu=2p+q+1$ and

$$\chi_f(T)=\left(T^{(q+1)/2q}\frac{T-1}{T^{1/q}-1}+T^{(p+1)/3p}(1+T^{(p+1)/3p})\frac{T-1}{T^{1/p}-1}+T\right)\Big/\mu$$

and

$$r(f)=\frac{p+1}{3p}, \qquad s(f)=\frac{4p-2}{3p}.$$

Let us examine the simplest case when $p=q=11$. Then

$$\chi_f(T)=T^{4/11}(1+T^{1/11}+2T^{2/11}+2T^{3/11}+3T^{4/11}+3T^{5/11}+3T^{6/11}$$
$$+3T^{8/11}+3T^{9/11}+3T^{10/11}+2T+2T^{12/11}+T^{13/11}+T^{14/11})$$

and

$$s(f)=14/11, \qquad r(f)=4/11.$$

Choose $d_0=11$ and set $\chi_f=X^4P(X)$ for a polynomial $P(X)$ of degree 14.

The program (3.5) shows that $P(X)=0$ has 10 roots on $|X|=1$ and therefore

$$s(f, 0)=10/11. \qquad \text{(see Figure 7)}$$

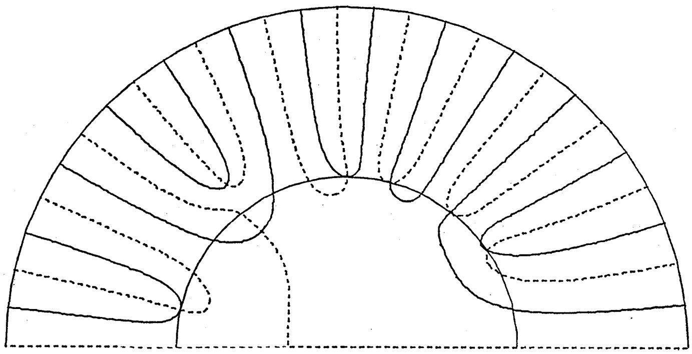

Figure 7. example 5. 1)
$f=x^{11}+y^{11}+x^2y^3$ $d_0=11$, $\deg P=14$, $\sharp=10$

2) Let $f=x^7+y^7+x^3y^3$. Then $\mu=29$ and

$$\chi_f(T)=\left(\frac{T^2-T^{1/3}}{T^{1/3}-1}+2\frac{T-T^{1/7}}{T^{1/7}-1}\cdot\frac{T-T^{1/3}}{T^{1/3}-1}\right)\Big/\mu$$

$$r(f)=1/3, \qquad s(f)=28/21.$$

Choose $d_0=21$ and put $\chi_f=T^{1/3}P(X)$ for a polynomial $P(X)$ of degree 28.

The program (3.5) shows that $P(X)=0$ has 16 roots on $|X|=1$, and therefore one gets

$$s(f, 0)=16/21. \quad \text{(See Figure 8)}$$

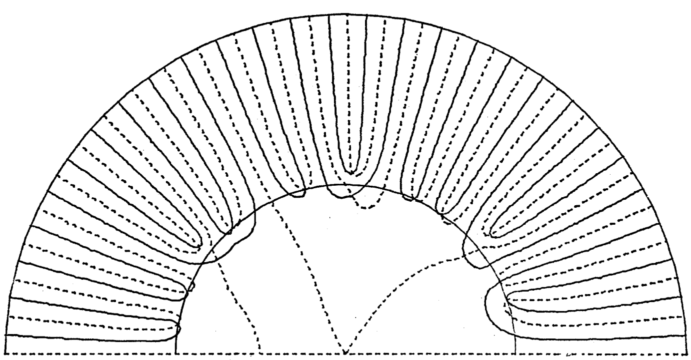

Figure 8. example 5. 2)
$f=x^7+y^7+x^3y^3$ $d_0=21$, $\deg P=28$, $\sharp=16$

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