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Minimal Models of Canonical 3-Folds

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§$-\infty$. Abstract

This paper introduces a temporary definition of *minimal models* of 3-folds (0.7), and studies these under extra hypotheses. The main result is Theorem (0.6), in which I characterise the singularities which necessarily appear on a minimal model, and prove the existence of a minimal model S of a 3-fold of f.g. general type, by blowing up the canonical model X studied in [C3–f], imitating closely the minimal resolution of Du Val surface singularities.

Apart from techniques familiar from [C3–f] (computations of the valuations of differentials; cyclic covers; crepant blow-ups of index 1 points which are not cDV), the main new element (Theorem (2.6)) is a method of blowing up the 1-dimensional singular locus, based on the Brieskorn-Tyurina result on the existence of simultaneous resolutions of a family of Du Val surface singularities, together with the elementary transformations in (-2)-curves of Burns and Rapoport. Part II is devoted to an exposition of these elementary transformations; much of this is folklore material, but it seems worthwhile to give a detailed account of what seems to be a key phenomenon of higher-dimensional birational geometry.

The canonical and terminal singularities introduced in [C3–f] and here have strong inductive properties, and there is some reason for believing that terminal singularities will provide the natural category for an inductive extension of Mori's results: elementary contractions (when these exist) specified by extremal faces of the $K<0$ part of the Mori cone are always discrepant. I have included in § 4 conjectures as to what the theory of minimal models and classification of 3-folds will look like in 3 or 4 years' time, and a section of conjectures in § 8 attempting to pin down the non-uniqueness of minimal models in the non-ruled case.

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Part I. Minimal Models

§ 0. Definitions and statements of the main results

(0.1) **Introduction.** Given a surface of general type, one can usefully consider 3 different models:

$$V \xrightarrow{g} S \xrightarrow{f} X.$$

Here V is an arbitrary non-singular model, S the minimal model, and X the canonical model; in this set-up:

(a) X has at worst Du Val singularities, and ω_X is ample; f is the minimal resolution of the singularities, and is *crepant*, that is $\omega_S = f^*\omega_X$, or $K_S = f^*K_X$.

(b) S is non-singular and K_S is *nef* (this word is defined and discussed in (0.12)).

(c) g is a composite of blow-ups of points, and so by the adjunction formula for a blow-up $K_V = g^*K_S + \Delta$, where Δ is a divisor made up of the exceptional curves of g with strictly positive coefficients; that is, g is *totally discrepant*.

In the surface case, it is well-known that S can be constructed from V by successively contracting (-1)-curves (the exceptional curves of the 1st kind; on a surface with $\kappa \geqq 0$ these are just the curves $C \subset V$ with $K_V C < 0$). And X can be constructed from S by contracting all of the finitely many (-2)-curves (the nodal curves, or Du Val curves; on a minimal surface of general type these are precisely the curves $C \subset S$ with $K_S C=0$).

It is not known how to carry out similar contractions in higher dimensions, although see [Mori] for a spectacular start on this problem. The results of [Mori], of [C3–f], § 5, and of the present paper make it quite clear that there is no hope of getting a non-singular model S for which K_S is nef.

I will be interested in starting from the other end, that is in generalising the minimal resolution of Du Val singularities. I will show how to construct a certain model S, which I will define to be a minimal model, by blowing up the canonical model X studied in [C3–f]; of course, this approach involves assuming for the moment that we are working with a 3-fold of f.g. general type. The main purpose of this paper is to determine the singularities which necessarily appear on S, the *quick* singularities.

(0.2) **Definition.** Let X be a variety of dimension $\geqq 2$ with canonical singularities [C3–f], and let $f\colon Y \to X$ be a partial resolution (see (0.12) below for a definition of partial resolution, and for f^* applied to a $\boldsymbol{Q}$-Cartier divisor).

(a) From the definition of canonical singularities,

$$K_Y = f^* K_X + \Delta,$$

where $\Delta \geqq 0$ is a Weil divisor with coefficients in $\boldsymbol{Q}$, called the *discrepancy* of f.

(b) I will say[1] that f is *crepant* if $\Delta=0$, and *totally discrepant* if

(i) f has at least one exceptional prime divisor;

and (ii) every exceptional prime divisor of f appears in Δ with strictly positive coefficient.

(c) $P \in X$ is a *terminal singularity* if it has a totally discrepant resolution $f\colon Y \to X$; it is equivalent to say that every partial resolution is either totally discrepant or an isomorphism in codimension 1 over P.

(0.3) **Remarks.** (a) A point $P \in X$ of a surface is terminal if and only if it is non-singular.

(b) Suppose that X is a canonical variety, that is, X is projective and K_X is ample; then a partial resolution $f\colon Y \to X$ is crepant if and only if $|mK_Y|$ is free for all m sufficiently large and divisible. I will show below (0.15) that f is crepant if and only if Y is Cohen-Macaulay and K_Y is nef, partially generalising [Mori], (3.44).

(c) "Terminal" is a natural strengthening of "canonical": if $P \in X$

[1] Linguistic conservatives may prefer "non-discrepant".

is a normal point such that $s \in \omega_X^{[r]}$ is a local basis, and $f: Y \to X$ is a resolution, then canonical is the condition that for every prime divisor $\Gamma \subset Y$ exceptional for f, and such that $f(\Gamma) \ni P$, we have $v_\Gamma(s) \geqq 0$; terminal is the same condition with a strict inequality.

(0.4) The following is a trivial consequence of the definitions, together with [C3–f], (1.2), (I) and (1.14), but is constantly used in inductive arguments.

Proposition. *Let $P \in X$ be canonical, and let $f: Y \to X$ be a crepant morphism; then Y has canonical singularities over $f^{-1}P$. If X has terminal singularities, so does Y; if in addition* $\dim X = 3$, *then* $\operatorname{Sing} Y \cap f^{-1}P$ *is a finite set.*

(0.5) **Definition.** A 3-fold point $P \in X$ is a *quick* singularity if it is a canonical singularity of index r such that the local r-fold cyclic cover $f: Y \to X$ defined in [C3–f], (1.19) is an isolated cDV point.

Recall [C3–f], (2.1), that a cDV(=compound Du Val) point $P \in X$ is a hypersurface singularity given locally analytically by $f + tg = 0$, where $f = f(x, y, z)$ defines a Du Val singularity, and $g = g(x, y, z, t)$ is arbitrary. I will say that $P \in X$ is a cA_n, cD_n, cE_6, cE_7 or cE_8 point to specify the general section through P.

Here the word "quick" is an acronym of "*qu*otient of an *i*solated *c*DV point, which is a *k*anonical singularity".

The reader interested in having a more concrete description of quick singularities is encouraged to try his or her hand at Problems (4.13–15).

(0.6) The main purpose of this paper is to prove the following result.

Main Theorem. (I) *Let $P \in X$ be a 3-fold point; then $P \in X$ is terminal if and only if it is quick.*

(II) *Let X be a 3-fold with canonical singularities; then there exists a partial resolution $f: S \to X$ such that*

(i) *f is crepant;*

and (ii) *S has quick singularities.*

Furthermore, this f can be chosen as the composite of certain elementary steps (blow-ups), which are intrinsic to X, and is then uniquely determined and projective.

Using cyclic coverings and the computations of valuations of differentials, as in [C3–f], we will see in § 3 that the case of index $r \geqq 1$ is a fairly mechanical consequence of the results for $r = 1$. For $r = 1$, (I) of (0.6) is proved in § 1, and (II) is reduced in § 2 to Theorem (2.12) on elementary transformations in (-2)-curves; (2.12) is finally proved in § 7.

(0.7) **Definition.** A projective 3-fold S will be called a *minimal model with* $\kappa_{\mathrm{num}} \geqq 0$ if S has only quick singularities and K_S is nef. If X is a 3-fold with canonical singularities, and $f: S \to X$ is a partial resolution satisfying (i) and (ii) of (0.6), I will say that S is a *terminal partial resolution* of X, or is a *minimal model over* X; the uniquely determined model $f: S \to X$ referred to in the final clause of (0.6) will be called *my choice*.

(0.8) **Remarks.** (a) If X is a canonical 3-fold, and S is a minimal model over X, then $|mK_S|$ is free for all m which are sufficiently large and divisible; this makes it clear that S is unique up to isomorphism in codimension 1, since the prime divisors appearing on S are just the prime divisors which are not fixed components of $|mK_V|$ for all m on some resolution V of X.

(b) S is not unique, and is certainly not an absolute minimal model in general. See [13] for a counter-example even if $S=X$ and is non-singular; this example is incidentally a counter-example to naive attempts at extending Mori's elementary contractions. The reader will also meet it if he works at Exercise (5.14). See also [43], (3.8).

(c) I believe that there should only be finitely many minimal models S over a given X, and that they can be obtained from one another by "elementary transformations"; the cases in which this is known arise from simultaneous resolutions of cDV points, and the different models are related to the combinatorics of the root systems A_n, D_n, E_6, E_7, E_8 (see (8.2) below, as well as [9], § 3, [34], [35] and [26], § 8).

(d) Terminal singularities may admit further partial resolutions which introduce only exceptional curves; the two small resolutions of the ordinary double point [5] is a famous instance. Shepherd-Barron has shown (see (8.4) below) that this relates to the (algebraic or analytic) local class group, and that "extra-terminal" singularities are the factorial points. However, it is not clear that one should make these extra resolutions, since we loose out on both projectivity and unicity.

(e) An easy example due essentially to Shepherd-Barron ((6.8) below) shows that 3-folds with $0 \leqq \kappa \leqq 2$ can have infinitely many distinct non-singular models for which $|K|$ is free.

(f) It is an open problem to characterise my choice of S among all minimal models, maybe in terms of the numerical properties of the isolated curves $C \subset S$ for which $K_S C = 0$.

(0.9) **Woffle.** In the background to (0.6) is a general principle, which I will not attempt to state formally (see § 8 below), to the effect that there are finite obstructions to the existence and uniqueness of a *non-singular* minimal model over a canonical 3-fold X. If S is such a model, then the isolated curves C with $K_S C=0$ seem somehow to form a system of simple roots (in some vector space birationally associated with X?), and the obstructions seem to relate to this choice.

(0.10) **Remark.** I should say that this paper makes no use of the strong numerical consequences of ampleness, which is the backbone of Mori's New Technology. The difficulties mentioned in (0.8) concerning non-uniqueness are substantially different in the projective category and the category of algebraic spaces, in which I mainly work.

(0.11) **Prejudice.** This paper has many relations with the "factorisation problem" (is every birational map $f: X \dashrightarrow Y$ between smooth complete 3-folds a composite of blow-ups and blow-downs ?); see especially § 5 and § 8. There are now many people working on this problem (see [11], [12], [18], and the references in [26]), and I would like to express here my opinion that the solution to the factorisation problem is more likely to be a consequence of a working understanding of the birational geometry of 3-folds than a particularly useful step towards this understanding. Compare also [43].

(0.12) **Conventions and abuse of language.** (a) I will use the word "variety" to mean an algebraic variety over $k=\boldsymbol{C}$, or an algebraic space over $k=\boldsymbol{C}$ in the sense of M. Artin; there is nothing against identifying this with the notion of Moishezon space (=analytic space birational to an algebraic variety). My interest is primarily in the case of a projective variety, but complex analytic constructions will turn up, in particular as partial resolutions; analytic manifolds is clearly the right context for the material of Part II. I will always write "birational" instead of "bimeromorphic".

(b) A *partial resolution* is a proper birational morphism $f: Y \to X$, in which Y is *always assumed normal.*

(c) If $f: Y \to X$ is a partial resolution, an *exceptional prime divisor* of f is any prime divisor $\Gamma \subset Y$ such that $\operatorname{codim} f(\Gamma) \geqq 2$; even if $P \in X$ is an isolated singularity, this should not be taken to mean that $\Gamma \subset f^{-1}P$.

(d) The phrase "$f: X_1 \dashrightarrow X_2$ is an isomorphism in codimension 1" means that f is a birational map, and deleting subsets of X_1 and X_2 of codimension $\geqq 2$, f becomes an isomorphism. Another way of saying the same thing is that f and f^{-1} are isomorphisms at the generic point of every prime divisor. This usage is exactly as in the well-established usage "non-singular in codimension 1".

(e) The *index* r of a canonical point $P \in X$ is defined in [C3-f], (1.1) to be the smallest r for which the Weil divisor rK_X is principal at P; rather than specifying in every instance the index, it is convenient to work consistently with $\boldsymbol{Q}$-Weil divisors, that is, Weil divisors with coefficients in $\boldsymbol{Q}$; such a divisor D is a *$\boldsymbol{Q}$-Cartier divisor* if for some integer $r>0$, rD is a Cartier divisor. Note that if $f: X \to Y$ is a morphism, and D is a $\boldsymbol{Q}$-Cartier divisor on Y, then $f^*D = (1/r)f^*(rD)$ is a well-defined $\boldsymbol{Q}$-Cartier divisor on X; even if D is an integral Weil divisor, f^*D may not be. For future use, I will say that a point of a normal variety $P \in X$ is *$\boldsymbol{Q}$-Gorenstein* if it is Cohen-Macaulay and K_X is $\boldsymbol{Q}$-Cartier at P.

(f) A $\boldsymbol{Q}$-Cartier divisor D is *nef* if for every irreducible curve $C \subset X$,

$$DC = \frac{1}{r}((rD)C) \geqq 0;$$

this condition is the numerical consequence of the condition that for some $m>0$, the linear system $|mD|$ is effective and free; thus nef = "numerically (effective and free)". It should be noted that, despite the fact that its use is sanctified by Zariski, Kleiman and Mori, the term "numerically effective" is anomalous, since the condition is logically independent of "effective".

(g) In [C3-f], (1.14), it was shown that a 3-fold X has at most a 1-dimensional Du Val locus (where X is analytically (Du Val surface singularity) $\times A^1$), and finitely many *dissident* points, which are not of this type. I will continue to use this terminology, which is satisfying for the following reasons: there are only finitely many dissidents and they represent much of the interest of the situation; and I will have to use barbaric methods (in § 2) to resolve them.

(0.13) **Acknowledgements.** It is a pleasure to thank R. Barlow, V. Danilov, Mohan Kumar and N. Shepherd-Barron for useful conversations, and D. Morrison for a stimulating (if at first somewhat enigmatic) letter. I trust that the reader will credit me with any of their original ideas, and attribute any inaccuracies to their influence. I am indebted to M. Miyanishi, S. Mukai and S. Tsunoda for instruction in Mori theory, and to N. Shepherd-Barron and H. Pinkham for sending me corrections to the preprint version, and for a number of ideas which I have assimilated into Part II. Example (5.15), which contradicts an earlier version of the string of conjectures (8.6–8), is stolen from Pinkham's preprint [26], § 8.

Appendix to § 0. Numerical characterisation of discrepancy

(0.14) **Weak Index Theorem.** *Let $f: Y \to X$ be a proper birational morphism between normal varieties, of which Y is supposed to be Cohen-*

Macaulay. Let Δ be a Cartier divisor made up of exceptional prime divisors. Then there exists a component F of Δ such that

(i) *$C\Delta \leqq 0$ for all curves $C \subset F$ contracted by g, and not lying in a proper closed subvariety of F;*

(ii) *the curves $C \subset F$ contracted by g and such that $C\Delta < 0$ fill up a dense subset of F.*

Roughly speaking, "$C\Delta < 0$ for most contracted curves in F".

Proof. By taking general sections of $f(\Delta)$ it is easy to reduce to the case $f(\Delta) = P \in X$. Let H be a Cartier divisor on X such that $P \in H$. Then

$$\operatorname{Supp} \Delta \subset f^{-1}P \subset f^*H;$$

now if $f^*H = \sum a_F F$, $\Delta = \sum b_F F$, setting $a/b = \min_F\{a_F/b_F\}$, I get

$$bf^*H = a\Delta + M,$$

where $M > 0$ is a Cartier divisor, and does not contain all the components of Δ. There must now exist some component F of Δ not contained in M, and such that $F \cap M$ contains a codimension 1 set of F. (i) and (ii) follow at once.

(0.15) **Corollary.** *Let X be a 3-fold with canonical singularities, and let $f: Y \to X$ be a partial resolution. Then f is crepant if and only if Y is Cohen-Macaulay and K_Y is relatively nef (that is, $K_Y C \geqq 0$ for every C contracted by f).*

Proof. The "only if" is clear by (0.4). To prove the "if", write

$$K_Y = f^*K_X + \Delta;$$

where Δ is $\boldsymbol{Q}$-Cartier. If K_Y is relatively nef, then so is

$$\Delta = K_Y - f^*K_X,$$

so that $\Delta = 0$ by (0.14). Q.E.D.

(0.16) **Problem.** It is probably true, though I have not checked it, that a birational map $f: X \dashrightarrow Y$ between two 3-folds with terminal singularities, and such that K_X and K_Y are both nef, is necessarily an isomorphism in codimension 1; this would be a strengthening of (0.8), (a).

§ 1. Characterisation of isolated cDV points

(1.1) **Theorem.** *Let $P \in X$ be a point of a 3-fold. Then $P \in X$ is an isolated cDV point if and only if $P \in X$ is terminal of index 1.*

(1.2) *Proof of "if"*. This is an easy consequence of [C3–f], (1.14) and (2.11). Let $P \in X$ be an index 1 canonical singularity. If $P \in X$ is not isolated, and $C \subset \operatorname{Sing} X$ is a 1-dimensional component with $P \in C$, then at the general point of C, X is analytically isomorphic to (Du Val singularity)$\times A^1$; the blow-up of C then provides at least one crepant prime divisor lying over C. Similarly, if $P \in X$ is not cDV, then (2.11) gives a crepant blow-up $\sigma : X_1 \to X$; thus if $P \in X$ is terminal, it must be isolated and cDV.

(1.3) Up till now I have considered $P \in X$ just as a 3-fold singularity; from now on I adopt a strategy suggested to me by D. Morrison: choose some element $t \in \mathcal{O}_{X,P}$, and use it to define a morphism of a neighbourhood of P (which I still denote X) to A^1, $t : X \to T \subset A^1$. I now regard X as a family of surface singularities. The advantage of this in a general setting is that after extracting a root of t, we can build a semi-stable resolution of the family, and use Kulikov methods ([17], [21], [24], [29]).

(1.4) This will be particularly effective for cDV points; I now fix some notation which will be used many times in the sequel:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ T' & \longrightarrow & T \\ t' & \longmapsto & t = t'^m. \end{array} \qquad (*)$$

here X is given by $X : (f + tg = 0) \subset A^4$, with $f = f(x, y, z)$ a Du Val polynomial, $t : X \to T \subset A^1$ is the morphism given by t; $m > 0$ is some integer, and X' is the pull-back of X by the cyclic branched cover $T' \to T$. Obviously X' is again a cDV point, given by $f + t'^m g(x, y, z, t'^m)$.

(1.5) **Theorem** (Brieskorn-Tyurina, [4], [8], [9], [25], [31]). *For a suitable integer m, there exists a simultaneous resolution of the family $X' \to T'$, that is a morphism of complex analytic spaces $h : Y \to X'$ such that for each t',*

$$\begin{array}{ccc} Y & \supset & Y_{t'} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h_{t'}} \\ X' & \supset & X'_{t'} \\ \downarrow & & \downarrow \\ T' & \ni & t' \end{array} ,$$

$h_{t'} : Y_{t'} \to X'_{t'}$ is the minimal resolution of the Du Val singularities of $X'_{t'}$.

(1.6) **Proposition.** *h is small and crepant.*

Here a birational morphism between 3-folds is *small* if every fibre has dimension $\leqq 1$.

Proof. This is obvious from the analogous properties of $h_{t'}$.

(1.7) **Proposition.** *Let $P \in X$ be a cDV point, and let $f: V \to X$ be a partial resolution. Then every prime divisor $\Gamma \subset f^{-1}P$ is discrepant.*

(1.8) First of all I prove this statement with X replaced by X' (as in (1.5)). For this, compare $f: V' \to X'$ with the small resolution $h: Y' \to X'$ of (1.5); the argument is exactly as in [C3–f], (2.3): $f: V' \to X'$ can be housed under a blow-up $\sigma: \tilde{Y}' \to Y'$,

$$\begin{tikzcd} & \tilde{Y}' \arrow[dl, "\sigma"'] \arrow[dr, "g"] & \\ Y' \arrow[dr, "h"'] & & V' \arrow[dl, "f"] \\ & X' & \end{tikzcd}$$

If $\Gamma \subset f^{-1}P \subset V'$, and $\tilde{\Gamma} \subset \tilde{Y}'$ is its proper transform, then $\tilde{\Gamma}$ is exceptional for σ, because h is small. Since σ is a sequence of blow-ups, $\tilde{\Gamma}$ is discrepant, by the adjunction formula for a blow-up, and Γ is discrepant for f.

(1.9) Now let $f: V \to X$ be a partial resolution, and let $V' = (V \times_X X')^{\sim}$ be the normalised pull-back. For $\Gamma \subset f^{-1}P \subset V$, let Γ' be a prime divisor of V' lying over Γ.

$$\begin{tikzcd} \Gamma' \subset V' \arrow[r] \arrow[d, "f'"'] & V \supset \Gamma \arrow[d, "f"] \\ X' \arrow[r, "\pi"] & X \end{tikzcd} \qquad (**)$$

I now fix the following: local bases $s \in \omega_X$, $s' \in \omega_{X'}$; local parameters x_Γ and $x_{\Gamma'}$ for the d.v.r.'s $\mathcal{O}_{V,\Gamma}$ and $\mathcal{O}_{V',\Gamma'}$; and bases s_Γ and $s_{\Gamma'}$ for the modules $\omega_{V,\Gamma}$ and $\omega_{V',\Gamma'}$.

I will estimate $v_{\Gamma'}(\pi^* s)$ in two different ways from the diagram (**). Firstly,

$$\pi^* s = (\text{unit}) \cdot t'^{m-1} \cdot s', \qquad (1)$$

so that

$$v_{\Gamma'}(\pi^* s) = (m-1) v_{\Gamma'}(t') + v_{\Gamma'}(s'); \tag{2}$$

now $v_{\Gamma'}(t') \geqq 1$ (because $f'(\Gamma') = P$ and $t' \in m_{X',P}$), and $v_{\Gamma'}(s') \geqq 1$ by the fact, proved in (1.8), that Γ' is discrepant for f'. Thus

$$v_{\Gamma'}(\pi^* s) \geqq m. \tag{3}$$

On the other hand, let e be the ramification index of $\mathcal{O}_{V,\Gamma} \subset \mathcal{O}_{V',\Gamma'}$; obviously, $e \leqq m$.

Now writing

$$\pi^* s_\Gamma = (\text{unit}) \cdot x_{\Gamma'}{}^{e-1} \cdot s_{\Gamma'} \tag{4}$$

and

$$s = (\text{unit}) \cdot x_\Gamma^{v_\Gamma(s)} \cdot s_\Gamma, \tag{5}$$

we get

$$v_{\Gamma'}(\pi^* s) = e \cdot v_\Gamma(s) + e - 1; \tag{6}$$

hence

$$m \leqq v_{\Gamma'}(\pi^* s) \leqq e \cdot v_\Gamma(s) + m - 1, \tag{7}$$

which implies that $v_\Gamma(s) > 0$. Q.E.D.

(1.10) *Proof of "only if" in* (1.1). Let $P \in X$ be an isolated cDV point, and $f: Y \to X$ a resolution; for $\Gamma \subset Y$ an exceptional prime divisor for f, either $f(\Gamma) = P$, in which case Γ is discrepant by (1.7); or $f(\Gamma) = C$ is a curve passing through P, but whose general point is non-singular for X. Above the general point of C, f is a composite of blow-ups in non-singular centres [11], and Γ is discrepant by the adjunction formula for a blow-up. Q.E.D.

The remainder of § 1 is concerned with tidying up some facts about cDV points, and is not used for the proof of (0.6).

(1.11) **Corollary.** *cDV singularities are canonical.*

The proof of this fact given in [C3–f], (2.6) was cohomological, and I had asked for a geometric proof ([C3–f], (6.9)).

(1.12) **Corollary.** *Let $P \in X$ be a Gorenstein* 3*-fold singularity having a small resolution $f: Y \to X$; then $P \in X$ is cDV.*

Special cases of these have been studied by Laufer and Pinkham [19], [26].

(1.13) It is convenient to introduce some notation for the following results. Let $P \in S$ be a Du Val singularity, and let $f\colon T \to S$ be its minimal resolution; if Λ is any subset of the (-2)-curves of $f^{-1}P$, then there is a normal surface T_Λ obtained by contracting out exactly the curves of Λ, and f factorises as

$$T \longrightarrow T_\Lambda \xrightarrow{f_\Lambda} S.$$

One sees at once that the morphisms $f_\Lambda : T_\Lambda \to S$ thus obtained exhaust all the crepant partial resolutions of $P \in S$.

(1.14) **Theorem.** *Let X be a 3-fold with cDV singularities, and let $f\colon Y \to X$ be a partial resolution. Then equivalent conditions:*

(i) *f is crepant;*

(ii) *f is small, and crepant above the general point of every 1-dimensional component of* Sing X;

(iii) *for every $P \in X$, and every $t \in m_P \subset \mathcal{O}_{X,P}$ for which the (local) surface singularity $P \in S \subset X$ defined by t is a Du Val singularity, $T = f^*S$ is a normal surface, and $f|_T : T \to S$ is crepant. Thus T is one of the T_Λ in the notation of* (1.13).

Proof. The implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) are trivial. Assume that $f\colon Y \to X$ is crepant; from (1.7), f is small. Now (0.4) together with (1.1) implies that Y has cDV singularities.

First note that it will be enough to show that $T = f^*S \subset Y$ is non-singular along any curve of $f^{-1}P$; for then T, as a Cartier divisor in a Cohen-Macaulay variety, satisfies S_2 and is then normal by Serre's criterion. The fact that $f|_T : T \to S$ is crepant is obvious from the adjunction formula.

Consider again the diagram of branched coverings

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X, \end{array}$$

where $X' \to X$ is obtained by taking an mth root of t, for some choice of $m \geq 2$, and Y' is the pull-back. Then obviously X' is again cDV, and $Y' \to X'$ is crepant. Both Y and Y' are non-singular at the general point of any curve of $f^{-1}P$, as follows from (1.7) (or alternatively, by (0.4) and (1.1)). But Y' can also be described as the m-fold cyclic cover of Y branched in T, and the fact that Y and Y' are both non-singular implies that also T is non-singular at the general point of any curve of $f^{-1}P$.

Q.E.D.

(1.15) The significance of (1.14) is that monodromy in any family $t: X \to T$ is now an obstruction to the existence of certain partial resolutions $f: Y \to X$ of the 3-fold point $P \in X$. Some extensions of this idea are discussed in (8.4) below.

(1.16) **Corollary.** *The singularity $P \in X$ given by $x^2+y^2+z^2+t^n$ has a resolution $f: Y \to X$ which is small and an isomorphism outside P if and only if n is even.*

Proof. For the "if" construction, originally due to Atiyah [5], see [8], (2.7). Let $f: Y \to X$ be a small partial resolution which is an isomorphism outside P; the section $S: (t=0)$ is an A_1 point, and there are only two possibilities for $f|_T: T \to S$ as in (1.14), (iii): either $T=S$, and f is an isomorphism; or T is the minimal resolution, in which case the family $X \to T$ given by t admits a simultaneous resolution. In the second case, n is even by an easy monodromy argument (compare [9], § 2).

§ 2. Resolving the Du Val locus

(2.1) Let X be a quasi-projective 3-fold with only cDV points. I write $\Sigma = \Sigma^1 X$ for the union of 1-dimensional components of $\operatorname{Sing} X$, and give Σ the reduced subscheme structure; $\mathscr{I} = \mathscr{I}_\Sigma \subset \mathcal{O}_X$ is the corresponding sheaf of ideals.

Outside a finite number of dissident points, X is analytically equivalent to $A^1 \times$(Du Val singularity); I want to construct a partial resolution $f: Y \to X$ having the properties:

(i) f coincides with $A^1 \times$(minimal resolution) along the Du Val locus;

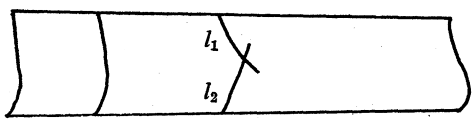
and (ii) f is small, that is there are no exceptional prime divisors over dissident points.

(2.2) It is clear from (0.4), (1.1) and (1.14) that then f is crepant, and Y has only isolated cDV singularities.

(2.3) **Woffle.** My solution to this problem is essentially simply to blow-up the reduced singular locus $\Sigma \subset X$ (see (2.6) for the correct statement), and proceed inductively. My proof of the fact that this gives a small partial resolution, although easy given the detailed information available concerning simultaneous resolution of Du Val points, is rather weird, and should perhaps be regarded as a temporary expedient. There are several other objections to my procedure: (a) Σ may have quite nasty singularities, so that blowing it up is disgusting from both the aesthetic and computational viewpoints; (b) since X can be considered (1.3) as the total space of a deformation of a Du Val singularity, there are good reasons for wanting a construction which is a pull-back from the versal deformation space; my construction is not of this kind, since $\Sigma \subset X$ does not commute with base change. Nash transform on the fibres of t (the blow-up of $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$) has the good universal property, but is not suitable for my purpose since it is not always crepant (for example on A_n points with n even).

The peculiarities of the construction are illustrated by two simple cases:

(2.4) **Examples.** (i) $X : xy = z^2 t$; this is a cA_2 point (Definition (0.5)), and $\Sigma : x = y = z = 0$. Blowing up Σ leads to a resolution $\sigma : Y \to X$, where the exceptional surface is a conic fibration:



$\sigma^{-1}P = l_1 \cup l_2.$

There are two other resolutions $Y_i \to X$ which can be obtained by elementary transformations in the components l_i, as well as two partial resolutions obtained by contracting out one of the l_i.

(ii) $X : xy = z^2 t^2$; this is a cA_3 point, and $\Sigma : x = y = zt = 0$ is the union of two non-singular branches C_1 and C_2.

In this case the blow-up $Y \to X$ of Σ introduces an ordinary double point on Y; an alternative method consists of blowing up first C_1 and then C_2 (or the other way round). This leads to a non-singular variety Y_1 which dominates Y. However, my choice of Y has the two advantages that it is uniquely specified by X, and is projective; if C_1 and C_2 are branches of the same irreducible curve, then Y_1 is essentially Hironaka's standard example of a Moishezon space (algebraic space) which is not an algebraic variety.

(2.5) Let $\Sigma \subset X$ be as in (2.1); write $\mathscr{I}^{[m]}$ for the ideal of functions vanishing to multiplicity $\geqq m$ at each generic point of Σ. The essential thing for my purpose is that $\mathscr{I}^{[m]}$ does not have dissident primary components.

(2.6) **Theorem.** *There exists an m_0 such that for every sufficiently large m with $m_0 | m$, the blow-up $\sigma = \sigma_m : X_1 \to X$ of $\mathscr{I}^{[m]}$ in X is small, X_1 is normal, and σ is independent of m.*

In (2.9) and (2.10) I will reduce the proof to the case when the family $X \to T$ of (1.3) has a simultaneous resolution $Y \to X \to T$; the essential point of the proof (2.12) will be to modify Y by elementary transformations [10] so that $\mathscr{I}^{[m]} \cdot \mathcal{O}_Y$ becomes an invertible ideal.

(2.7) **Exercise.** If $P \in X$ is a cA_n or cD_n point (0.5), one can write down explicit generators for the ideal $\mathscr{I}$, and check that the blow-up of $\mathscr{I}$ is small. I sketch the computation in the most entertaining case, and leave the reader to play with the others.

Certain cD_n points are of the form $X: (F = x^2 + y^2z + zG(z, t) = 0)$, where $z \nmid G(z, t)$. Write $G = g_1^2 g_2$, where g_1 is the greatest reduced divisor such that $g_1^2 | G$. Then

$$\Sigma = (x = y = g_1 = 0) \cup (x = z = y^2 + G = 0).$$

One sees that

$$\mathscr{I} = (x, y, g_1) \cap (x, z, y^2 + G) = (x, yz, y^2 + G, g_1 z).$$

The blow-up of A^4 along $\mathscr{I}$ is got by choosing a $\boldsymbol{P}^3$ with homogeneous coordinates (s, u, v, w), and setting

$$A^4 \times \boldsymbol{P}^3 \supset B : \operatorname{rk}\begin{pmatrix} x & yz & y^2+G & g_1 z \\ s & u & v & w \end{pmatrix} \leqq 1.$$

Some of the forms vanishing on the proper transform of X are

$$yF : ys^2 + uv$$
$$zF : zs^2 + u^2 + w^2 g_2$$
$$g_1 F : g_1 s^2 + vw.$$

Above $(0, 0, 0, 0) \in X$ we get $y = z = g_1 = 0$, so that the fibre of σ is contained in the 1-dimensional scheme

$$uv = u^2 + w^2 g_2(0, 0) = vw = 0;$$

(this is true regardless of whether $g_2(0, 0) = 0$). One can in fact check that the scheme-theoretic fibre of σ above 0 is a subscheme of degree 3 in $\boldsymbol{P}^3$, whereas the fibre along the Du Val locus is a conic in $\boldsymbol{P}^2 \subset \boldsymbol{P}^3$. In particular, the ideal $\mathscr{I}$ is not normally flat.

(2.8) **Problem.** I do not know how to do the corresponding computation for E_6, E_7, E_8. Although it is easy to write down the set-theoretic singular locus Σ of

$$X : (F = x^2 + y^3 - 3yf_2(z, t) + 2f_3(z, t) = 0)$$

in terms of multiple factors of f_2, f_3 and $f_2^3 - f_3^2$, it does not seem possible to write down in a uniform way a set of generators for $\mathscr{I}$.

If this could be done we could take $m_0 = 1$ in Theorem (2.6) (and scrap the remainder of this paper).

(2.9) **Proposition.** *For some $m>0$, suppose that the blow-up $s: Z\to X$ of $\mathscr{I}^{[m]}$ is a small morphism; set $X_1=\tilde{Z}$ for the normalization, and write σ for the composite $\sigma: X_1\to X$. Then for all sufficiently large integers a, the blow-up of $\mathscr{I}^{[am]}$ coincides with σ.*

Proof (projective normalization). By definition of the blow-up the sheaf of ideals $\mathscr{J}=\mathscr{I}^{[m]}\cdot\mathcal{O}_Z\subset\mathcal{O}_Z$ is invertible and relatively ample for s; since the normalisation $\pi: X_1\to Z$ is finite, the same holds for

$$\mathscr{J}_1=\mathscr{I}^{[m]}\cdot\mathcal{O}_{X_1}=\pi^*\mathscr{J}.$$

Thus for any sufficiently large a, $\mathscr{J}_1^a$ is relatively very ample, that is the blow-up of $\sigma_*\mathscr{J}_1^a$ is $\sigma: X_1\to X$. On the other hand, $\sigma_*\mathscr{J}_1^a=\mathscr{I}^{[am]}$, since both sides are the intersections of the same primary components at generic points of Σ. Q.E.D.

(2.10) **Proposition.** *Let $P\in X$ be a cDV point, and let X' be the cyclic m-fold cover as in diagram* (*) *of* (1.4); *write Σ' for the* 1*-dimensional components of X', and $\mathscr{I}'=\mathscr{I}_{\Sigma'}\subset\mathcal{O}_{X'}$. Suppose that the blow-up Z of $\mathscr{I}'$ in X' is small, and write $\sigma': Y'\to X'$ for the normalisation of Z, as in* (2.9). *Then $\boldsymbol{\mu}_m$* (the cyclic group of mth roots of 1) *acts on Y', such that the quotient $Y=Y'/\boldsymbol{\mu}_m$ is the blow-up of $\mathscr{I}^{[am]}$ in X for all $a\gg 0$.*

In particular, if the statement of Theorem (2.6) *holds for X', with $m_0=1$, then it holds for X with $m_0=m$.*

Proof. The blow-up of $\mathscr{I}'$ in X' is intrinsic to X', so that the action of $\boldsymbol{\mu}_m$ on X' extends to an action on Y'; the quotient $Y=Y'/\boldsymbol{\mu}_m$ fits into the commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

Now $Y\to X$ is certainly small, and coincides with the blow-up of $\mathscr{I}$ except possibly over a finite set of X, so that there is a Weil divisor $E>0$ on Y such that $\mathscr{I}\cdot\mathcal{O}_Y=\mathcal{O}_Y(-E)$ outside a finite number of fibres. By definition of the blow-up $Y'\to X'$, there is a Cartier divisor E' on Y' such that $\mathscr{I}'\cdot\mathcal{O}_{Y'}=\mathcal{O}_{Y'}(-E')$. I want to deduce from this that mE is a Cartier divisor on Y, and $\mathscr{I}^{[m]}\cdot\mathcal{O}_Y=\mathcal{O}_Y(-mE)$; then also $mE'=f^*(mE)$, so that $\mathcal{O}_Y(-mE)$ is relatively ample on Y, proving the result.

The cyclic cover $X'\to X$ is branched exactly in the fibre of $X'\to T'$ over $(t'=0)$, and the covering group $\boldsymbol{\mu}_m$ fixes this fibre pointwise, and acts freely outside $(t'=0)$. The action on Y' thus has the same property. I only need to prove my assertion at points of the branch locus, so let $P \in Y$ be in the $(t=0)$ fibre, and let $P' \in Y'$ be its unique inverse image.

By hypothesis on Y', there exists an $f' \in \mathscr{I}'$ such that f' is a basis of $\mathcal{O}_{Y'}(-E')$ in a neighbourhood of P'. The same is then true of $\varepsilon^*(f')$ for any $\varepsilon \in \boldsymbol{\mu}_m$, and the product $f = \prod_{\varepsilon \in \mu_m} \varepsilon^*(f')$ is invariant under $\boldsymbol{\mu}_m$, and thus is an element of $\mathscr{I}^{[m]} \subset \mathcal{O}_X$, and bases $\mathcal{O}_{Y'}(-mE')$ near P'. It follows that f bases $\mathcal{O}_Y(-mE)$ near P. Q.E.D.

(2.11) I assemble the notation and hypotheses for the key result (2.12) below. In the diagram

$$\begin{array}{ccccc} & Y_0 & \subset Y \supset & Y_t \\ & \downarrow{\scriptstyle h_0} & \downarrow{\scriptstyle h} & \downarrow \\ P \in & X_0 & \subset X \supset & X_t \\ & \downarrow & \downarrow & \downarrow \\ & 0 & \in T \ni & t \end{array} ,$$

h is a simultaneous resolution of the family of Du Val singularities $X \to T$. I can suppose that $P \in X$ is the only dissident, and that $P \in X_0$ is the only singularity; the existence of the simultaneous resolution h implies that the monodromy on $H_2(Y_t, \boldsymbol{Z})$ is trivial, and this implies that each component Γ_i of Σ maps isomorphically to T.

Now $h^{-1}\Sigma = \bigcup F_j$ is a union of surfaces, each of which is a Cartier divisor on Y (since Y is smooth); each F_j meets the general fibre Y_t in a (-2)-curve. There is a unique positive divisor

$$Z = \Sigma\, n_j F_j$$

such that Z meets each general fibre Y_t in a cycle Z_t which is the sum of the Artin fundamental cycles ([2], p. 132) of the Du Val singularities of X_t. It then follows that $\mathscr{J} = \mathcal{O}_Y(-Z) \subset \mathcal{O}_Y$ is an invertible ideal sheaf such that $\mathscr{J} = \mathscr{I} \cdot \mathcal{O}_Y$ except possibly over $h^{-1}P$.

If Y' is some other simultaneous resolution of the same family $X \to T$, then Z' and $\mathscr{J}'$ denote the same objects constructed from $Y' \to X$; in this case, since $f\colon Y \dashrightarrow Y'$ is an isomorphism in codimension 1, we can identify $\operatorname{Div} Y = \operatorname{Div} Y'$ and $\operatorname{Pic} Y = \operatorname{Pic} Y'$ (taking a divisor into its proper transform, see (6.2) below), and under this identification $Z = Z'$, $\mathscr{J} = \mathscr{J}'$.

I will say that an invertible sheaf $\mathscr{L} \in \operatorname{Pic} Y$ is (relatively) *nef* on Y if $\mathscr{L}E \geqq 0$ for every curve E contracted by h (every such curve is of course a (-2)-curve on some Y_t); $\mathscr{L}$ is, *nef over the Du Val locus* (or over $X - P$) if $\mathscr{L}E \geqq 0$ for every (-2)-curve on Y_t, $t \neq 0$. The case of special interest is $\mathscr{L} = \mathscr{J} = \mathcal{O}_Y(-Z)$, which is nef over the Du Val locus by construction of the Artin fundamental cycle.

For the rest of the paper, I will use the index j *exclusively* to run through the components of $h^{-1}\Sigma = \bigcup F_j$, and k to run through the components of $h^{-1}P = \bigcup E_k$.

(2.12) **Theorem.**[1] *The hypotheses are as in* (2.11). *Let* $\mathscr{L} \in \operatorname{Pic} Y$ *be nef over the Du Val locus; then there exists a simultaneous resolution* Y' *of the family* $X \to T$, *with* $Y' \dashrightarrow Y$ *an isomorphism in codimension* 1, *such that the proper transform* $\mathscr{L}'$ *of* $\mathscr{L}$ *on* Y' *is nef.* Y' *can be obtained from* Y *by a composite of elementary transformations in the* (-2)*-curves of* $h^{-1}P$, *as defined in* (5.6) *below.*

(2.13) **Lemma.** *If* $\mathscr{L} \in \operatorname{Pic} Y$ *is nef then* $h^*h_*\mathscr{L} \to \mathscr{L}$ *is surjective*; (*that is,* $\mathscr{L}$ *is "relatively generated by its* H^0".)

Proof. Write $\mathscr{L}_0 = \mathscr{L}_X \otimes \mathcal{O}_{Y_0}$; then $\mathscr{L}_0$ is nef on Y_0, so that by the methods of Artin [1], [2], one can easily show that $\mathrm{R}^1 h_* \mathscr{L}_0 = 0$ and $h^*h_*\mathscr{L}_0 \to \mathscr{L}_0$ is surjective. The lemma then follows from the base-change theorem in coherent cohomology.

(2.14) **Corollary.** *The blow-up* $\sigma : X_1 \to X$ *of* X *along* $\mathscr{I}$ *is small.*

Proof. Let $Y \to X$ be a simultaneous resolution as given by (2.12), such that $\mathscr{J} = \mathcal{O}_Y(-Z)$ is nef. Now consider $h_*\mathscr{J}$; by definition of Z, this coincides with $\mathscr{I} = \mathscr{I}_\Sigma$ at the generic points of Σ, and since h is small $h_*\mathscr{J}$ has no other primary components. Hence $h_*\mathscr{J} = \mathscr{I}$; thus by (2.13), $h^*\mathscr{I} \to \mathscr{J}$ is surjective, so that $\mathscr{I} \cdot \mathcal{O}_Y = \mathscr{J}$, and in particular, $\mathscr{I} \cdot \mathcal{O}_Y$ is invertible. By the universal property of a blow-up (Hartshorne, (II.7.14)), the blow-up of $\mathscr{I}$ is dominated by the small morphism h. Q.E.D.

(2.15) **Remarks.** (a) The same method can be used to blow up some choice $\Sigma' \subset \Sigma$ of the components of Σ, although the result will not be uniquely specified by X (see (2.4)) unless Σ' can be chosen intrinsically. There are of course other natural choices, for example blow up first all the E_8 locuses, $\cdots$.

(b) I believe, without having checked the details, that the same method can be used to resolve the codimension 2 singular locus of an n-fold having cDV singularities.

(2.16) We have seen above that (2.14) implies (2.6). The proof of (2.12) is divided into 3 steps: (i) In § 5 I define the elementary transformation ρ_C in an isolated (-2)-curve $C \subset Y$. (ii) In § 6 I study the effect

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[1] Compare D. Morrison's proof of a closely related result in [30], (3.5), p. 261.

of ρ_C on the homology of $C \subset Y$; if C lies on a smooth surface S, with $(C^2)_S = -2$, then there is a certain sense in which ρ_C acts on $H_2(S, Z)$ by reflection in the class of C. (iii) In the notation of (2.11), $h^{-1}P = \bigcup E_k \subset Y_0$ is a collection of (-2)-curves, which span a root system in the homology of Y_0; the effect of the elementary transformation ρ_{E_k} is a reflection in the corresponding root; in § 7 I use the usual proof that the Weyl group acts transitively on the Weyl chambers to show that a composite of admissible reflections knock the linear form corresponding to $\mathscr{L}$ into the (closed) Weyl chamber of linear forms non-negative on the primitive roots.

§ 3. Proof of the main theorem for index $r \geqq 1$

(3.1) The following works in all dimensions, and together with (1.1) implies (I) of (0.6):

Proposition. (I) *Let $\varphi : Y \to X$ be a morphism between normal varieties, etale in codimension* 1 *on Y; if $P \in Y$ is such that $Q = \varphi(P) \in X$ is canonical* (*or terminal*), *then so is $P \in Y$.*

(II) *Let $Q \in X$ be an index r canonical point, and $\varphi : Y \to X$ its local index* 1 *cover, as in* [C3-f], (1.9); *then $\varphi^{-1}Q = P$* (*a single point*), *and if $P \in Y$ is terminal, then so is $Q \in X$.*

(3.2) Form a commutative diagram

$$\begin{array}{ccccc} \Delta \subset \tilde{Y} & \xrightarrow{g} & Y \ni P & & \\ \psi \downarrow & & \downarrow \varphi \quad \downarrow & & (*) \\ \Gamma \subset \tilde{X} & \xrightarrow[f]{} & X \ni Q & & \end{array}$$

with f and g partial resolutions.

(3.3) *Proof of* (I) (compare [C3-f], (1.7)). By hypothesis on φ and X, and using Convention (0.12), (e),

$$K_Y = \varphi^* K_X$$

and $$K_{\tilde{X}} = f^* K_X + \Delta_f, \quad \text{with} \quad \Delta_f \geqslant 0.$$

By the adjunction formula for ψ,

$$K_{\tilde{Y}} = \psi^* K_{\tilde{X}} + R_\psi,$$

where the ramification divisor R_ψ contains the exceptional prime divisors of ψ. Hence

$$K_{\tilde{Y}}=\psi^*f^*K_X+\psi^*\Delta_f+R_\psi$$
$$=g^*K_Y+\psi^*\Delta_f+R_\psi,$$

and (I) follows at once.

(3.4) *Proof of* (II). Now let $\Gamma\subset\tilde{X}$ be a prime divisor exceptional for f, and $\Delta\subset\tilde{Y}$ a prime divisor lying over Γ; write e for the ramification index of $\mathcal{O}_{\tilde{X},\Gamma}\subset\mathcal{O}_{\tilde{Y},\Delta}$.

If $s\in\omega_X^{[r]}$ is a basis near Q, and $\varphi: Y\to X$ is the construction of [C3–f], (1.9), then $\varphi^*s=t^r$, with $t\in\omega_Y$ a basis, and from

$$v_\Delta(\varphi^*s)=ev_\Gamma(s)+r(e-1) \tag{1}$$

I get

$$v_\Delta(t)=\frac{e}{r}v_\Gamma(s)+e-1. \tag{2}$$

(3.5) *Claim.* $(v_\Delta(t)+1, e)=1$.

This claim proves (II), because if Γ is crepant for f, $v_\Gamma(s)=0$, whence $v_\Delta(t)+1=e=1$, so that Δ is crepant for g and ψ etale at Δ. Note that this proof depends on a congruence, rather than an inequality: if $P\in Y$ is terminal, and $\boldsymbol{\mu}_r$ acts on $P\in Y$, then there is no reason why the quotient $Q\in X$ should be canonical; all that is asserted here is that if it is canonical, then it must be terminal.

(3.6) *Proof of* (3.5). $k(X)\subset k(Y)$ is Galois with group $\boldsymbol{\mu}_r$; let $\boldsymbol{\mu}_e\subset\boldsymbol{\mu}_r$ be the ramification group of the d.v.r. $\mathcal{O}_{\tilde{Y},\Delta}\subset k(Y)$ (geometrically, $\boldsymbol{\mu}_e$ is the subgroup of automorphisms of Y which, as rational maps on $\tilde{Y}$, fix Δ pointwise). Write $B=(\mathcal{O}_{\tilde{Y},\Delta})^{\mu_e}$ for the fixed subring. Then by the Galois theory of d.v.r.'s, a local parameter $x_\Delta\in\mathcal{O}_{\tilde{Y},\Delta}$ can be chosen so that

$$1, x_\Delta, \cdots, x_\Delta^{e-1} \text{ is a } B\text{-basis of } \mathcal{O}_{\tilde{Y},\Delta}, \tag{**}$$

and $\boldsymbol{\mu}_e$ acts on x_Δ by

$$\boldsymbol{\mu}_e\ni\varepsilon: x_\Delta\mapsto\varepsilon^a x_\Delta,$$

for some integer a (coprime to e). A basis $s_\Delta\in\omega_{\tilde{Y},\Delta}$ can be chosen in the form

$$s_\Delta=dx_\Delta\wedge dx_2\wedge\cdots\wedge dx_n,$$

with $x_i\in B$ for $i=2,\cdots,n$; thus also

$$\boldsymbol{\mu}_e\ni\varepsilon: s_\Delta\mapsto\varepsilon^a s_\Delta,$$

with the same value of a.

Now because of the relation $\varphi^*s=t^r$, $t\in\omega_{Y,P}$ is an eigenvector of the action of $\boldsymbol{\mu}_r$; since

$$t=(\text{unit})\cdot x_\Delta^{v_\Delta(t)}\cdot s_\Delta,$$

the unit is also an eigenvector, so by (**) must belong to B, so that

$$\boldsymbol{\mu}_e\ni\varepsilon : t\mapsto\varepsilon^{a(v_\Delta(t)+1)}t.$$

The hypothesis that

$$\text{index}(P\in Y)=1;\ \text{index}(Q\in X)=r$$

implies that $\boldsymbol{\mu}_r$ acts faithfully on t, so that in particular

$$(a(v_\Delta(t)+1),\ e)=1.$$

Q.E.D.

(3.7) I now start on the proof of (0.6), (II).

Proposition. *Let $Q\in X$ be canonical of index r, and let $\varphi : Y\to X$ be its index* 1 *cover.*

(A) *If $P\in Y$ is not cDV, let $\tau : Y_1\to Y$ be the crepant blow-up of* [C3–f], (2.11);

(B) *If $P\in Y$ is cDV but not isolated, let $\tau : Y_1\to Y$ be the crepant blow-up of $\Sigma\subset Y$ as in* (2.6) *above.*

Then the action of $\boldsymbol{\mu}_r$ on $P\in Y$ extends to an action on Y_1, and letting $X_1=Y_1/\boldsymbol{\mu}_r$ we get

$$\begin{tikzcd} Y_1 \arrow[r, "\tau"] \arrow[d, "\tilde\varphi"'] & Y \arrow[d, "\varphi"] \\ X_1 \arrow[r, "\sigma"'] & X \end{tikzcd}$$

where σ is crepant and has at least one exceptional prime divisor, and $\tilde\varphi$ is etale in codimension 1.

(3.8) *Proof.* If τ is intrinsic to Y, then the action of $\boldsymbol{\mu}_r$ on Y extends to Y_1; the remaining assertions follow easily from (1) in (3.4) above, after relabelling the diagrams.

The blow-up of $\Sigma\subset Y$ in (2.6) is obviously intrinsic; I now justify the fact that the blow-up of [C3–f], (2.11) is intrinsic, referring freely to the notation of [C3–f], (2.10).

If $k \geq 3$ then τ is the blow-up of the closed point $P \in Y$, which is obviously intrinsic. In the cases $k=2$ or $k=1$, then τ is defined as the weighted blow-up, which is the Proj of an $\mathcal{O}_X$-algebra $\mathscr{A}=\bigoplus_{n \geq 0} I_n$, where the I_n are the ideals generated by the monomials x^m of weight $\alpha(x^m) \geq n$. Now it is easy to verify that in the case $k=2$, $\mathscr{A}$ is generated by I_1 and I_2, and that these ideals do not depend on the choice of coordinates: $I_1 = m_P$, and

$$I_2=(x)+m_P=\left\{h \in \mathcal{O}_{Y,P} \,\middle|\, \begin{array}{l}\text{the divisor } (h) \subset Y \\ \text{has mult.} \geq 4 \text{ at } P\end{array}\right\}.$$

In the $k=1$ case, $\mathscr{A}$ is generated by I_1, I_2 and I_3, and these ideals can be characterised by similar equalities. This proves that the construction of [C3–f], (2.11) is intrinsic.

(3.9) *Proof of* (0.6), (II). If X is a 3-fold with canonical singularities, I can carry out a sequence $X_a \xrightarrow{\sigma_a} \cdots X_1 \xrightarrow{\sigma_1} X$, where each σ_i is the construction of (3.7), (A) at some point $P \in X_{i-1}$ whose index 1 cover is not cDV; if at some stage all points $P \in X_a$ have cDV points as their index 1 covers, I can carry out a sequence $X_b \xrightarrow{\sigma_b} \cdots X_{a+1} \xrightarrow{\sigma_{a+1}} X_a$, where each σ_i is the construction of (3.7), (B); note that outside the finitely many dissidents, σ_i just consists of blowing up the entire Du Val locus of X, so that the construction of (B) performed in Zariski neighbourhoods of different points coincide on the intersections.

Now each σ_i introduces at least one new crepant prime, so that each of these two sequences must terminate after a finite number of leaps, by [C3–f], (2.3). This proves (0.6), (II).

(3.10) **Remark.** The fact that cyclic quotients $A^3/\boldsymbol{\mu}_r$ are terminal if and only if they have index exactly r, and the fact that canonical toric singularities have crepant partial resolutions with only these terminal singularities, was suggested by some computations of R. Barlow, and discovered independently by V. Danilov. Anyone who is curious about extensions of these kind of results to higher dimensions ("Is 3 a Big Number?" seems to be a standard tea-time question) would be well advised to experiment around with toric 4-folds.

§ 4. Problem list in minimal models and classification of 3-folds

(4.0) Open problems remaining from [C3–f]: (3.11), (4.2), (5.7), (6.3), (6.4), (6.6), (6.7), (6.10), (6.11), (6.12).

(4.1) **Woffle.** To give the flavour of the conjectures (4.3–7) below, consider the following classes of non-singular 3-folds V of general type:

$A=\{V|K_V \text{ is ample}\}$;
$\cap$
$B=\{V||nK_V| \text{ is free for some } n>0\}$;
$\cap$
$C=\{V|R(V, K_V) \text{ is f.g.}\}$;
$\cap$
$D=\{V|\text{there exists a minimal model } X \text{ of } V \text{ as in (0.7)}\}$;
$\cap$
$E=\{\text{all 3-folds of general type}\}$.

Roughly, I have the feeling that B is about twice as complicated as A, and C about 5 times as complicated as B (see [C3-f], § 5). The inclusion $C \subset D$ is (0.6), (II). The conjectures (4.3–7) can be summed up in the case of varieties of general type by saying that $C=D=E$; of these, the 2nd seems to be much the deeper, but either could still be wrong. The only evidence for them is indirect: the special case of the "degeneration situation" [17], [21], [29]; and the very special cases of $C=D$ proved by Wilson [36], [37].

Added in Proof. Y. Kawamata has a proof which I am confident will give $C=D$; one key ingredient is the Kawamata-Viehweg "strong vanishing" [41]; for $D=E$?, see (4.18) below.

It should also be mentioned that whether or not these conjectures are right, they raise a colossal number of new geometrical phenomena, the study of which looks like keeping us busy for decades to come.

(4.2) In (0.7) I defined minimal models with $\kappa_{\text{num}} \geqq 0$; it seems rather unlikely that one can prove the existence of minimal models for 3-folds of general type without having some firm control over all phenomena concerned with curves C such that $K_V C < 0$, which is why I widen the scope in (4.3) below. For the cases which are understood, and on which my guesswork is based, see [Mori], Ch. 2 for surfaces, and [Mori], Ch. 3 for a very substantial start on the 3-fold case.

(4.3) **Conjecture** (minimal models). *Let V be a projective 3-fold. Then there exists a projective variety X birationally equivalent to V such that X has only quick singularities, and*
either (i) *K_X is nef;*
or (ii) *there exists a morphism $\varphi : X \to Y$ with Y projective, $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$,* $\dim Y = 0$, 1 *or* 2, *such that*

(a) *$-K_X$ is relatively ample for φ,*
and (b) $\operatorname{Pic} X \otimes \boldsymbol{Q} = \operatorname{Pic} Y \otimes \boldsymbol{Q} \oplus \boldsymbol{Q} \cdot (-K_X)$.

One hopes that the varieties in (ii) are uniruled, and I define these temporarily to be *minimal models with* $\kappa_{\mathrm{num}}=-\infty$. The most essential case of $\boldsymbol{Q}$-Fano 3-folds is discussed in (4.16) below.

(4.4) The hope for proving (4.3) is that Mori's Theorem on the Cone can be extended to 3-folds X with quick singularities, to prove that if K_X is not nef then X has certain special curves (corresponding to extremal rays R of the cone $\overline{NE}(X)$ with $K_X R<0$), and that these will specify either a birational contraction of X (but not necessarily by a morphism, see [13], unless X satisfies some very carefully chosen inductive hypothesis), or a morphism φ_R as in (4.3), (ii). See also (4.18).

(4.5) **Definition** (Mumford, [22]). Let X be a projective 3-fold with ample $H \in \operatorname{Pic} X$, and suppose that K_X is nef. Define

$$\kappa_{\mathrm{num}}(X)=\max\{k \mid H^{3-k}\cdot K_X^k>0\};$$

then $\kappa_{\mathrm{num}}=0$, 1, 2 or 3, and

$$\kappa_{\mathrm{num}}=0 \Longleftrightarrow K_X\equiv 0 \text{ (numerical equivalence)},$$
$$\kappa_{\mathrm{num}}=3 \Longleftrightarrow K_X^3>0 \Longleftrightarrow \kappa(X)=3.$$

It is likely that all the essential difficulties of the conjectures discussed in (4.6–12) below are already present in the case X is non-singular, and it would seem to be a good strategy to work with this extra hypothesis pending a more complete study of quick singularities.

(4.6) **Conjecture** (classification, 1st version). *Let X be a minimal model with $\kappa_{\mathrm{num}}\geqq 0$. Then for some $m>0$, $|mK_X|$ is free.*

Considering the morphism $\varphi=\varphi_{mK_X}$, this implies

(4.7) **Conjecture** (classification, 2nd version). *Let X be a minimal model with $\kappa_{\mathrm{num}}\geqq 0$. Then there exists a morphism*

$$\varphi : X \longrightarrow Y$$

with Y projective, $\varphi_*\mathcal{O}_X=\mathcal{O}_Y$, $\dim Y=\kappa_{\mathrm{num}}(X)=0$, 1, 2 or 3 such that a curve $C\subset X$ is contracted by φ if and only if $K_X C=0$.

It is possible that (4.7) could be proved first, by studying deformations of curves $C\subset X$ with $K_X C=0$, and (4.6) deduced by an appropriate "canonical bundle formula". Compare [22], and note that (4.6–7) corresponds to the subtle part of the classification of surfaces in Mumford's treatment.

(4.8) **Problem.** Let X be a minimal model with $\kappa_{\text{num}} \geqq 0$. Prove that $\kappa \geqq 0$.

(4.9) **Conjecture.** *Let X be a non-singular 3-fold with K_X nef and ample H; then*

$$Hc_2 \geqq \left(\frac{2n}{n-1}\right) Hc_1^2.$$

This statement is a limiting case of inequalities of Bogomolov and Yau, and can probably be proved using the ideas of Bogomolov outlined in [27], § 3, with an appropriate amount of hard work.

(4.10) I now outline a proof, due to S. Tsunoda, that (4.9), together with Iitaka's statements $C_{3,1}$ and $C_{3,2}$, proved by Kawamata and Viehweg, implies (4.8) for smooth X. Indeed, (4.9) gives $Hc_2 \geqq 0$ for every ample H on X; by the assumption that K_X is nef, and by Kleiman's ampleness criterion, this implies $K_X c_2 \geqq 0$, and hence $\chi(\mathcal{O}_X) = (1/24) c_1 c_2 \leqq 0$. But

$$\chi(\mathcal{O}_X) = 1 - q + h^{0,2} - p_g \leqslant 0$$

gives either $p_g \neq 0$, or $q \neq 0$; in the second case, applying $C_{3,1}$ or $C_{3,2}$ to the fibres of the Albanese morphism gives $\kappa \geqq 0$.

(4.11) In the case $\kappa_{\text{num}} = 1$, to prove (4.7) it is enough to prove that $\kappa = 1$, that is $h^0(mK_X) \geqq 2$ for some $m > 0$.

(4.12) *Discussion of* (4.6–7) *in the case* $\kappa = 3$. By analogy with the surface case, there are two lines of attack, both of which lead to technical difficulties, but neither of which is exhausted:

(i) following the approach of Zariski and Mumford [23], one can study the deformation theory of curves $C \subset X$ with $K_X C = 0$ and try either to construct the canonical model directly by contracting these, or to get enough information on the cohomology of infinitesimal neighbourhoods to prove that $|mK_X|$ is free near C (see [36] and [37] for partial results using this method).

(ii) following the approach of Kodaira and Bombieri-Ramanujam [6], one can attempt to find suitable numerical conditions on divisors D_1 and D_2 (assuming $P \in D_1 \cap \operatorname{Sing} D_2$) which imply that P is not a base point of $|D_1 + D_2 + K_X|$. Straightforward chasing exact sequences leads to problems with H^0's of sheaves on the intersection of D_1 and D_2, although it's not clear exactly how to proceed even if D_1 and D_2 intersect properly.

The next set of problems (4.13–15) are concerned with the detailed study of quick singularities.

(4.13) **Problem.** It should be possible to give a much more explicit description of quick singularities: let $P \in Y$ be an isolated cDV singularity, and let $\boldsymbol{\mu}_r$ (the cyclic group of rth roots of 1) act on Y such that P is fixed, and the action is free on $Y-P$. Then the condition that the quotient $Q \in X=(P \in Y)/\boldsymbol{\mu}_r$ is canonical is a very strong restriction. If $P \in Y$ is non-singular it is known that $Q \in X$ is analytically the quotient singularity $(1/r)(1, a, -a)$, that is $A^3/\boldsymbol{\mu}_r$, where

$$\boldsymbol{\mu}_r \ni \varepsilon : (x, y, z) \longmapsto (\varepsilon x, \varepsilon^a y, \varepsilon^{-a} z)$$

for some a coprime to r; this has been proved independently be D. Morrison and by Danilov and Frumkin. The proof consists of putting together [C3–f], (3.1) and a non-trivial combinatoric fact proved in [44], Theorem 1.

The canonical condition gets stronger if $P \in Y$ is singular and r large, so it's quite likely that there are very few quick singularities apart from these quotients and the cDV points themselves.

(4.14) *Hint for* (4.13). It is easy to see that, analytically, $\boldsymbol{\mu}_r$ acts on $P \in Y \subset A^4$ by an action on A^4; and thus the quotient $Q \in X \subset A^4/\boldsymbol{\mu}_r$ is a subvariety of a toric variety, defined by the equations $x^m F=0$ for monomials x^m such that $x^m F$ is $\boldsymbol{\mu}_r$-invariant (here $F=0$ is the equation of $Y \subset A^4$). The condition that $Q \in X$ is a canonical singularity imposes strong restrictions on the Newton polyhedron of F, so that (4.13) should be accessible to direct computations using toric methods.

(4.15) **Problem.** For a quick singularity $P \in X$, calculate the invariants of [C3–f], (5.6), and especially $c_2 \Delta$.

These invariants relate to $\chi(\mathcal{O}_V(nf^*K_X))$, and hence to $\chi(\mathcal{O}_X(nK_X))$, where $f: V \to X$ is a resolution, so that an understanding of them is likely to be important for (4.6–8) and (4.17). Some computations of R. Barlow on the cyclic quotients $(1/r)(1, a, -a)$ suggest that there should be nice clean formulas.

(4.16) *$\boldsymbol{Q}$-Fano 3-folds.* Assuming the truth of Conjecture (4.3), a basic step in the birational study of preruled ($\kappa=-\infty$) 3-folds is the *biregular* study of $\boldsymbol{Q}$-Fano 3-fold with $\operatorname{Pic} \cong \boldsymbol{Z}$; that is, X has only quick singularities, and $-K_X$ is an ample $\boldsymbol{Q}$-Cartier divisor. Write $r \in \boldsymbol{Q}$ for the smallest positive number such that $-rK_X \in \operatorname{Pic} X$; in the case X is smooth, $r=1/k$ is the reciprocal of the index as defined by Iskovskikh, and I call it the *reciprocal index*. It's easy to construct examples of X for which $r=2$ or 3 (compare [C3-f], (3.10)); for example, the quotient $\boldsymbol{P}^3/\boldsymbol{\mu}_7$, where

$$\mu_7 \ni \varepsilon : (x_1, \cdots, x_4) \longmapsto (x_1, \varepsilon x_2, \varepsilon^2 x_3, \varepsilon^5 x_4)$$

has $r=7/4$. I don't know of any systematic construction giving $r \to \infty$.

(4.17) **Problem.** (a) What are the possibilities for r and the function $P_X(n)=h^0(\mathcal{O}_X(n))$? Are there only finitely many possibilities?

(b) Is X uniruled?

(4.18) *Added in proof.* In the last year a proof of (4.3) has emerged in two special cases: the very special case of toric varieties [43], and the much more substantial case of semi-stable degenerations of minimal surfaces with $\kappa \geqq 0$ being worked out by S. Tsunoda, I outline below the proof in these two cases; the detailed proofs side-step all the genuine difficulties of working with 3-folds by using on the one hand toric techniques, and on the other the theory of minimal models of "open" surfaces [42]. Incidentally, the Kawamata-Tsunoda treatment of open surfaces is a vindication of Iitaka's extraordinary intuition that the "log catagory" of pairs (X, D) consisting of a normal variety X, and a reduced divisor D, giving X the log canonical divisor K_X+D, should be treated on an equal footing with the complete catagory: [42] gives a treatment of log canonical and log minimal models of surfaces, in complete analogy with (0.1) above.

I believe that the program presented below will also go through in the 3-fold case—although the proofs will require a further large slice of Mori's amazing technical virtuosity.

Step 1. Establish the inductive catagory of projective varieties X with $\boldsymbol{Q}$-factorial terminal singularities; here "$\boldsymbol{Q}$-factorial" (every Weil divisor is a $\boldsymbol{Q}$-Cartier divisor) means that Weil divisors and 1-cycles modulo numerical equivalence give two dual vector spaces $N^1(X)$ and $N_1(X)$.

Step 2. Theorem on the Cone: $NE(X)$ is locally polyhedral in the half-space $(K<0) \subset N_1(X)$.

Step 3. Given an extremal ray R of $NE(X)$ with $K_X R<0$, there exists an elementary contraction ϕ_R: $X \to Y$ corresyonding to R; ϕ_R belongs to one of 3 cases;

(a) ϕ_R is a fibre space as in (4.3), (ii);

or (b) ϕ_R is birational and contracts exactly one prime divisor of X;

or (c) ϕ_R is a isomorphism in codimension 1.

In (a), we're home; in (b) it is easy to check that Y has again $\boldsymbol{Q}$-factorial terminal singularities. In (c) the group (Cl Y)/(Pic Y) has rank 1, so that by Hironaka's theory of characteristic cones, there can exist at most 2 projective small partial resolutions ϕ_R: $X \to Y$ and ϕ_1: $X_1 \to Y$; $-K_X$ is ample for ϕ_R, so that K_{X_1} is relatively ample for ϕ_1 if ϕ_1 exists.

Step 4, In case (c), there exists a morphism ϕ_1: $X_1 \to Y$ such that K_{X_1} is relatively ample for ϕ_1.

Tsunoda seems to prove this by an enumeration of cases, systematising [21], § 4.

Step 5. Finally, we have to prove that the induction terminates.

Part II. Elementary Transformations

§ 5. (-2)-curves

In this section X will be an analytic 3-fold non-singular along a curve C; for most purposes we could replace X by a tubular neighbourhood of C, or by the formal completion of X along C. By *surface* I will mean a germ of a surface near C.

(5.1) **Definition.** A curve $C\subset X$ is a (-2)-*curve* if $C\cong P^1$, and $N_{X|C}\cong\mathcal{O}_{P^1}(a)\oplus\mathcal{O}_{P^1}(b)$, with $(a, b)=(-1, -1)$ or $(0, -2)$.

(5.2) **Remarks.** (a) If $K_XC=0$, and C is contained in a non-singular surface S, $C\subset S\subset X$ with $(C^2)_S=k$, then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{P^1}(k) \longrightarrow N_{X|C} \longrightarrow \mathcal{O}_{P^1}(-k-2) \longrightarrow 0,$$

from which we can deduce: (i) if C is a (-2)-curve, then $k\leqq 0$; (ii) if $0\geqq k\geqq -2$, then C is a (-2)-curve.

(b) If C is a (-2)-curve, then $\mathrm{H}^1(\mathrm{S}^m(N^*_{X|C}))=0$ for all m, and it follows that C satisfies

$$(*)\qquad \begin{cases} K_XC=0, \text{ and } \mathrm{H}^1(\mathcal{O}_Z)=0 \text{ for every subscheme} \\ Z\subset X \text{ with } \operatorname{Supp} Z=C. \end{cases}$$

(c) The condition (*) implies that $\mathrm{H}^1(N^*_{X|C})=0$, so that $(a, b)=(-1, -1)$, $(0, -2)$ or $(1, -3)$; in [19], an example is given to show that the 3rd case can occur, even if C can be contracted to an isolated cDV singularity. Compare (5.15) below. It is clear that if C can be contracted to a point $P\in Y$ by a morphism $f\colon X\to Y$, then (*) is equivalent to $P\in Y$ being Cohen-Macaulay. It would be interesting to know if (*) is either necessary or sufficient for the existence of a contraction.

(5.3) **Definition.** The *width* of a (-2)-curve $C\subset X$ is given by

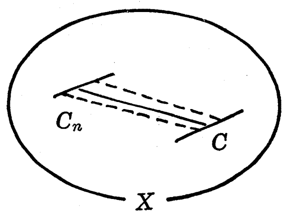
$$n=\operatorname{width}(C\subset X)=\sup\left\{n\ \middle|\ \begin{matrix}\text{there exists a scheme } C_n \text{ with} \\ C\subset C_n\subset X \text{ and } C_n\cong C\times \operatorname{Spec} k[\varepsilon]/\varepsilon^n\end{matrix}\right\};$$

if $n<\infty$, C is *isolated.*

I will study this invariant in some detail in the course of proving Theorem (5.4). We will see that

$$n=1 \Longleftrightarrow (a, b)=(-1, -1);$$

and $n=\infty \Leftrightarrow C$ moves in a scroll S, so is non-isolated. In any case it will be useful to picture C as moving n infinitesimal steps on a stump of a scroll:



(5.4) **Theorem.** *Let $C \subset X$ be a (-2)-curve of width $n \geqq 2$; in the diagram*

$$\begin{tikzcd} C' \arrow[r, phantom, "\subset"] \arrow[dr, "\sim"] & F \arrow[d] \arrow[r, phantom, "\subset"] & X' \arrow[d, "\sigma"] \\ & C \arrow[r, phantom, "\subset"] & X, \end{tikzcd}$$

σ is the blow-up of $C \subset X$; then the exceptional scroll $F \cong \boldsymbol{F}_2$, and letting C' be the negative section of F, $C' \subset X'$ is a (-2)-curve with width $(C' \subset X')=n-1$.

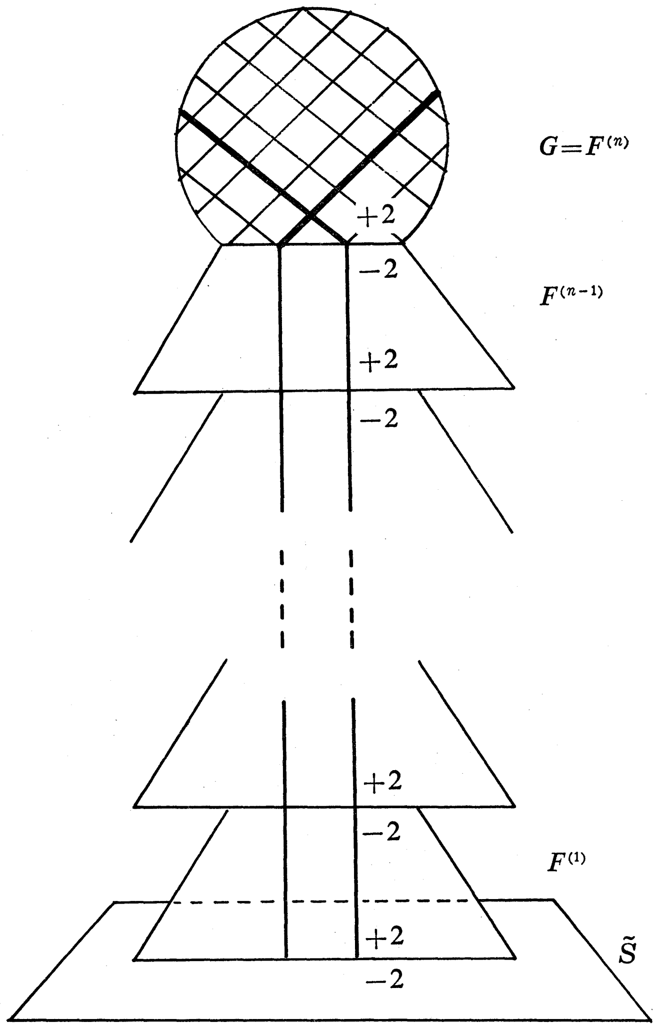
(5.5) It is well-known (see for example [5], or [17], (4.2)) that if $(a, b)=(-1, -1)$, then there is a diagram

$$\begin{tikzcd} & F \subset X_1 \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ C \subset X \arrow[rr, "\rho_C"] & & X^{\rho_C} \supset C' \end{tikzcd}$$

in which the same surface $F \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \subset X_1$ serves in two different ways as the exceptional locus for the two blow-ups σ and σ'.

(5.6) **Corollary.** *There is an elementary transformation $X \xrightarrow{\rho_C} X^{\rho_C}$ defined for any isolated (-2)-curve $C \subset X$.*

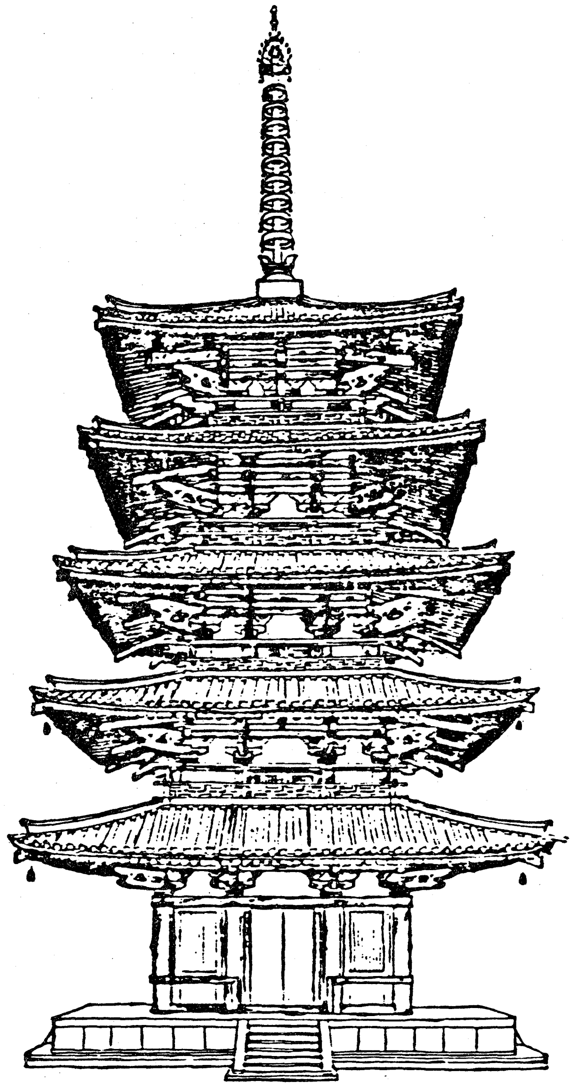
(5.7) To see the corollary, use (5.4) to blow-up $C \subset X$, leading to a curve $C' \subset X'$ of width $(n-1)$; proceeding inductively, after blowing-up $C^{(n-1)}$, I get Pagoda (5.8). This has the two rulings indicated, and by [3],



Pagoda (5.8)

Each of the lower layers $F^{(i)}$ is a copy of $\boldsymbol{F}_2$, meeting $F^{(i+1)}$ in the negative section, and $F^{(i-1)}$ in a disjoint section of self-intersection $+2$. The topmost layer G is a copy of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, with normal bundle of type $(-1, -1)$, intersecting $F^{(n-1)}$ in a curve of type $(1, 1)$. The thick black lines that look like lightning conductors are fibres of the two rulings.

The base $\tilde{S}$, which is optional, is the proper transform of a surface $S \subset X$ with $C \subset S$ and $(C^2)_S = -2$.



The pagoda of Hōryūji, near Nara, Japan
(from Horyuji by T. Nishioka and S. Miyakami, illustrated by K. Hozumi, published by Soshisha, Tokyo, 1980).

(6.10) or [14] can be blown down along either of them. The blowing-down can also be done step-by-step, starting from the top, and this involves nothing more complicated than the Castelnuovo-Moishezon-Nakano criterion for contractions of geometrically ruled surfaces on analytic 3-folds. The reader who has not met this kind of thing before is encouraged to work out for himself the normal bundles at the various stages.

(5.9) To prove Theorem (5.4), I have to explain how the width n is determined; I am indebted to Mohan Kumar for instruction in these matters.

Let $I=J_1$ be the ideal defining $C\subset X$. Then $I/I^2=N^*_{X|C}$, and obviously $(a, b)=(0, -2)$ if and only if there exists a surjection $I/I^2\to\mathcal{O}_C$, or equivalently an ideal J_2 satisfying

$$I^2\subset J_2\subset I, \quad I/J_2\cong\mathcal{O}_C \quad \text{and} \quad J_2/I^2\cong\mathcal{O}_C(2).$$

Then $\mathcal{O}_X/J_2\cong\mathcal{O}_C[\varepsilon]/\varepsilon^2$, so that the width of C is $\geqq 2$ if and only if $(a, b)=(0, -2)$.

Now suppose by induction that there is a sequence of ideals

$$J_k\subset J_{k-1}\subset\cdots\subset J_2\subset J_1\subset\mathcal{O}_X, \tag{1_k}$$

satisfying

$$IJ_i\subset J_{i+1}\subset J_i, \quad J_i/J_{i+1}\cong\mathcal{O}_C \quad \text{and} \quad J_{i+1}/IJ_i\cong\mathcal{O}_C(2)$$

for all $i\leqq k-1$.

In suitable analytic coordinates (x, y, z) around a point of C, $J_k=(x, y^k)$; several assertions in what follows, notably the fact that J_k/IJ_k is a locally free $\mathcal{O}_C$-module of rank 2, and the inclusion $J_2J_{k-1}\subset IJ_k$, can be most conveniently proved using these local coordinates.

For J_k/IJ_k there is the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IJ_{k-1}/IJ_k & \longrightarrow & J_k/IJ_k & \longrightarrow & J_k/IJ_{k-1} & \longrightarrow & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & I/J_2\otimes J_{k-1}/J_k & & & & \mathcal{O}_C(2) & & \\ & & \wr\| & & & & & & \\ & & \mathcal{O}_C. & & & & & & \end{array} \tag{2_k}$$

The equality in the left-hand column is proved by noting that the surjection

$$I\otimes J_{k-1}\longrightarrow IJ_{k-1}/IJ_k$$

kills $J_2\otimes J_{k-1}$ and $I\otimes J_k$.

Thus the chain (1_k) can be extended to a chain (1_{k+1}) if and only if (2_k) splits. This proves:

(5.10) **Proposition.** *C has width n if and only if there exists a chain (1_n) as in (5.9) such that $J_n/IJ_n\cong\mathcal{O}(1)\oplus\mathcal{O}(1)$.*

(5.11) The proof of (5.4) is now just an extended exercise in understanding the definition of the blow-up $\sigma\colon X' \to X$. The exceptional surface is

$$\sigma^{-1}C = F = \boldsymbol{P}_C(I/I^2) \cong \boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{O} \oplus \mathcal{O}(2)) = \boldsymbol{F}_2,$$

with $C' \subset F$ the section of $F \to C$ corresponding to the projection of $\mathcal{O} \oplus \mathcal{O}(2)$ onto its first factor. Thus

$$I \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F) \tag{3_1}$$

and

$$J_2 \cdot \mathcal{O}_{X'} = I_{C'} \cdot \mathcal{O}_{X'}(-F). \tag{3_2}$$

$\mathcal{O}_F(-F)$ is the tautological bundle of $\boldsymbol{P}_C(I/I^2)$, and one checks that

$$C'F = 0 \text{ and } C'K_{X'} = C'(\sigma^* K_X + F) = 0, \tag{4}$$

and C' is a (-2)-curve by (5.2), (a).

For $2 \leqq r \leqq n$ define ideals $J'_{r-1} \subset \mathcal{O}_{X'}$ by setting

$$J_r \cdot \mathcal{O}_{X'} = J'_{r-1} \cdot \mathcal{O}_{X'}(-F). \tag{3_r}$$

Then

$$J'_{n-1} \subset \cdots \subset J'_2 \subset J'_1 = I_{C'} \subset \mathcal{O}_{X'},$$

and Theorem (5.4) obviously follows from (5.10) and the following claim.

(5.12) **Proposition.** *There are isomorphisms of $\mathcal{O}_C$-modules*

(A) $\mathcal{O}_C \cong J_r/J_{r+1} \xrightarrow{\approx} (J'_{r-1}/J'_r) \otimes \mathcal{O}_{C'}(-F)$, *for all* $r \leqq n-1$;

and

(B) $J_r/IJ_r \xrightarrow{\approx} (J'_{r-1}/I_{C'}J'_{r-1}) \otimes \mathcal{O}_{C'}(-F)$, *for all* $r \leqq n$.

To be more precise, the r.h.s. is an $\mathcal{O}_{C'}$-module, and the isomorphism involves identifying C and C'; of course, $\mathcal{O}_{C'}(F) \cong \mathcal{O}_{C'}$, by (4) above.

Proof. The image of $J_r \to J_r \cdot \mathcal{O}_{X'} = J'_{r-1}(-F)$ generates the r.h.s. as $\mathcal{O}_{X'}$-module. In (A) the map

$$J_r/J_{r+1} \longrightarrow J'_{r-1}(-F)/J'_r(-F)$$

is obviously well-defined, and to show that it's onto it's enough to show that the r.h.s. is an $\mathcal{O}_{C'}$-module. But this follows from the diagram

$$\begin{array}{ccc} I_{C'}\cdot J_r\cdot \mathcal{O}_{X'} & & J_{r+1}\cdot \mathcal{O}_{X'} \\ \| & & \| \\ J_2J_r\cdot \mathcal{O}_{X'}(F) & \subset & IJ_{r+1}\cdot \mathcal{O}_{X'}(F), \end{array}$$

where the vertical equalities come from (3_1) and (3_2), and the horizontal inclusion from $J_2J_r \subset IJ_{r+1}$.

For (B) the image is obviously an $\mathcal{O}_{C'}$-module, and the problem is to show that the map is well-defined, that is, to show that

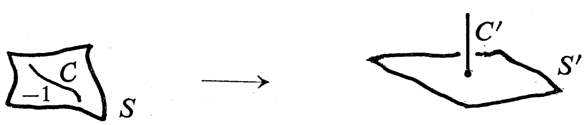
$$IJ_r\cdot \mathcal{O}_{X'} \subset J_2J_r\cdot \mathcal{O}_{X'}(F);$$

but I can deduce this by multiplying both sides by the invertible ideal sheaf $I\cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$, and noting that

$$I^2J_r\cdot \mathcal{O}_{X'} \subset J_2J_r\cdot \mathcal{O}_{X'},$$

because $I^2 \subset J_2$. Q.E.D.

(5.13) **Remarks.** (a) If the (-2)-curve C is contained in a non-singular surface $S\subset X$, then $\rho_{C|S}$ extends to a morphism $S\to S'$. If $(C^2)_S=-1$, then $S\to S'$ contracts the (-1)-curve $C\subset S$ to the point $C'\cap S'$:



If $(C^2)_S=-2$, then $S\to S'$ is an isomorphism, which in § 6 I will use to identify S and S', following Burns and Rapoport. If $(C^2)_S=-k$, then S' has multiplicity $(k-1)$ along C', and $S\to S'$ is the blow-up of S' in C', which coincides with the normalisation.

(b) Pagoda (5.8) can be contracted to a point by a birational morphism; indeed, one can construct a positive cycle $Z=\sum a_iF^{(i)}$ (for example, $a_i=n^2-(n-i)^2$) such that $\mathcal{O}_{X^{(n)}}(-Z)$ is ample on a neighbourhood, so that the assertion follows from [3], (6.10) or [14].

One sees easily that the point is analytically a singularity $(xy=z^2-t^{2n})$, so that X and X^{ρ_C} can be thought of as the two distinct Atiyah-Brieskorn small resolutions corresponding to the graphs of the rational functions $x/(z+t^n)=(z-t^n)/y$ and $x/(z-t^n)=(z+t^n)/y$. This is the Burns-Rapoport definition of ρ_C.

It follows that the analytic neighbourhood of $C\subset X$ is determined by n; hence, for example, every isolated (-2)-curve $C\subset X$ is a (-2)-curve on a smooth surface S, $C\subset S\subset X$.

(c) If X is an open subvariety of a compact Moishezon space, then deformations on C in X are parametrised by a Hilbert scheme or Douady space which is of finite type; this shows that if C is a (-2)-curve of infinite width, then C moves in an analytic scroll S.

(d) There is a sense in which the width 1 curve $C\subset X$ with $N_{X|C}\cong\mathcal{O}_{P^1}(-1)\oplus\mathcal{O}_{P^1}(-1)$ is a universal (-2)-curve; every other fits into a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \cup & & \cup \\ D & \longrightarrow & C, \end{array}$$

(in which $Y\to X$ is a branched cover if $n\neq\infty$, and maps to a surface if $n=\infty$). On the contracted level, these are the maps $\bar{Y}\to\bar{X}$, where $\bar{X}$ is $(xy=z^2-t^2)$, and $\bar{Y}$ is the cover correponding to an nth root of t if $n<\infty$; if $n=\infty$, $\bar{Y}$ is the product $A^1\times(xy=z^2)$ mapping to the $(t=0)$ section of $\bar{X}$.

Then ρ_D can be regarded as a pull-back of the standard elementary transformation of (5.5). This makes clear that an analog in higher dimension is a (-2)-*centre*, that is a codimension 2 subvariety $C\subset X$ which is a P^1-bundle $C\to B$, such that the restriction to each fibre $C_b\cong P^1$ of $N_{X|C}$ is $\mathcal{O}(a)\oplus\mathcal{O}(b)$, with $(a, b)=(-1, -1)$ or $(0, -2)$. Elementary transformations in these (-2)-centres have important application to versal deformations of resolutions of Du Val singularities, and to versal deformations of Kodaira elliptic curves.

(5.14) **Exercise.** Let $C\subset X$ with $C\cong P^1$, and $N_{X|C}\cong\mathcal{O}(a)\oplus\mathcal{O}(b)$, with $a\geqq b$. Show that if $K_XC\geqq 0$ (that is $a+b\leqq -2$), and $(a, b)\neq(-1, -1)$ or $(0, -2)$, then the pagoda constructed from C as in (5.7) cannot terminate in $P^1\times P^1$ meeting the previous layer $F^{(n-1)}$ is a curve of type $(1, 1)$, so that there is no elementary transformation corresponding to C which factorises into blow-ups and blow-downs in such a simple way.

Hint. If $C'\subset F\subset X'\to X$ are as in (5.4), then $F\cong F_{a-b}$, and $FC'=a$; we have $N_{X'|C'}\cong\mathcal{O}(a')\oplus\mathcal{O}(b')$, where *either* $a\geqq a'\geqq b'\geqq(-a+b)$, and $a'+b'=b$, *or* $(a', b')=(-a+b, a)$.

When can $(a', b')=(-1, -1)$?

(5.15) **Example** (Laufer, Pinkham, D. Morrison [19], [26], § 8). The following leads to an example of a $(1, -3)$-curve which gives rise to an elementary transformation.

Start with the 5-fold cD_4 singularity $P\in\mathcal{Y}$ given (at the origin of A^6) by

$$f = x^2 + y^2u + z^2v + t^2uv.$$

The same equation f can be thought of as a quadratic form in the 4 variables (x, y, z, t) over $K = k(u, v)$, defining an ordinary double point $p \in \mathcal{Q}$ over K; because the discriminant u^2v^2 of this quadratic form is a square in K, the two families of generators of the tangent cone to $p \in \mathcal{Q}$ are each defined separately over K, and there are two distinct small resolutions of $p \in \mathcal{Q}$ defined over K.

Over $K(w)$, where $w^2 + u = 0$, one of these resolutions can be described as the blow-up of the ideal

$$J = (x - wy, z - wt) \subset K(w)[x, y, z, t];$$

J is obviously generated by the 4 elements

$$(-w, 1)M, \text{ where } M = \begin{pmatrix} x & y & z & t \\ -uy & x & -ut & z \end{pmatrix}.$$

(5.16) **Lemma.** *Let $I = J \subset k[u, v, x, y, z, t]$ be the ideal generated by the 2×2 minors of M; then the blow-up of $I \cdot \mathcal{O}_{\mathcal{Y}}$ defines a small resolution $\varphi \colon \mathcal{X} \to \mathcal{Y}$. If $Y \subset \mathcal{Y}$ is a general 3-fold section through P, and $X = \varphi^{-1}Y$, then $\varphi \colon X \to Y$ is a small resolution, and $\varphi^{-1}P = C \subset X$ is a $(1, -3)$-curve.*

The final assertion concerning C will follow because $C = f^{-1}P$ is contained in the scheme-theoretic inverse image Z of P, the subscheme $Z \subset \mathcal{X}$ defined by $I_Z = m_{\mathcal{Y},P} \cdot \mathcal{O}_{\mathcal{X}}$, and by the computation below Z is isomorphic to a double line $2l \subset \boldsymbol{P}^2$; thus $N_{X|C}$ has $\mathcal{O}_C(1)$ as a quotient sheaf.

The reader can check that making "the other" resolution of the ordinary double point $p \in \mathcal{Q}$ gives a distinct small resolution $X' \to Y$, and hence there is an elementary transformation $X \xrightarrow{\rho_C} X' = X^{\rho_C}$ defined in C; compare [26], §8, where it is shown how to decompose ρ_C into a sequence of blow-ups and blow-downs.

(5.17) **Problem.** Let $P \in S$ be a Du Val surface singularity, and $f \colon S_1 \to S$ a crepant partial resolution for which $C = f^{-1}P \cong \boldsymbol{P}^1$ (so that $S_1 = T_\Delta$, with $\Delta = \{1 \text{ curve}\}$ in the notation of (1.13)). Now let

$$\begin{tikzcd} C \subset S_1 \arrow[d, "f"'] & \subset X \arrow[d] \arrow[dd, bend left, "g"] \\ S \arrow[d] & \subset Y \arrow[d] \\ 0 & \in T \end{tikzcd}$$

be a 1-parameter deformation of $S_1 \to S$ such that $C \subset X$ is isolated. It should be true that there exist exactly two different morphisms $g_i\colon X_i \to T$ with the same fibres as g; if so, what is the effect on the homology of $C \subset X$ of the corresponding elementary transformation? When can the total space X be nonsingular?

This problem seems to be solved implicitly in [26], §8, using the methods of [34], [35], but it would be nice to have more explicit information, in particular for applications in (8.8) below. Note that factorising these elementary transformations into conventional blow-ups and blow-downs in non-singular centres is not necessarily the best way to understand them.

(5.18) *Proof of* (5.16). The ideal I is generated by

$$\begin{aligned} p&=x^2+y^2u,\\ q&=z^2+t^2u,\\ r&=xt-yz,\\ s&=xz+ytu. \end{aligned}$$

One can check that these verify the identities

$$\begin{aligned} &p+vq=f,\\ &s^2+ur^2+vq^2=qf; \end{aligned} \tag{A}$$

and trivial matrix identities give

$$\begin{aligned} -xs+yur+zp&=0,\\ xr+ys-tp&=0,\\ -xq+zs+tur&=0,\\ yq+zr-ts&=0. \end{aligned} \tag{B}$$

The blow-up of $I\cdot\mathcal{O}_{\mathscr{Y}}$ is the graph $\mathscr{X}\subset\mathscr{Y}\times\boldsymbol{P}^3$ of the correspondence $(p:q:r:s)=(p_1:q_1:r_1:s_1)$, where (p_1, q_1, r_1, s_1) are homogeneous coordinates in $\boldsymbol{P}^3$. Above any point of $\mathscr{Y}$, (A) imply that the fibre of φ is contained in the plane conic

$$(p_1+vq_1=0=s_1^2+ur_1^2+vq_1^2)\subset\boldsymbol{P}^3;$$

in particular, φ is small. Finally, one sees easily from (B) that $\mathscr{X}$ is a non-singular 5-fold. For example, on the affine piece given by $s_1=1$, $\mathscr{X}$ is given by the following 4 relations in the 9 variables $x, \cdots, v, p_1, q_1, r_1$:

$$\begin{aligned} &p_1=-vq_1;\ 1+ur_1^2+vq_1^2=0;\\ &x=yur_1+zp_1;\ t=yq_1+zr_1. \end{aligned}$$

The other affine pieces are if anything easier. Q.E.D.

§ 6. Effect of ρ_C on the homology

(6.1) Let $C \subset X$ be an isolated (-2)-curve, and $C \subset S \subset X$ a smooth surface such that $(C^2)_S = -2$. Then we have the diagram

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ X \arrow[rr, "\rho_C"] & & X^{\rho_C} = X' \\ S \arrow[u, hook, "i"] \arrow[rr, "\approx"] & & S' \arrow[u, hook, "i'"'] \end{tikzcd}$$

Identify S with its image $S' \subset X'$ by means of $\rho_{C|S}$ extended over its removable singularities along C.

(6.2) **Proposition.** *If $f\colon X \dashrightarrow Y$ is an isomorphism in codimension* 1 *between smooth varieties, then proper transform induces a commutative diagram of isomorphisms f':*

$$\begin{tikzcd} \mathrm{Div}\, X \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "f'"] & \mathrm{Div}\, Y \arrow[d] \\ \mathrm{Pic}\, X \arrow[r, "\simeq"] & \mathrm{Pic}\, Y. \end{tikzcd}$$

Proof. X and Y have the same prime divisors and the same function field, so this follows trivially from the definitions.

(6.3) **Theorem** (Burns-Rapoport). *Let $C \subset S \subset X$ be as in* (6.1). *Then the diagram*

$$\begin{tikzcd} \mathrm{Pic}\, X \arrow[r, "\rho'_C"] \arrow[d, "i^*"] & \mathrm{Pic}\, X' \arrow[d, "i'^*"] \\ \mathrm{Pic}\, S \arrow[r, "r_C"] & \mathrm{Pic}\, S \end{tikzcd}$$

commutes, where

$$r_C\colon M \longmapsto M + (MC)C \qquad \text{for } M \in \mathrm{Pic}\, S. \qquad (*)$$

Here and below Pic *is written additively. If S is compact,* Pic S *has a bilinear pairing, and* $(*)$ *just means that r_C is the reflection in the class of C; however,* $(*)$ *is meaningful in any case, since $MC = \deg_C M_{|C}$,* and $\mathcal{O}_S(C) \in \mathrm{Pic}\, S$.

(6.4) To prove this I first "resolve" the proper transform map; since σ and σ' are composites of blow-ups,

$$\operatorname{Pic} \tilde{X} = \sigma^* \operatorname{Pic} X \oplus \bigoplus_{i=1}^{n} \boldsymbol{Z} \cdot F^{(i)}$$

and

$$\operatorname{Pic} \tilde{X} = (\sigma')^* \operatorname{Pic} X' \oplus \bigoplus_{i=1}^{n} \boldsymbol{Z} \cdot F^{(i)},$$

where the exceptional components $F^{(i)}$ are the layers of Pagoda (5.8).
Hence if $L \in \operatorname{Pic} X$ and $L' \in \operatorname{Pic} X'$ are such that $L' = \rho'_C L$, then

$$\sigma^* L = (\sigma')^* L' + D, \text{ with } D = \sum r_i F^{(i)}.$$

(6.5) *Claim.*

$$D = -(LC) D_0, \quad \textit{where} \quad D_0 = \sum i F^{(i)}.$$

Indeed, letting f_i be the ruling of $F^{(i)}$ for $1 \leqq i \leqq n-1$; f_i is contracted by both σ and σ', so that

$$0 = (\sigma^* L - (\sigma')^* L') f_i = D f_i = r_{i-1} - 2r_i + r_{i+1}.$$

This implies that D is a multiple of D_0.

Now let h and h' be the two rulings of $F^{(n)} \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, with h contracted by σ and h' by σ'. Then $\sigma_{|h'} \colon h' \xrightarrow{\approx} C$, so that

$$\left.\begin{array}{r} \sigma^* L h' = LC \\ \sigma^* L h = 0 \end{array}\right\}, \text{ and similarly } \left\{\begin{array}{l} (\sigma')^* L' h' = 0 \\ (\sigma')^* L' h = L' C; \end{array}\right.$$

the claim follows.

(6.6) *Proof of* (6.3). If $\tilde{S} \subset \tilde{X}$ is the proper transform of S, (the base of Pagoda (5.8)), then identifying $S = \tilde{S} = S'$, I can calculate i^* and i'^* from the diagram

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\
X & \tilde{S} \arrow[u, hook, "\tilde{i}"'] & X' \\
S \arrow[u, hook, "i"] \arrow[ur, equal] & & S' \arrow[u, hook, "i'"'] \arrow[ul, equal]
\end{tikzcd}$$

I get

$$i^* L = (\sigma \circ \tilde{i})^* L = \tilde{i}^* ((\sigma')^* L' - (LC) D_0) = i'^* L' - (LC) C,$$

since $\tilde{S}$ (the base of Pagoda (5.8)) meets D_0 just in the curve C. Q.E.D.

The reader interested only in the proof of (2.12) should proceed directly to § 7.

(6.7) An entirely similar result holds on replacing the functor Pic by $H^2(\ ,Z)$, $H^2_c(\ ,Z)$ or $H_2(\ ,Z)$; the class C on the r.h.s. of $(*)$ has just to be interpreted pragmatically as the class of C in $H_2(S,Z)$ or $H^2_c(S,Z)$. Compare [10], (7.8).

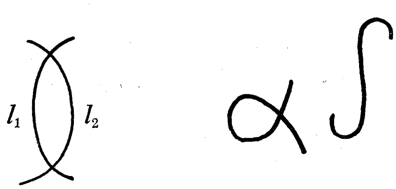
An advantage of working with homology is that if X is a tubular neighbourhood of a configuration of (-2)-curves on a surface S, then X is homotopy equivalent to S, and identifying $H_2(S,Z)$ with $H_2(X,Z)$, $(*)$ can be interpreted as the action of r_C on the whole of $H_2(S,Z)$.

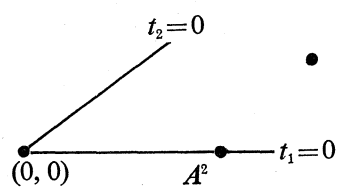
(6.8) **Example** (based on an earlier more complicated example of Shepherd-Barron).

Let $f\colon X\to A^2$ be the 2-parameter family of elliptic curves given by

$$y^2=((x-a)^2-t_1)((x-b)^2-t_2),$$

where $a\neq b$ are constants, and (t_1,t_2) are coordinates in A^2. Near $(0,0)$, f can be regarded as the versal deformation of the Kodaira elliptic curve I_2; that is, $f^{-1}(0,0)=l_1\cup l_2$ is a pair of P^1's meeting transversally in 2 points; $f^{-1}P$ is a non-singular elliptic curve if $P\notin(t_1t_2=0)$, and is a nodal curve if $P\in(t_1t_2=0)\backslash(0,0)$. Each l_i is an isolated (-2)-curve.





Define the abstract reflection group G by

$$G=\langle r_1,r_2\,|\,r_1^2=r_2^2=1\rangle;$$

then G is infinite, and for $g\in G$ we can construct a model X^g birational to X:

$$X\dashrightarrow X^g=X^{\rho_{l_1}\rho_{l_2}\cdot\,\cdot\rho_{l_1}}$$

if $g=r_1r_2\cdots r_1$, by successive elementary transformations in l_1 and l_2. For $g\neq g'$, $X^g \dashrightarrow X^{g'}$ is not an isomorphism.

It is easy to construct global 3-folds X of Kodaira dimension 0, 1 or 2, having elliptic fibrations over a surface with fibres as in this example; indeed, this is a generic case. Thus such a 3-fold will usually have infinitely many distinct non-singular models with $|K|$ free.

(6.9) To prove the assertions in (6.8), set $t=t_1+t_2$, and consider the smooth surface $S\subset X$ given by $t=0$. Then S has an elliptic pencil $S\to \boldsymbol{A}^1$ given by t_1, and this pencil has an I_2 fibre over 0, consisting of $l_1\cup l_2$; hence each $l_i\subset X$ is a (-2)-curve by (5.2), (a), and is isolated because $f\colon X\to \boldsymbol{A}^2$ has no other reducible fibres. ρ_{l_1} induces an isomorphism on S by (5.13), (a). The group G acts faithfully on $H_2(S, \boldsymbol{Z})$, proving the final assertion.

§ 7. Proof of (2.12)

(7.1) I return to the notation and hypotheses of (2.11). Let V be the $\boldsymbol{R}$-vector space spanned by symbols e_k, in 1-to-1 correspondence with the components of $h^{-1}P=\bigcup E_k\subset Y_0$; V is given the usual negative definite pairing

$$e_k^2=-2,\qquad e_ke_{k'}=\begin{cases}0 & \text{if } E_k\cap E_{k'}=\phi,\\ 1 & \text{if } E_k\cap E_{k'}=\{pt\}.\end{cases}$$

V contains a root system R for which the e_k are the simple roots:

$$R=\{r=\textstyle\sum n_ke_k \mid n_k\in \boldsymbol{Z},\ r^2=-2\}.$$

V can be considered as a subspace of $H_2(Y_0, \boldsymbol{R})$, or of $H^2_c(Y_0, \boldsymbol{R})$; its dual space V^* can be considered as a quotient of $H^2(Y_0, \boldsymbol{R})$. Since the pairing on V is negative-definite, we have a canonical identification of V and V^*, and V^* contains a root system R^*, with simple roots e_k^*, and has a metric defined by

$$d(a, b)^2=-\langle a-b, a-b\rangle.$$

(7.2) Now let $K^*\subset V^*$ denote the closed Weyl chamber defined by

$$K^*=\{a\in V^* \mid a(e_k)\geqq 0 \text{ for all } k\}.$$

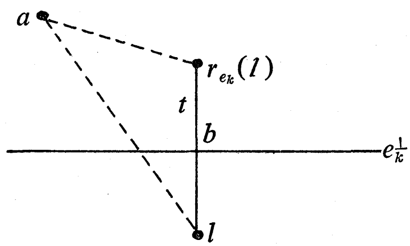
Now fix, once and for all, an interior point $a\in (K^*)^{\mathrm{int}}$, that is a point such that $a(e_k)>0$ for all k.

(7.3) **Lemma** (Bourbaki, see [7], Ch. V, § 3, n$^{\mathrm{o}}$. 3. 1. 2). *Let*

$l \in V^* \backslash K^*$; then there exists an e_k such that $l(e_k) < 0$, and

$$d(a, r_{e_k}(l)) < d(a, l).$$

Proof. Set $l = b - t$, $r_{e_k}(l) = b + t$, with $b(e_k) = 0$, and $2t = l(e_k) e_k^*$; then



$$\begin{aligned} d(a, l)^2 - d(a, r_{e_k}(l))^2 &= -\langle a-b+t, a-b+t \rangle + \langle a-b-t, a-b-t \rangle \\ &= -4\langle a, t \rangle = -2l(e_k)a(e_k) > 0. \end{aligned}$$

Q.E.D.

(7.4) **Corollary.** *For any $l \in V^*$ there exists a sequence $e_{k_1}, \cdots, e_{k_n}$ such that, setting*

$$l_0 = l, \quad l_i = r_{e_{k_i}}(l_{i-1}) \quad \textit{for } i = 1, \cdots, n,$$

we have

$$l_{i-1}(e_{k_i}) < 0, \tag{1}$$

and

$$l_n \in K^*, \quad \textit{that is } l_n(e_k) \geqq 0 \quad \textit{for each } k. \tag{2}$$

Here condition (2) will mean that I eventually end up with a model Y' for which $\mathscr{L}$ is nef whereas (1) will guarantee that I never need to make an elementary transform in a non-isolated (-2)-curve.

Proof. At each stage the distance to a decreases strictly; and we know that the Weyl group generated by the r_{e_k} is finite (because it acts faithfully on the finite set of roots R), so that l has at most finitely many translates. Q.E.D.

(7.5) *Proof of* (2.12). In the notation of (2.11), let $\mathscr{L} \in \operatorname{Pic} Y$ be such that $\mathscr{L}$ is nef over the Du Val locus. Let V and V^* be as in (7.1), and let $l \in V^*$ be given by $l(e_k) = \mathscr{L} E_k$.

Let $e_{k_1}, \cdots, e_{k_n}$ be as in (7.4); I claim that

$$Y \longrightarrow Y' \longrightarrow \cdots \longrightarrow Y^{(n)}$$

can be defined in such a way that $Y^{(i-1)} \dashrightarrow Y^{(i)}$ is the elementary transformation in

$$E_{k_i} \subset Y_0 \xhookrightarrow{i^{(i-1)}} Y^{(i-1)},$$

which is an isolated (-2)-curve.

Indeed, on identifying $Y_0 \subset Y$ with its proper transform under $Y \dashrightarrow Y^{(i-1)}$, all the curves E_k of $h^{-1}P$ remain (-2)-curves, by (5.2), (a); and if E_k is a non-isolated curve, then E_k has to be of the form $F_j \cap Y_0$, for some $F_j \subset Y^{(i-1)}$ an exceptional surface over a component of Σ. Then E_k is numerically equivalent to a curve $E'_k = F_j \cap Y_t$ for $t \neq 0$, so that $l_{i-1}(e_k) = \mathscr{L}E_k = LE'_k \geqslant 0$ by the hypothesis that $\mathscr{L}$ is nef over $X - P$.

Thus the curve E_i in which I intend to carry out the next elementary transformation $Y^{(i-1)} \dashrightarrow Y^{(i)}$ is isolated so that (6.3) applies. By construction, $\mathscr{L}$ is nef on $Y^{(n)}$. Q.E.D.

§ 8. Comments and problems on elementary transformations

(8.1) Let $X \to T$ be a family of Du Val surface singularities which admits a simultaneous resolution $Y \to X \to T$; in (8.2) I use the notation of (2.11) and (7.1). Recall the convention on j and k made at the end of (2.11). For any surface $F_j \subset Y$ contracted to a component of $\sum$, let $F_j \cap Y_0 = \sum n_{jk} E_k$. Then $f_j = \sum n_{jk} e_k \in V$ is a root of R, and the f_j form the primitive roots of a subroot system $R_t \subset R$, corresponding to the Du Val singularities of X_t for $t \neq 0$. Write $V_0^* \subset V^*$ for the following union of Weyl chambers of V^*:

$$V_0^* = \{l \in V^* \mid l(f_j) \geqq 0 \text{ for all } j\}.$$

(8.2) **Theorem** (Brieskorn [9]). *The hypotheses are as in* (8.1).

(I) *Simultaneous resolutions of the family $X \to T$ are in bijection with the set $\mathscr{S}$ of Weyl chambers $K^* \subset V^*$ contained in V_0^*.*

(II) *Let $Y \to X \to T$ be a simultaneous resolution, and $K^* \in \mathscr{S}$ the corresponding Weyl chamber. Then for any k, the (-2)-curve E_k of Y_0 is isolated in Y if and only if $r_{e_k}(K^*) \subset V_0^*$.*

(III) *Any two simultaneous resolutions can be obtained from one another by a sequence of elementary transformations in isolated (-2)-curves.*

Sketch proof. (I) is proved in [9], § 3 under the extra hypothesis that $P \in X$ is an isolated singularity (that is, there are no F_j); however, the proof goes through automatically in the present slightly more general case. First of all, assuming the existence of one simultaneous resolution, V and its root system R can be identified with constructions made from the class group of $P \in X$. The correspondence one way just associates to $Y \to X \to T$ the Weyl chamber $K^* \subset V^*$ given by

$$K^* = \{l \in V^* \mid l(e_k) \geqq 0 \text{ for all } k\}.$$

Conversely, given a Weyl chamber $K^* \in \mathscr{S}$, a rational point $l \in (K^*)^{\mathrm{int}} \cap V_Q^*$ corresponds via Hironaka's theory of characteristic cones to an ideal of $\mathcal{O}_X$ whose blow-up is a simultaneous resolution.

(II) By the usual properties of Weyl chambers, K^* and $r_{e_k}(K^*)$ are separated by a single hyperplane $e_k^\perp$; since V_0^* is a union of Weyl chambers, given $K^* \subset V_0^*$,

$$\begin{aligned} r_{e_k}(K^*) \not\subset V_0^* &\Leftrightarrow e_k^\perp \text{ is a wall of } V_0^* \\ &\Leftrightarrow e_k = f_j \qquad \text{for some } j. \end{aligned}$$

This is statement (II).

(III) follows from (2.12): given Y and Y' with corresponding Weyl chambers K^* and K'^*, an interior rational point of K'^* corresponds to a relatively ample invertible sheaf on Y'; by (2.12) a succession of elementary transformations in isolated (-2)-curves will knock it into K^*.

Q.E.D.

(8.3) **Remarks.** (a) The family $Y \to T$ is differentiably locally trivial, so that it provides an identification

$$\alpha_Y : H_2(Y_0, \boldsymbol{Z}) \longrightarrow H_2(Y_t, \boldsymbol{Z}),$$

where $t \neq 0$ is some fixed base-point. If $Y' \to X \to T$ is another simultaneous resolution, then fibre-by-fibre we have $Y'_t \cong Y_t$ (for all $t \in T$); this is the traditional enigmatic assertion that "Y and Y' are different families with the same members". (Compare the equally bizarre pronouncement: "the moduli stack of surfaces is locally non-separated".) The new identification

$$\alpha_{Y'} : H_2(Y_0, \boldsymbol{Z}) \longrightarrow H_2(Y_t, \boldsymbol{Z})$$

is *a priori* different, and an assertion implicit in (8.2) is that the homological picture is faithful: different families Y and Y' give rise to different identifications α_Y and $\alpha_{Y'}$. Thus we can think of passing from Y to Y' as moving $H_2(Y_0, \boldsymbol{Z})$ around with respect to the fixed $H_2(Y_t, \boldsymbol{Z})$.

(b) In the case that $P \in X$ is an isolated singularity, (8.2) can be paraphrased by saying that the set of simultaneous resolutions of $X \to T$

form a principal homogeneous space under the Weyl group $W(R)$, with the generators r_{e_k} of $W(R)$ acting as elementary transformations in the corresponding (-2)-curves E_k. I was amazed to discover that such a substantial case of the "factorisation problem", with such a beautiful proof, has been implicit in the literature for so long.

(c) There seems to be a notion of "obstructed principal homogeneous space" implicit in (8.2), and I would be grateful if someone would inform me whether this occurs in other contexts.

(8.4) **Problem.** Let $P \in X$ be any cDV point, and $t \in m_P \subset \mathcal{O}_{X,P}$ a general elephant; by (1.5), after taking an appropriate root of t, the covering X' can be thought of as the total space of a family of Du Val singularities admitting a simultaneous resolution, so that (8.2), together with (1.14) can be used to interpret any questions concerned with partial resolutions of $P' \in X'$ in terms of the root system $R \subset V$ corresponding to the section S: $(t=0)$. What about the original point $P \in X$? It seems likely that there should be an identification of the following 3 types of object:

(i) small partial resolutions of $P \in X$;

(ii) suitable cones in V satisfying covariance properties with respect to the monodromy of the family t': $X' \to T'$;

(iii) suitable cones in the class group of $P \in X$.

There is also the question of extending these ideas to all quick singularities.

The simplest case of this problem has been seen in (1.16) above; a direct relationship between (algebraic or analytic) small partial resolutions and the (algebraic or analytic) local class group of $P \in X$ is given by a purity theorem of Van der Waerden (see E.G.A. IV_4, 21.12.12). Some general results in this direction have been obtained by Shepherd-Barron.

(8.5) According to (0.8), (a) and (0.15), we have the right to expect that if X_1 and X_2 are two birationally equivalent minimal models with $\kappa_{\mathrm{num}} \geqq 0$, the birational map $X_1 \dashrightarrow X_2$ is an isomorphism in codimension 1. In the remainder of this section I want to suggest that a much more precise statement concerning $X_1 \dashrightarrow X_2$ can be expected, and to indicate how the ambiguity in the choice of minimal models may eventually be reduced to combinatorics, in the spirit of (0.9).

In order to have a compact statement, it is convenient in (8.6–8) to restrict attention to non-singular minimal models X, although as in the corresponding part (4.6–12) above one could hope for precise results in the general case of quick singularities.

(8.6) **Conjecture.** *Let f: $X \dashrightarrow X'$ be a birational map, where X and*

X' are both non-singular 3-folds with K nef. Then f is a composite ofe lementary transformations in isolated rational curves C with $K_X C=0$.

Finding the precise notion of elementary transformation is part of the problem; one could hope to get by with the candidates given implicitly in (5.17).

It should be clear from (8.2), from [10], § 7, and from [26], § 8 that (8.6) is true in many interesting cases, and the program outlined in (8.8) below is based on an extrapolation of these known cases.

Paradoxically, perhaps one of the most convincing reason for believing in a statement like (8.6) in the $\kappa_{\mathrm{num}} \geqq 0$ case is given by the Fano-Manin-Iskovskikh results (surveyed in [38]) on factorisation of birational maps between Fano 3-folds, which is by far the deepest (and most neglected, outside Moscow) aspect of work on the Lüroth problem.

(8.7) **Conjecture.** *Let f: $V \to C$ be a fibre space from an algebraic 3-fold V to a curve C, and suppose that the general fibre is a surface with $\kappa=0$; then there exists a model g: $X \to C$ birational to V such that X is a relatively minimal model (that is, X has only quick singularities, g is a morphism, and K_X is nef on the fibres of g).*

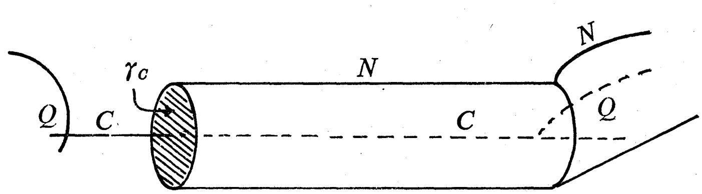
(8.8) **Subconjectures of** (8.6–7). A suitable form of statements (a)—(e) below will prove (8.6) and (8.7).

(a) *In* (8.6), *f is an isomorphism in codimension* 1, *and if $Q \subset X$ is the exact locus of indeterminacy of f, the components of Q are rational curves C with $K_X C=0$.*

(b) *Let N be a tubular neighbourhood of Q, that is, for a suitable metric and $\varepsilon > 0$,*

$$N=\{x \in X \mid d(x, Q) < \varepsilon\}.$$

With each component C of Q, associate the class $\gamma_C \in H^2(N-Q, \mathbf{Z})$ corresponding to the intersection of N with a hypersurface cutting C transversally:



Then as C runs through the components of Q, $\gamma_C \in H^2(N-Q, \mathbf{Z})$ form the simple roots of a root system in $H^2(N-Q, \mathbf{Z}) \otimes \mathbf{R}$.

Note that *a priori* there is no inner product on $H^2(N-Q, Z)$, and when there is an inner product it need not be definite, so that "root system" has to be understood in the general sense of [7], Ch. V and [20].

Let Q' be the locus of indeterminacy of f^{-1}, and $N' \subset X'$ the tubular neighbourhood of N' such that f induces an isomorphism

$$f\colon N-Q \xrightarrow{\approx} N'-Q';$$

the point of (b) is that the lattice $H^2(N-Q, Z) \cong H^2(N'-Q', Z)$ now contains two distinct sets of simple roots, so that it should be possible to pass from one to the other by a sequence of reflections, as in the proof of (2.12). It should be noted that by examples such as (6.8), we know that the combinatorics of elementary transformations is at least as complicated as the word problem in a (possibly infinite) reflection group, and (b) is an attempt at getting a representation of this combinatoric problem in homology.

(c) *An elementary transformation ρ_C can be defined in some curve $C \subset Q$, and the effect of this on the set of simple roots $\gamma_{C_i} \in H^2(N-Q, Z)$ is to move the classes γ_{C_i} by the reflection τ_{γ_C}.*

(d) *An isomorphism*

$$f\colon N-Q \xrightarrow{\approx} N'-Q'$$

extends to an isomorphism $N \xrightarrow{\approx} N'$ if and only if $Q \xrightarrow{\approx} Q'$ and the classes γ_C correspond under f^*.

(e) *Let $g\colon X \dashrightarrow X$ be a birational transformation of finite order, where X is a minimal model with $\kappa_{\mathrm{num}} \geqq 0$. Then if $Q \subset X$ is the indeterminacy locus of g, Q can be contracted to a finite set by a birational morphism $h\colon X \to \bar{X}$ (such that h is an isomorphism outside Q; $\bar{X}$ should have only isolated cDV points, and g acts biregularly on $\bar{X}$).*

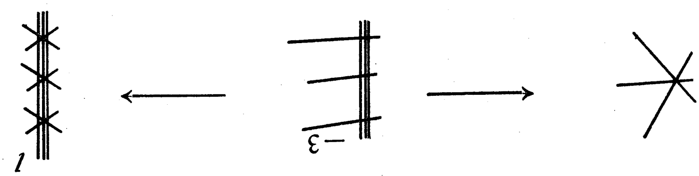
The point of (e) is that the condition that g be of finite order should exclude cases such as (6.8) above, and imply that Q is negative in a suitable sense.

(e), together with known results concerning the degeneration of surfaces with $\kappa=0$ (see [17], [24], [21]) implies (8.7). For example in the $K3$ case, Kulikov's epoch-making result is that after pulling back $f\colon V \to C$ by a cyclic cover $C' \to C$ we can replace V' by a *Kulikov model*:

$$\begin{tikzcd}
X' \arrow[dr, "f'"'] & V' \arrow[l, dashed] \arrow[r] \arrow[d] & V \arrow[d, "f"] \\
& C' \arrow[r] & C
\end{tikzcd}$$

X' is a smooth model birational to V' and such that $K_{X'}$ is numerically (in fact analytically) trivial on the fibres of f'. The cyclic group Gal(V'/V) now acts on X' by birational transformations, and (e) tells us how to get a model $\bar{X}'$ on which it acts biregularly, so that a quotient $X=\bar{X}'/(g)$ birational to the original V can be constructed.

(8.9) *Added in proof.* The above assertion that (e) implies (8.7) is false, because the singularities of the quotient $X=\bar{X}'/(g)$ will not in general be canonical. This is an interesting point even for degeneration of elliptic curves. For example, the Kodaira elliptic of type IV is a case of potentially good reduction: after a triple cover, the semi-stable model is a smooth elliptic curve on which the covering group $Z/3$ acts with 3 fixed points. The quotient is an elliptic surface whose special fibre $\cong 3l \subset \boldsymbol{P}^2$, with 3 rational triple points; this is obtained from the Kodaira model by a blow-up, and a contraction of 3 curves of self-inteasection -3:



This model (X, l), considered as a log surface, is a minimal model in the sense of [42], and is more closely related to the semi-stable model than the conventional minimal model!

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