Advanced Studies in Pure Mathematics 1, 1983
Algebraic Varieties and Analytic Varieties
pp. 69–99

Algebraic Methods in the Theory of Algebraic Threefolds

—surrounding the works of Iskovskih, Mori and Sarkisov—

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This is an expository report on the recent developments in the theory of algebraic threefolds, with emphasis on algebraic (not transcendental) methods. One of the works which this report is based on is Mori's theory [24, 25], which, the reporter believes, is a real break-through in the theory of algebraic threefolds and will be published soon in a complete paper. Though Mori's theory treats mainly the case where the base field is an algebraically closed field of characteristic zero and the canonical divisor of an algebraic threefold is not numerically effective, he succeeded in giving a clear-cut view of this case and gave significant directions in pointing out divisors which should be contracted first in order to construct minimal models of algebraic threefolds. Moreover, we expect that his theory can be extended, to some extent, to the case of positive characteristic, since his approach is similar to that of Mumford [29], in which Mumford renovated some parts of the arguments in the classification theory of algebraic surfaces, for the purpose of extending the theory to the case of positive characteristic. This expectation is one of the reasons why we stick to the algebraic methods in this report.

In the first half of this report, we shall give a concise presentation of the theory of algebraic threefolds along the line set by Mori's theory. In the second half, several basic results will be given on conic bundles, which, on the analogy of vector bundles, have more to be worked out algebraically.

The plan of this report is as follows:

Received August 30, 1981.

§ 1. Outline of the theory

An algebraic threefold X is a nonsingular projective variety of dimension 3 defined over an algebraically closed field k of characteristic zero, which we fix as the base field throughout this report. Let K_X be the canonical divisor. Mori considers an algebraic threefold X with K_X not numerically effective. Then, let $\overline{NE}(X)$ be the closed convex cone spanned by all effective 1-cycles on X modulo numerical equivalence (denoted by $\equiv$). If K_X is not numerically effective then the side of $\overline{NE}(X)$ with $-K_X > \varepsilon L > 0$, (a bit away from the hyperplane $-K_X = 0$, where L is an ample divisor on X), is a convex polyhedral cone, which is spanned by finitely many rays R_i $(1 \leqq i \leqq n)$ such that

(1) $R_i = \boldsymbol{R}_+[l_i]$ with a rational curve l_i, where $\boldsymbol{R}_+$ is the set of non-negative real numbers, and

(2) $z_1 + z_2 \in R_i$ and $z_1, z_2 \in \overline{NE}(X)$ imply $z_1, z_2 \in R_i$; such rational curves l_i are called *extremal rational curves* and rays R_i are called *extremal rays*. Mori shows that if K_X is not numerically effective, there exists an extremal rational curve l.

If the Picard number $\rho(X)$ equals 1, then $-K_X$ is ample and hence X is a Fano threefold with $\rho(X)=1$. Under an additional hypothesis that $\rho(X) \geqq 2$, one may choose a numerically effective divisor H such that $H^{\perp} \cap \overline{NE}(X) = R = \boldsymbol{R}_+[l]$, where $H^{\perp} = \{z \in \overline{NE}(X); (H \cdot z) = 0\}$. It can be shown that $(H^3) > 0$ if and only if R is not numerically effective.

On the other hand, the Riemann-Roch theorem and a variation of the Kodaira vanishing theorem imply the following equalities:

$$h^0(mH) = \chi(mH) = \tfrac{1}{6}(H^3)m^3 + \tfrac{1}{4}(c_1(X) \cdot H^2)m^2 + \tfrac{1}{12}(H \cdot c_1(X)^2 + c_2(X))m + \tfrac{1}{24}(c_1(X) \cdot c_2(X))$$

for a sufficiently large integer m. Then Mori's theory asserts that the results (1)$\sim$(4) below hold true. Note that if R is not numerically effective there exists an irreducible, reduced divisor D on X such that $(D \cdot l) < 0$. Throughout the assertions (1) $\sim$ (4), the morphism $\Phi_{|mH|}: X \to \Phi_{|mH|}(X)$ is called *the contraction of* R and denoted by cont_R, while $\Phi_{|mH|}(X)$ is denoted by $\mathrm{cont}_R(X)$.

(1) If R is not numerically effective and $H \cdot D \equiv 0$ on D, then the divisor D is uniquely determined by the extremal ray $R = \boldsymbol{R}_+[l]$ and D is isomorphic to either $\boldsymbol{P}^2$ or a possibly singular quadric surface V in $\boldsymbol{P}^3$. If $D \cong \boldsymbol{P}^2$ then $\mathcal{O}_D(D) \cong \mathcal{O}_{\boldsymbol{P}^2}(-1)$ or $\mathcal{O}_{\boldsymbol{P}^2}(-2)$. If $D \cong V$ then $\mathcal{O}_D(D) \cong \mathcal{O}_V(-1)$, where $\mathcal{O}_V(1)$ is the sheaf of hyperplane sections. Moreover, if m is sufficiently large, the mapping $\Phi_{|mH|}: X \to X_m \subset \boldsymbol{P}^{\dim |mH|}$ is a birational morphism onto a normal threefold X_m such that $P := \Phi_{|mH|}(D)$ is a point,

$\Phi_{|mH|}$ induces an isomorphism $X-D \xrightarrow{\sim} X_m-\{P\}$, and

(i) P is a smooth point if $D \cong \boldsymbol{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_{\boldsymbol{P}^2}(-1)$,

(ii) P is an ordinary double point of the analytic type $\hat{\mathcal{O}}_P \cong k[[x, y, z, u]]/(x^2+y^2+z^2+u^2)$ if $D \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, i.e., D is a nonsingular quadric in $\boldsymbol{P}^3$,

(iii) P is a double point of the analytic type $\hat{\mathcal{O}}_P \cong k[[x, y, z, u]]/(x^2+y^2+z^2+u^3)$ if D is a quadric cone in $\boldsymbol{P}^3$,

(iv) P is a quadruple point of the analytic type $\hat{\mathcal{O}}_P \cong k[[x, y, z]]^{(2)}$ if $D \cong \boldsymbol{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_{\boldsymbol{P}^2}(-2)$, where $k[[x, y, z]]^{(2)}$ is the invariant subring of $k[[x, y, z]]$ under the $\boldsymbol{Z}/2\boldsymbol{Z}$-action defined by $(x, y, z) \mapsto (-x, -y, -z)$.

The divisor D is called *the exceptional divisor associated with R.*

(2) If R is not numerically effective and $H \cdot D \not\equiv 0$ on D, then D is uniquely determined by the extremal ray R and D is a $\boldsymbol{P}^1$-bundle over a nonsingular complete curve Y. Moreover, for a sufficiently large integer m, $\varphi_m := \Phi_{|mH|}: X \to X_m \subset \boldsymbol{P}^{\dim|mH|}$ is a birational morphism onto a nonsingular projective threefold X_m, φ_m contracts D to a curve Y_m in X_m isomorphic to Y, and φ_m induces an isomorphism $X-D \xrightarrow{\sim} X_m - Y_m$. As in the assertion (1), D is called *the exceptional divisor associated with R.*

(3) If $(H^3)=0$ and $(c_1(X) \cdot H^2)>0$, then $f := \Phi_{|mH|}: X \to Y \subset \boldsymbol{P}^{\dim|mH|}$ is a morphism onto a nonsingular projective surface Y, and $f: X \to Y$ is *a conic bundle*, i.e., every fiber X_y of f is isomorphic to a possibly singular conic in $\boldsymbol{P}^2$, where m is a sufficiently large integer. Moreover, we have:

(i) $[C] \in R=\boldsymbol{R}_+[l]$ if and only if $\dim f(C)=0$, where C is an irreducible curve,

(ii) $\rho(X)=\rho(Y)+1$,

(iii) $f^{-1}(Z)$ is irreducible for any irreducible curve Z on Y.

(4) If $(H^3)=0$ and $(c_1(X) \cdot H^2) \leqq 0$, then $H^2 \equiv 0$, $(c_1(X)^2+c_2(X) \cdot H)>0$ and $\rho(X)=2$. Moreover, for a sufficiently large integer m, $f := \Phi_{|mH|}: X \to Y \subset \boldsymbol{P}^{\dim|mH|}$ is a surjective morphism onto a nonsingular complete curve Y whose arbitrary fiber X_y is an irreducible and reduced surface with $\omega_{X_y}^{-1}$ ample, where ω_{X_y} is the dualizing invertible sheaf of X_y; $f: X \to Y$ is called *a del Pezzo fiber space.* For any irreducible curve C on X, $[C] \in R=\boldsymbol{R}_+[l]$ if and only if $\dim f(C)=0$.

We have an analogue for a nonsingular projective surface S with K_S not numerically effective. As in the case of algebraic threefolds, there exists an extremal rational curve l on S. If $\rho(S)=1$ then $-K_S$ is ample and hence S is isomorphic to $\boldsymbol{P}^2$. So, assume that $\rho(S) \geqq 2$. Then there exists a numerically effective divisor H such that $H^\perp \cap \overline{NE}(S)=R:=\boldsymbol{R}_+[l]$. The extremal ray R is not numerically effective, i.e., $(l^2)<0$, if and only if $(H^2)>0$. In this case, l is an exceptional curve of the first kind, and, for a sufficiently large integer m, $\varphi_m := \Phi_{|mH|}: S \to S' \subset \boldsymbol{P}^{\dim|mH|}$ is a birational morphism onto a nonsingular projective surface S' such that P

$:=\varphi_m(l)$ is a point and $\varphi_m: S-l\to S'-\{P\}$ is an isomorphism. Conversely, if l is an exceptional curve of the first kind on S, then l is an extremal rational curve such that $R:=\boldsymbol{R}_+[l]$ is not numerically effective. Thus the absence of extremal rational curves which are not numerically effective is equivalent to the condition that S is relatively minimal. On the other hand, if R is numerically effective, then $f:=\Phi_{|mH|}: S\to C\subset \boldsymbol{P}^{\dim|mH|}$ is a surjective morphism onto a nonsingular complete curve C for a sufficiently large integer m, and S is, in fact, a $\boldsymbol{P}^1$-bundle over C. Moreover, an irreducible curve Z is a fiber of f if and only if $[Z]\in R$.

However, in the case of algebraic threefolds, the assertion (1) shows that we cannot necessarily contract D to a smooth point or a smooth curve. Reid remarked that the point P has only canonical (hence rational) singularity of index 1 (in the cases (ii) and (iii)) and index 2 (in the case (iv)). The fact that the new threefold X' $(:=X_m)$ obtained from X by contracting D acquires a canonical singularity seems to be an obstacle (or a key result) to finding (or defining) a relatively minimal model of X in the correct sense.

Meanwhile, Mori's theory contains the following result: Let $f: X\to Y$ be a birational morphism of nonsingular projective threefolds. If f is not an isomorphism, then X contains an extremal rational curve l such that $f_*l=0$. The extremal ray $R:=\boldsymbol{R}_+[l]$ is not numerically effective and f factors through $\mathrm{cont}_R: X\to X_1:=\mathrm{cont}_R(X)$, i.e., $f=g\cdot\mathrm{cont}_R$ with $g: X_1\to Y$, and the exceptional set of f contains the exceptional divisor D associated with R. This result implies the following, which is a special case of Danilov's result [6]:

Let f: $X\to Y$ be a *small* birational morphism of nonsingular projective threefolds; namely, $\dim f^{-1}(y)\leqq 1$ for every point $y\in Y$. Then f is decomposed into a product of blowing-ups along nonsingular curves.

In general, we have no further results.

In view of Mori's theory, when we consider algebraic threefolds whose canonical divisors are not numerically effective, the following three classes remain to be studied further.

(I) A threefold X having a conic bundle structure f: $X\to Y$ over a nonsingular projective surface Y such that $\rho(X)=\rho(Y)+1$;

(II) A threefold X which is a del Pezzo fiber space f: $X\to Y$ over a nonsingular projective curve Y, and $\rho(X)=2$;

(III) $\rho(X)=1$. In this case, $-K_X$ is ample, hence X is a Fano threefold with $\rho(X)=1$; this class was thoroughly investigated by Iskovskih [16, 18], as we mention again later.

(I) A conic bundle f: $X\to Y$ is said to be *standard* if $\mathrm{Pic}\,X\cong f^*\,\mathrm{Pic}\,Y \oplus \boldsymbol{Z}K_X$; this is equivalent to saying that $\rho(X)=\rho(Y)+1$ and the generic fiber of f is a conic defined over the function field $k(Y)$ without any $k(Y)$-

rational points. Concerning the rigidity of a given conic fibration $f: X \to Y$, Sarkisov [37] proved the following result:

Let Y be a nonsingular projective surface with $\chi(\mathcal{O}_Y) \geqq 1$ and let $f: X \to Y$ be a standard conic bundle such that $|4K_Y + \Delta_f| \neq \phi$, where Δ_f is the degeneracy locus of f. Then the given conic fibration $f: X \to Y$ is rigid in the following sense: Let $f': X \to Y'$ be a rational mapping whose general fibers are rational curves; then there are birational mappings $\xi: X \to X$ and $\eta: Y \to Y'$ such that $f' \cdot \xi = \eta \cdot f$. Therefore X is irrational.

Indeed, Sarkisov succeeded in constructing an example of a unirational, irrational threefold X with a standard conic bundle structure such that $H^3(X, \boldsymbol{Z}) = (0)$; note that the torsion subgroup of $H^3(X, \boldsymbol{Z})$ is a birational invariant [1].

His proof depends on an extension of the arguments of Iskovskih-Manin [21] used to prove that a smooth quartic hypersurface in $\boldsymbol{P}^4$ is not rational. The condition $\chi(\mathcal{O}_Y) \geqq 1$ implies that either Y is not ruled or Y is rational. So, one has to care about this latter case.

On the other hand, if $f: X \to \boldsymbol{P}^2$ is a conic bundle such that Δ_f is nonsingular, Beauville [4] established an isomorphism between the Chow group $(A_1)^0(X) = \{$1-cycles algebraically equivalent to $0\}/$(rational equivalence) and the k-rational points of the Prym variety of the pseudo-covering $G(f)/\Delta_f$ associated with $f: X \to \boldsymbol{P}^2$. Thereby, he proved the irrationality of certain conic bundles.

(II) We have little knowledge on del Pezzo fiber spaces $f: X \to Y$ with $\rho(X) = 2$. Let η be the generic point of Y and let F be the generic fiber X_η of f. Since $\rho(X) = 2$, F is a relatively minimal model defined over $k(\eta)$ in the sense of Iskovskih [20], in view of which we have the following result.

Let $f: X \to Y$ be as above. Then $-K_F$ is ample on F, Pic $F \cong \boldsymbol{Z}$, and one of the following cases takes place:

(1) $-K_F$ is divisible by 3 in Pic F; then X is a $\boldsymbol{P}^2$-bundle over Y;

(2) $-K_F$ is divisible by 2 in Pic F; then X is a quadric bundle over Y, i.e., all fibers of f are isomorphic to a possibly singular quadric in $\boldsymbol{P}^3$;

(3) $-K_F$ is a generator of Pic F; then a general fiber X_y of f contains finitely many exceptional curves of the first kind; there exists an irreducible reduced divisor Z on X such that $Z \cap X_y$ is the union of all exceptional curves of the first kind contained in X_y; moreover, $1 \leqq (K_F^2) \leqq 6$.

(III) Before stating relevant results in the last case, we shall make several comments on Fano threefolds. An algebraic threefold X is called *a Fano threefold* (or *a Fano variety*) if $-K_X$ is ample; this notion is, of course, a generalization of that of a del Pezzo surface in the two-dimensional case. The largest (positive) integer r, which divides $-K_X$ in Pic X is called *the index* of X; then $r \leqq 4$, and we may write $-K_X \sim rH$ with $H \in$

Pic X. Iskovskih [16, 17, 18] succeeded in modernizing the old theory of Fano threefolds originated by G. Fano. More precisely, he classified all Fano threefolds with index $r \geqq 2$ or with Picard number $\rho = 1$, under the assumption that the linear system $|H|$ contains a smooth member. However, Shokurov [40] soon removed this restriction by proving that $|H|$ contains a smooth member.

Let X be a Fano threefold with $\rho(X)=1$. The index r of X is not larger than 4. If $r=4$ then X is isomorphic to $\boldsymbol{P}^3$. If $r=3$ then X is isomorphic to a quadric hypersurface Q_2 in $\boldsymbol{P}^4$. If $r=2$ there are five different classes V_d $(1 \leqq d \leqq 5)$ up to flat deformations, for which $(-K_{V_d})^3 = 8d$:

V_1: a double covering of the Veronese cone W_4,
V_2: a double covering of $\boldsymbol{P}^3$ branched in a quartic hypersurface,
V_3: a cubic hypersurface in $\boldsymbol{P}^4$,
V_4: a complete intersection of type (2, 2) in $\boldsymbol{P}^5$,
V_5: a section by a linear space of codimension 3 of the Grassmann variety $G(1, 4) \subset \boldsymbol{P}^9$ (Plücker embedding).

If $r=1$, they are divided into ten classes, for which the following assertions hold:

(1) $(-K_X)^3$ is even, $2 \leqq (-K_X)^3 \leqq 22$ and $(-K_X)^3 \neq 20$.
(2) $-K_X$ is very ample except in the following two classes:
V_2': a double covering of $\boldsymbol{P}^3$ branched in a sextic hypersurface, for which $(-K_X)^3 = 2$,
V_4': a double covering of $Q_2 \subset \boldsymbol{P}^4$ branched in a surface of degree 8.
(3) Except in V_2' and V_4', X (identified with the anti-canonical model in $\boldsymbol{P}^{\dim|-K_X|}$) contains a line (cf. Shokulov [41], Reid [34]).

More precisely, Shokurov proved the following result.

Let X be a Fano threefold. An irreducible curve C on X is called a line if $(-K_X \cdot C)=1$. Then just one of the following holds: (i) X contains a line; (ii) index $(X) \geqq 2$; (iii) $X \cong \boldsymbol{P}^1 \times \boldsymbol{P}^2$.

Recently, Mori and Mukai [28] gave a complete classification of Fano threefolds with the Picard number $\geqq 2$. According to their results, there are eighty seven different classes up to flat smooth deformations. We only mention that Mori's theory plays a quite natural and significant role in this work (cf. Mukai's article in this volume).

In the subsequent sections, we shall look into the details of these results. The third section where we attempt to account for a part of Mori's theory is based on various preprints of Mori's, among which we refer to [25, 26, 27].

§ 2. Preliminary results

2.1. *Blowing-ups.* Let X be a nonsingular projective threefold and let $f: X' \to X$ be the blowing-up along a nonsingular subvariety Y, where $\dim Y=0$ or 1. Let $Y'=f^{-1}(Y)$. Then we have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle i} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

If Y is a point then $Y' \cong \boldsymbol{P}^2$, and if Y is a curve then Y' is a $\boldsymbol{P}^1$-bundle over Y. Let l be a line on $\boldsymbol{P}^2$ (or a fiber of the $\boldsymbol{P}^1$-fibration of Y' over Y, resp.) if Y is a point (or a curve, resp.). For a nonsingular complete variety T, we denote by $A^i(T)$ the Chow group of codimension i cycles on T modulo numerical equivalence, and by $A(T)=\bigoplus_i A^i(T)$ the Chow ring graded by codimensions. Then the following assertions hold (cf. [18]):

(1) $A(X') \cong f^*A(X) \oplus \boldsymbol{Z}Y' \oplus \boldsymbol{Z}l$ as an additive group, and $f_*(Y')=f_*(l)=0$ and $f_*f^*(\xi)=\xi$ for $\xi \in A(X)$;

(2) the multiplicative structure of $A(X')$ is given as follows;

(i) *Case where Y is a point*: $Y'^2=-l$, $(Y'^3)=-(Y' \cdot l)=1$ and $(Y' \cdot f^*\xi)=(l \cdot f^*\xi)=0$ for $\xi \in A(X)$;

(ii) *Case where Y is a curve*: $Y'^2=-f^*Y+c_1(N_{Y/X})l$, $(Y'^3)=-c_1(N_{Y/X})$, $(Y' \cdot l)=-1$, $Y' \cdot f^*D=(Y \cdot D)l$ and $(l \cdot f^*D)=0$ for $D \in A^1(X)$, $(Y' \cdot f^*C)=(l \cdot f^*C)=0$ for $C \in A^2(X)$, and $c_1(N_{Y/X})+2-2p_a(Y)=(c_1(X) \cdot Y)$, where $c_1(N_{Y/X})$ is the first Chern class of the normal bundle of Y in X.

Let $c_i(X)$ and $c_i(X')$ be the i-th Chern classes of X and X', respectively. Then the following assertions hold (cf. Porteous [32]):

(i)$'$ *Case where Y is a point*: $c_1(X')=f^*c_1(X)-2Y'$, $c_2(X')=f^*c_2(X)$, and $c_3(X')=c_3(X)+2$;

(ii)$'$ *Case where Y is a curve*: $c_1(X')=f^*c_1(X)-Y'$, $c_2(X')=f^*c_2(X)-Y'^2-j_*g^*c_1(Y)$, and $c_3(X')=c_3(X)+2c_1(N_{Y/X})+2-2p_a(Y)$.

2.2. *Blowing-downs.* Let X' be a nonsingular, projective threefold and let Y' be a codimension 1 subvariety of X' such that either Y' is isomorphic to $\boldsymbol{P}^2$ or Y' is a $\boldsymbol{P}^1$-bundle over a nonsingular complete curve Y. As above, let l be a line on $\boldsymbol{P}^2$ if $Y' \cong \boldsymbol{P}^2$ and a fiber of a $\boldsymbol{P}^1$-bundle if Y' is a $\boldsymbol{P}^1$-bundle over Y. Assume that $(Y' \cdot l)=-1$. Then there exist a complete nonsingular algebraic space X and a birational morphism $f: X' \to X$ such that f is the blowing-up of X along a nonsingular subvariety Y and $Y'=f^{-1}(Y)$. As to when X is projective we have few criteria except in the following special cases:

(1) Let φ': $X' \to S'$ be a conic bundle over a nonsingular projective surface. Assume that $(Y' \cdot Z) = 0$ for a general fiber Z of φ'. Then X has a conic bundle structure φ: $X \to S$ such that $\varphi \cdot f = \sigma \cdot \varphi'$, where σ: $S' \to S$ is a birational morphism with S nonsingular. In particular, X is projective.

(2) Let φ': $X' \to C$ be a del Pezzo fiber space over a nonsingular complete curve, whose Picard number $\rho(X')$ is not necessarily 2. Assume that $\varphi'(Y') = C$ and $\varphi'_*(l) = 0$. Then X has a del Pezzo fiber space structure φ: $X \to C$ such that $\varphi' = \varphi \cdot f$. In particular, X is projective.

2.3. *Cone of effective curves.* Let X be a nonsingular projective variety of dimension n. We denote by $N(X)$ the vector space over the reals $\boldsymbol{R}$ of dimension $\rho(X)$,

$$(\{\text{1-cycles on } X\}/(\equiv)) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}^{(*)}.$$

The smallest convex cone in $N(X)$ containing all effective 1-cycles and closed under multiplications by the nonnegative reals $\boldsymbol{R}_+$ is denoted by $NE(X)$ and called *the effective cone of curves.* The closure $\overline{NE}(X)$ of $NE(X)$ with respect to the metric topology of $N(X)$ is called *the closed cone of curves.* Let D be a divisor on X. D is called *pseudo-ample* if $(D^s \cdot Y) \geqq 0$ for every closed subvariety Y of dimension s $(1 \leqq s \leqq n)$. On the other hand, D is said to be *numerically effective* if $(D \cdot C) \geqq 0$ for every irreducible curve C on X. Then we know:

(1) (Kleiman) A divisor D is pseudo-ample if and only if D is numerically effective.

We denote by $N(X)^*$ the real vector space of dimension $\rho(X)$,

$$(\{\text{Divisors on } X\}/(\equiv)) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}.$$

The convex cone generated by all pseudo-ample divisors on X is called *the pseudo-ample cone of* X and denoted by $P(X)$. Note that

(2) $P(X)$ is the dual cone of $\overline{NE}(X)$, i.e., $P(X) = \{D \in N(X)^* \mid (D \cdot C) \geqq 0$ for every $C \in \overline{NE}(X)\}$, and vice versa.

Let $P^0(X)$ be the convex cone generated by all ample divisors on X. Then we have:

(3) $P^0(X) = \operatorname{int} P(X)$.

In fact, this is equivalent to the following criterion of ampleness by Kleiman:

(4) Let $\|\ \|$ be any norm on the real vector space $N(X)$. Then a divisor D is ample if and only if there exists a real number $\varepsilon > 0$ such that $(D \cdot C) \geqq \varepsilon \|C\|$ for every irreducible reduced curve C on X.

(*) $Z_1 \equiv Z_2$ or $Z_1 \approx Z_2$: Z_1 is numerically equivalent to Z_2.

For these results, we refer to Kleiman [22] or Hartshorne [14].

§ 3. Several central lemmas in Mori's theory

3.1. Let X be a nonsingular projective variety of dimension n and let L be an ample divisor on X. We consider the metric topology on $N(X)$ and any norm $\| \ \|$ which accords with the given metric topology. Let ε be a positive number. Define

$$N_\varepsilon(X)=\{Z \in N(X) \mid (Z\cdot c_1(X))\leqq\varepsilon(Z\cdot L)\}, \quad \text{and}$$
$$\overline{NE}_\varepsilon(X)=\overline{NE}(X)\cap N_\varepsilon(X).$$

Then the fundamental theorem on the shape of $\overline{NE}(X)$ is the following:

Theorem. *For an arbitrary positive number ε, there exist a finite number of rational curves $l_1, \cdots, l_r$ on X such that $(l_i\cdot c_1(X))\leqq n+1$ for $1\leqq i\leqq r$ and*

$$\overline{NE}(X)=\boldsymbol{R}_+[l_1]+\cdots+\boldsymbol{R}_+[l_r]+\overline{NE}_\varepsilon(X).$$

In the proof of this result, essential roles are played by the following lemma and the reduction of the arguments to the case of positive characteristic.

Lemma (Mori [23]). *Let C be a nonsingular projective curve of genus g. Then, for any morphism f: $C\to X$, there exist a morphism h: $C\to X$ and an effective 1-cycle Z such that*

(a) $(h_*(C)\cdot c_1(X))\leqq ng$,

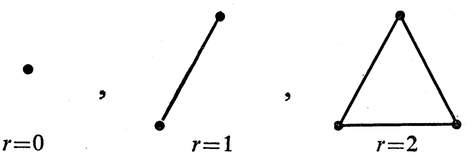
(b) *an arbitrary irreducible component Z' of Z is a possibly singular rational curve with $(Z'\cdot c_1(X))\leqq n+1$,*

(c) *$f_*(C)$ is algebraically equivalent to $h_*(C)+Z$, $f_*(C)\approx h_*(C)+Z$ by notation.*

A half line $R=\boldsymbol{R}_+[Z]$ in $\overline{NE}(X)$ is called *an extremal ray* if $(Z\cdot c_1(X))>0$ and $Z_1, Z_2 \in R$ whenever $Z_1+Z_2 \in R$ and $Z_1, Z_2 \in \overline{NE}(X)$. A rational curve l on X is *an extremal rational curve* if $(l\cdot c_1(X))\leqq n+1$ and $R:=\boldsymbol{R}_+[l]$ is an extremal ray. In view of the above theorem, $\overline{NE}(X)$ is a convex polyhedral cone on the side $c_1(X)>\varepsilon L$ for every positive number ε. Note that every extremal ray is spanned by an extremal rational curve. Therefore, X has an extremal rational curve if and only if K_X $(\equiv -c_1(X))$ is not numerically effective. If $-K_X$ is ample then X contains finitely many rational curves $l_1, \cdots, l_r$ such that $(l_i\cdot c_1(X))\leqq n+1$ for $1\leqq i\leqq r$ and $NE(X)=\overline{NE}(X)=\boldsymbol{R}_+[l_1]+\cdots+\boldsymbol{R}_+[l_r]$. Indeed, taking ε so that $1/\varepsilon$ is a positive integer and $\varepsilon^{-1}c_1(X)-L$ is ample, we have only to apply the above theorem.

Consider the case where $n=2$, i.e., X is a nonsingular projective surface. Suppose that K_X is not numerically effective. Hence X has an extremal rational curve l. If $\rho(X)=1$, then $-K_X$ is ample and X is hence a del Pezzo surface; X is, in fact, isomorphic to $\boldsymbol{P}^2$. So, we assume that $\rho(X)>1$. We know that $(l^2)\leqq 0$ and that if $(l^2)<0$ then l is an exceptional curve of the first kind, while if $(l^2)=0$ then $l\cong\boldsymbol{P}^1$, $(l^2)=0$ and $\dim|l|=1$. Conversely, let l be an exceptional curve of the first kind on X. Suppose that $Z_1+Z_2\equiv al$ with $Z_1, Z_2\in\overline{NE}(X)$ and $a>0$. Let $\sigma\colon X\to Y$ be the contraction of l. Then $\sigma_*(Z_1)+\sigma_*(Z_2)\equiv 0$ and $\sigma_*(Z_i)\in\overline{NE}(Y)$ $(i=1, 2)$. Hence $\sigma_*(Z_i)=0$, i.e., $Z_i\in\boldsymbol{R}_+[l]$, for $i=1, 2$. Thus l is an extremal rational curve and $R:=\boldsymbol{R}_+[l]$ is an extremal ray.

Let X_r be the nonsingular projective surface obtained from $\boldsymbol{P}^2$ by blowing up r points $P_1, \cdots, P_r$ in general position. It is known that if $0\leqq r\leqq 8$, X_r is a del Pezzo surface with $\rho(X)=r+1$. The shape of $NE(X)=\overline{NE}(X)$ is rather easy to describe. Let $\mathscr{P}(X)$ be the section of $NE(X)$ by a hyperplane not passing through the point of origin. Then $\mathscr{P}(X)$ is an r-dimensional convex polyhedron which has as many vertices as exceptional curves of the first kind on X if $r\geqq 2$;



However, if $r\geqq 9$, X_r is no longer a del Pezzo surface and X_r contains infinitely many exceptional curves of the first kind (cf. Nagata [31]; note also that X_r contains no rational curves C with $C\cong\boldsymbol{P}^1$ and $(C^2)=-2$). Thus, as we make a positive number ε smaller and smaller, the cone $\overline{NE}(X)$ on the side $c_1(X)>\varepsilon L$ gets more and more extremal rays.

Let X be a nonsingular projective variety of dimension n. X is said to be *uniruled* if the function field $k(X)$ of X has a finite algebraic extension which is a simple transcendental extension of some subfield. If X is uniruled, then K_X is not numerically effective. Indeed, if $f\colon Y\to X$ is a generically finite surjective morphism of nonsingular projective varieties, then $K_Y\sim f^*K_X+B$ with an effective divisor B on Y. If f is birational then $K_X\sim f_*K_Y$. Suppose first that $Y\cong\boldsymbol{P}^1\times T$. Then $(K_Y\cdot C_t)=-2$ for $C_t:=\boldsymbol{P}^1\times\{t\}$ with $t\in T$. More generally, if Y is birational to $\boldsymbol{P}^1\times T$, we can readily see that K_Y is not numerically effective and there exists an $(n-1)$-dimensional family of curves $\{C_s\}_{s\in S}$ such that $(K_Y\cdot C_s)<0$. Now suppose that X is uniruled. Then there exists a generically finite surjective morphism $f\colon Y\to X$ as above, where Y is birational to $\boldsymbol{P}^1\times T$. Then, for a curve C_s on Y with $C_s\not\subset\operatorname{Supp}(B)$, we have $(K_X\cdot f_*C_s)=(f^*K_X\cdot C_s)=(K_Y\cdot C_s)-(B\cdot C_s)<0$. Hence K_X is not numerically effective.

3.2. One of the results which are used effectively in Mori's theory is the following:

Lemma. *Let Y be an irreducible reduced projective Gorenstein surface defined over k such that ω_Y^{-1} is ample and $\chi(\mathcal{O}_Y)\geqq 0$, where ω_Y is the dualizing invertible sheaf on Y. Then $h^1(\mathcal{O}_Y)=0$ and hence $\chi(\mathcal{O}_Y)=1$.*

We shall give the following

Remark. Let Y be a surface as considered in the above lemma. If Y is normal, a precise structure of such a surface was clarified by Brenton [5] and Hidaka-Watanabe [15]. Let $\tilde{Y}$ be a minimal resolution of singularities of Y. According to their result, $\tilde{Y}$ is a rational surface or an elliptic ruled surface $\boldsymbol{P}(\mathcal{O}_C \oplus \mathcal{L})$, where C is a nonsingular elliptic curve and $\mathcal{L}$ is a line bundle with $\deg \mathcal{L} > 0$. If $\tilde{Y}$ is a rational surface, then $\tilde{Y}$ is isomorphic to F_n ($n=0, 2$), where $F_n = \boldsymbol{P}(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(n))$, or obtained from $\boldsymbol{P}^2$ by blowing up r points $P_1, \cdots, P_r$ ($r \leqq 8$) in such a position that

(i) no four of them are collinear,

(ii) no seven of them lie on a conic,

(iii) for each i, $\mathrm{dil}_{P_i} P_i$ ($=$the exceptional curve arising from the blowing-up of P_i) carries at most one P_j; in short, $P_1, \cdots, P_r$ are in such a position that, after blowing up $P_1, \cdots, P_r$, at most (-2)-curves appear on $\tilde{Y}$.

When $\tilde{Y}$ is rational, this is exactly the case which Demazure studied in [7]. The surface Y is obtained from $\tilde{Y}$ by contracting all (-2)-curves (when Y is rational) or the minimal section of $\boldsymbol{P}(\mathcal{O}_C \oplus \mathcal{L})$ (when Y is elliptic ruled). Consider first the case where $\tilde{Y} = \boldsymbol{P}(\mathcal{O}_C \oplus \mathcal{L})$ with $d := \deg \mathcal{L} > 0$. If $\mathcal{L}$ is very ample, i.e., $d \geqq 3$, then ω_Y^{-1} is very ample and Y is isomorphic to a cone over a nonsingular elliptic curve C of degree d in $\boldsymbol{P}^{d-1}$. If $d=1$ or 2, then Y is not embedded into a projective space as an elliptic cone. Consider next the case where Y is rational. If either $\tilde{Y}$ is isomorphic to F_n ($n=0, 2$) or $\tilde{Y} = \mathrm{dil}_{P_1,\dots,P_r} \boldsymbol{P}^2$ with $0 \leqq r \leqq 6$, then Y is embedded into the projective space $\boldsymbol{P}^d$ as a surface of degree d, where $d = (\omega_Y^2)$. Namely, ω_Y^{-1} is very ample. However, if $r=7$ or 8, ω_Y^{-1} is not very ample. In fact, ω_Y^{-2} is very ample if $r=7$ and ω_Y^{-3} is very ample if $r=8$. Moreover, in the last two cases, Y cannot be embedded into a projective space $\boldsymbol{P}^n$ as a surface of degree n.

Now consider the case where Y is not necessarily normal. We use the following notations: $\pi: \bar{Y} \to Y$ is the normalization of Y, $\mathcal{I}$ is the conductor ideal, $\bar{E} = V_{\bar{Y}}(\mathcal{I})$ and $E = V_Y(\mathcal{I})$. Then $E \cong \boldsymbol{P}^1$ and $\bar{E}$ is isomorphic to a conic. Goto and, then independently, Reid (cf. [35]) proved that $H^1(\omega_Y^{-n}) = 0$ for every integer n. Suppose that ω_Y^{-1} is very ample. Then Y is embedded into $\boldsymbol{P}^d$ as a surface of degree d, where $d := (\omega_Y^2)$. By virtue of Nagata [30; I. Th. 8 and Prop. 11], Y is obtained as a projection of a surface X in $\boldsymbol{P}^{d+1}$ with the center outside of the surface X, where $\deg X=d$. As is well-known [ibid.; Th. 7], such a surface X is isomorphic to one of the following:

(i) $d=1$ and $X=\boldsymbol{P}^2$,

(ii) $d=4$ and X is the Veronese transform of $\boldsymbol{P}^2$,

(iii) $d=2, 3, \cdots, X=F_s$, where $d-s-2$ is a non-negative even integer; X is embedded into $\boldsymbol{P}^{d+1}$ by $|M+\frac{1}{2}(d+s)l|$, where M (resp. l) is the minimal section (resp. a fiber) of F_s,

(iv) $d=2, 3, \cdots$, and X is a cone over a rational curve of degree d in $\boldsymbol{P}^d$.

In order to obtain Y from X, we have to exclude trivially unsuitable cases: (i), $d=2$ in (iii), $d=2$ in (iv). There are examples for which $\bar{E}\cong \boldsymbol{P}^1$, $\bar{E}\cong \boldsymbol{P}^1+\boldsymbol{P}^1$ and $\bar{E}\cong$a double line, respectively:

(1) $Y=$a point projection of the Veronese transform of $\boldsymbol{P}^2$ in $\boldsymbol{P}^5$; e.g., $(x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2)\mapsto(x_0^2, x_0x_1, x_1^2, x_0x_2, x_2^2)$;

(2) $Y=$a cone over a plane nodal cubic;

(3) $Y=$a cone over a plane cuspidal cubic.

However, we have no further informations as to when ω_Y^{-1} is not very ample. For example, we may ask the following questions:

(1) Does Sing (Y) coincide with E? Namely, does Y have isolated singular points?

(2) Is ω_Y^{-3} very ample?

3.3. The following result is used in Mori's theory to test whether a given curve (mostly belonging to an extremal ray) moves in a positive-dimensional family.

Lemma. *Let X be a nonsingular projective threefold and let C be an irreducible reduced curve on X such that $(K_X\cdot C)<0$ and that $\chi(\mathcal{O}_{C'})\geqq 0$ for every one-dimensional closed subscheme C' of X with $C'_{\mathrm{red}}=C$. Then $C\cong \boldsymbol{P}^1$ and one of the following cases takes place*:

(1) *$N_{C/X}\cong\mathcal{O}(-1)\oplus\mathcal{O}$ and hence $(K_X\cdot C)=-1$,*

(2) *$N_{C/X}\cong\mathcal{O}_C^{\oplus 2}$ and C is numerically effective,*

(3) *$N_{C/X}\cong\mathcal{O}_C(1)\oplus\mathcal{O}_C(-2)$ and C is numerically effective; moreover, let $\mathcal{J}$ be the ideal defined by $\mathcal{I}_C\supset\mathcal{J}\supset\mathcal{I}_C^2$ and $\mathcal{J}=\mathcal{O}_C(2)+\mathcal{I}_C^2$, $\mathcal{I}_C$ being the ideal defining C in X; then $\mathcal{J}/\mathcal{J}^2\cong\mathcal{O}_X/\mathcal{J}\oplus\mathcal{O}_X/\mathcal{J}$.*

Remark. Referring to the results stated in § 1 concerning Mori's theory, we consider a nonsingular rational curve C as chosen in each of the cases as follows.

(I) R is not numerically effective:

Case E_1: $H\cdot D\not\equiv 0$ on D and D is a $\boldsymbol{P}^1$-bundle over a nonsingular curve Y; C is a fiber of the $\boldsymbol{P}^1$-fibration of D;

Case E_2: $H \cdot D \equiv 0$ on D, $D \cong \boldsymbol{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_{\boldsymbol{P}^2}(-1)$; C is a line on $\boldsymbol{P}^2$;

Case E_3: $H \cdot D \equiv 0$ on D, $D \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $\mathcal{O}_D(D) \cong \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(-1, -1)$; $C = (s) \times \boldsymbol{P}^1$ with $(s) \in \boldsymbol{P}^1$;

Case E_4: $H \cdot D \equiv 0$ on D, $D \cong$ a quadric cone V in $\boldsymbol{P}^3$ and $\mathcal{O}_D(D) \cong O_V(-1)$; C is a generator of the cone V;

Case E_5: $H \cdot D \equiv 0$ on D, $D \cong \boldsymbol{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_{\boldsymbol{P}^2}(-2)$; C is a line on $\boldsymbol{P}^2$.

(II) R is numerically effective and f: $X \to Y$ is a conic bundle.

Case C_1: f is not a $\boldsymbol{P}^1$-fiber bundle; hence the conic fibration contains a degenerate fiber; C is an irreducible (reduced) component of a degenerate fiber and C_g is a general fiber of f;

Case C_2: f is a $\boldsymbol{P}^1$-fiber bundle; C is a fiber of f.

(III) R is numerically effective and f: $X \to Y$ is a del Pezzo fiber space.

Case D_1: f: $X \to Y$ is not a $\boldsymbol{P}^2$-bundle nor a quadric bundle; C is an exceptional curve of the first kind contained in a general fiber of f;

Case D_2: f: $X \to Y$ is a quadric bundle; C is a fiber of one of two distinct $\boldsymbol{P}^1$-fibrations on a general fiber of f;

Case D_3: f: $X \to Y$ is a $\boldsymbol{P}^2$-bundle; C is a line on a general fiber of f.

The curve C as chosen as above satisfies the condition:

$$-(K_X \cdot C) = \min \{-(K_X \cdot B) \mid B \text{ is a curve such that } [B] \in R = \boldsymbol{R}_+[l]\}.$$

We have the following list:

Type	$-(K_X \cdot C)$	$N_{C/X}$	
E_1	1	$\mathcal{O} \oplus \mathcal{O}(-1)$	(*)
E_2	2	$\mathcal{O}(1) \oplus \mathcal{O}(-1)$	not (*)
E_3	1	$\mathcal{O} \oplus \mathcal{O}(-1)$	
E_4	1	$\mathcal{O} \oplus \mathcal{O}(-1)$	
E_5	1	$\mathcal{O}(1) \oplus \mathcal{O}(-2)$	
C_1	1	$\mathcal{O} \oplus \mathcal{O}(-1)$	$C + C'$ is a fiber of f; $C \neq C'$; (*)
		$\mathcal{O}(1) \oplus \mathcal{O}(-2)$	$2C$ is a fiber of f; (*)
C_2	2	$\mathcal{O} \oplus \mathcal{O}$	(*)
D_1	1	$\mathcal{O} \oplus \mathcal{O}(-1)$	
D_2	2	$\mathcal{O} \oplus \mathcal{O}$	
D_3	3	$\mathcal{O}(1) \oplus \mathcal{O}$	not (*)

In the above list, $(*)$ means that the curve C enjoys the property that $\chi(\mathcal{O}_{C'}) \geqq 0$ for every 1-dimensional closed subscheme C' of X with $C'_{\mathrm{red}}=C$.

3.4. We only note that the following result from Fujita's theory of Δ-genus is used to determine D and $\mathcal{O}_D(D)$ in the case E_i ($i=2, 3, 4, 5$).

Lemma [9; Th. 4.8]. *Let D be an irreducible reduced scheme defined over k and let M be an ample Cartier divisor on D. Set* $\Delta(D, M) := \dim D + (M^{\dim D}) - h^0(D, M)$. *Then the following assertions hold*:

(1) $\Delta(D, M) > \dim Bs\,|M|$, *where $Bs\,|M|$ denotes the set of base points of $|M|$ and* $\dim Bs\,|M| = -1$ *if* $Bs\,|M| = \phi$;

(2) $\Delta(D, M)=0$ *implies that D is normal, Cohen-Macaulay and M is very ample.*

§ 4. Conic bundles

In this section, we shall gather together most of the basic results concerning the conic bundles over a nonsingular projective surface. On the analogy of relatively minimal ruled surfaces, we introduce the notion of a standard conic bundle and its degeneracy locus and explain how to produce a standard conic bundle from the given "general" conic bundle; this is done by contracting extra components of codimension 1 lying over the degeneracy locus. Then, in order to change a conic bundle within its birational class so that the degeneracy locus acquires the properties which match better for specific purposes, we introduce two operations called the φ-process and the β-process. Next, we consider a conical fibration over an irreducible projective surface S which, roughly speaking, corresponds to a plane conic defined over the function field $k(S)$. The notion of conical fibration plays an important role, when we consider the problems of birational nature on a given conic bundle, e.g., the problem of determining whether or not a conic bundle is irrational over the base field k. Concerning a conical fibration, there is a result due to Zagorskih [42], according to which a conical fibration is birationally equivalent to a (standard) conic bundle. Since the original proof of Zagorskih contains a gap, we present another proof based on his idea. Finally, depending on the paper by Artin-Mumford [1], we shall give a result which assures the existence of conic bundles in some special, though effective in practical use, cases.

4.1. A nonsingular projective threefold X is called *a conic bundle* over a nonsingular projective surface Y if there exists a surjective morphism $f: X \to Y$ such that every fiber X_y is isomorphic to a possibly singular conic in $\boldsymbol{P}^2$. A conic bundle $f: X \to Y$ is said to be *standard* if $\operatorname{Pic} X = f^*\operatorname{Pic} Y \oplus \boldsymbol{Z}K_X$.

Lemma (cf. [4; Prop. 1.2]). *Let f: $X \to Y$ be a conic bundle. Then the following assertions hold*:

(1) *There exists a vector bundle $\mathcal{E}$ of rank* 3 *over Y, a line bundle $\mathcal{M}$ over Y and a section $q \in H^0(Y, \mathcal{S}^2(\mathcal{E}) \otimes \mathcal{M}^2)$ such that X is isomorphic to the zero locus of q in $\boldsymbol{P}(\mathcal{E} \otimes \mathcal{M}) := \mathrm{Proj}\,(\mathcal{E} \otimes \mathcal{M})$.*

(2) *There exists a curve Δ_f on Y satisfying the properties*:

(i) *Δ_f has only normal crossing as singularities*;

(ii) *X_y is a smooth conic, a reducible conic or a double line according as $y \notin \Delta_f$, $y \in \Delta_f - \mathrm{Sing}\,\Delta_f$ or $y \in \mathrm{Sing}\,\Delta_f$, respectively.*

The curve Δ_f on Y is called *the degeneracy locus* (or *the discriminant locus*) of the conic bundle f: $X \to Y$. If Δ_f is nonsingular (singular, resp.), the conic bundle f: $X \to Y$ is called *ordinary* (*special*, resp.).

4.2. Let C be a connected reduced curve having only ordinary double points as singularities. A double covering π: $\tilde{C} \to C$ is called *a pseudo-covering* if the following conditions are satisfied:

(1) If $s \in C$ is a smooth point, there exists an open neighborhood U of s in C such that π_U: $\pi^{-1}(U) \to U$ is an étale covering of degree 2;

(2) if $s \in C$ is a singular point and if $\hat{\mathcal{O}}_{s,C} \cong k[[u, v]]/(uv)$ then $\pi^{-1}(s) = \{\tilde{s}\}$ and $\hat{\mathcal{O}}_{\tilde{s},\tilde{C}} \cong k[[x, y]]/(xy)$ with $\pi^* u = x^2$ and $\pi^* v = y^2$.

For a conic bundle f: $X \to Y$, define g: $G(f) \to \Delta_f \subset Y$ as follows:

$$G(f)_y := \{\text{lines in } X_y \subset \boldsymbol{P}^2_{k(y)}\}$$

$$g\colon G(f) \longrightarrow Y \text{ is the natural mapping.}$$

Then the image of g is apparently the discriminant locus Δ_f.

Lemma [4; Prop. 1.5]. *Let f: $X \to Y$ be a conic bundle and let Δ_f be the discriminant locus of f. If f is ordinary, then g: $G(f) \to \Delta_f$ is an étale covering of degree* 2; *if f is special, then g: $G(f) \to \Delta_f$ is a pseudo-covering.*

Remark. Assume that a conic bundle f: $X \to Y$ satisfies the condition: For every irreducible curve C on Y, $f^{-1}(C)$ is irreducible. Assume furthermore that $\Delta_f \neq \phi$. Then every nonsingular rational component of Δ_f (if it exists at all) meets other components of Δ_f in at least two points.

In the subsequent two paragraphs, we shall define, after Sarkisov [37], two fundamental transformations on conic bundles, the "φ-process" and the "β-process".

4.3. Let f: $X \to Y$ be a conic bundle. Let B be a nonsingular complete curve on X such that f: $B \to f(B)$ is an isomorphism and $f(B) \cap \Delta_f = \phi$. Let μ: $X' \to X$ be the blowing-up with center B, let $E := \mu^{-1}(B)$, let $Z := f^{-1}(f(B))$ and let $Z' := \mu'(Z)$ be the proper transform of Z, where $Z' \cong Z$.

Let l be a fiber of the ruled surface $Z' \to f(B)$. Since $0 = (\mu(l) \cdot Z) = (l \cdot Z') + (l \cdot E) = (l \cdot Z') + 1$, we have $(l \cdot Z') = -1$. Thus Z' is contractible. Namely, there exist a nonsingular projective threefold X_1 and a birational morphism $\mu_1 \colon X' \to X_1$ such that $B_1 := \mu_1(Z')$ is a nonsingular curve and μ_1 is the blowing-up with center B_1 (cf. Remark 2.2). Moreover, X_1 has a conic bundle structure over Y, $f_1 \colon X_1 \to Y$;

$$
\begin{tikzcd}
 & X' \arrow[dl, "\mu_1"'] \arrow[dr, "\mu"] & \\
X_1 \arrow[dr, "f_1"'] & & X \arrow[ll, dashed, "\mathrm{elm}"'] \arrow[dl, "f"] \\
 & Y &
\end{tikzcd}
$$

The process of obtaining X_1 from X is called *the elementary transformation with center B* (or *the φ-process with center B*, for short). The conic bundle $f_1 \colon X_1 \to Y$ is standard if $f \colon X \to Y$ is a standard conic bundle (cf. [37]).

4.4. Lemma. *Let $f \colon X \to Y$ be a conic bundle and let $\sigma \colon Y' \to Y$ be the blowing-up with center at a point $y \in Y$. Then there exist a conic bundle $f' \colon X' \to Y'$ and a birational mapping $\tau \colon X' \to X$ such that $\sigma \cdot f' = f \cdot \tau$. Furthermore, $f' \colon X' \to Y'$ is standard if so is $f \colon X \to Y$. Then $f' \colon X' \to Y'$ is said to be obtained from $f \colon X \to Y$ by the β-process with center at y.*

Proof. We consider separately the following three cases: (I) $y \notin \Delta_f$, (II) $y \in \Delta_f - \operatorname{Sing} \Delta_f$, (III) $y \in \operatorname{Sing} \Delta_f$.

(I) Suppose that $y \notin \Delta_f$. Let $X' = X \times_Y Y'$ and let $\tau \colon X' \to X$ be the first projection. Then τ is the blowing-up with center at $X_y \cong \boldsymbol{P}^1_k$. Hence X' is smooth, and the second projection $f' \colon X' \to Y'$ endows X' with a conic bundle structure. Let $E := \tau^{-1}(X_y) = f'^{-1}(\sigma^{-1}(y))$. Then $K_{X'} = \tau^* K_X + E$ and $\operatorname{Pic} X' = \tau^* \operatorname{Pic} X \oplus \boldsymbol{Z}E$. Then it is easy to see that $f' \colon X' \to Y'$ is standard if so is $f \colon X \to Y$.

(II) Suppose that $y \in \Delta_f - \operatorname{Sing} \Delta_f$. Then $X_y = L_1 \cup L_2$ with $L_1 \cong L_2 \cong \boldsymbol{P}^1_k$. Let $\mu_1 \colon X_1 \to X$ be the blowing-up with center at L_1 and let $\mu_2 \colon X_2 \to X_1$ be the blowing-up with center at the proper transform $\mu_1'(L_2)$. Let $E_1 := \mu_1^{-1}(L_1)$ and let $E_2 := \mu_2^{-1}(\mu_1' L_2)$. We show that E_2 is a quadric $F_0 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and E_2 is contractible onto a fiber of the ruling $E_2 \to \mu_1' L_2$.

Indeed, let T be a nonsingular complete curve on Y such that T meets Δ_f transversally at y and let $Z := f^{-1}(T)$. Then Z is nonsingular along $X_y \subset Z$, $N_{Z/X|L_1} \cong \mathcal{O}_{L_1}$ and $(L_1^2)_Z = -1$. From an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & N_{L_1/Z} \arrow[r] \arrow[d, phantom, "\wr\|"] & N_{L_1/X} \arrow[r] & N_{Z/X|L_1} \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
 & \mathcal{O}_{L_1}(-1) & & \mathcal{O}_{L_1} &
\end{tikzcd}
$$

we deduce that $N_{L_1/X} \cong \mathcal{O} \oplus \mathcal{O}(-1)$. Therefore, $E_1 = \boldsymbol{P}(N^*_{L_1/X}) \cong F_1$ (cf. 3.2), where $\boldsymbol{P}(N^*_{L_1/X}) = \operatorname{Proj}(S^{\cdot}_{\mathcal{O}_{L_1}}(N^*_{L_1/X}))$. Let $L_2' := \mu_1'(L_2)$ and $Z' := \mu_1'(Z)$. It is clear that $(Z' \cdot L_2') = -1$, $(E_1 \cdot L_2') = 1$ and $(L_2'^2)_{Z'} = -1$. Therefore we have an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & N_{L_2'/Z'} & \longrightarrow & N_{L_2'/X_1} & \longrightarrow & N_{Z'/X_1|L_2} & \longrightarrow 0 \\ & \wr\| & & & & \wr\| & \\ & \mathcal{O}_{L_2'}(-1) & & & & \mathcal{O}_{L_2'}(-1) & . \end{array}$$

Hence $N_{L_2'/X_1} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $E_2 = \boldsymbol{P}(N^*_{L_2'/X_1}) \cong F_0$. Let l_2 and s_2 be fibers of two distinct $\boldsymbol{P}^1$-fibrations of F_0. Then $E_{2|E_2} \sim -l_2 - s_2$. Indeed, $(E_2^3) = -c_1(N_{L_2'/X_1}) = 2$ and $(E_2 \cdot l_2) = (E_2^2 \cdot l_2)_{E_2} = -1$. Hence, writing $E_{2|E_2} \sim \alpha l_2 + \beta s_2$, we easily obtain $\alpha = \beta = -1$. Therefore, $(E_2 \cdot s_2) = -1$. This implies that E_2 is contractible onto the generator l_2. Denote by $\nu : X_2 \to X'$ the contraction of E_2, where X' is, for the moment, a nonsingular algebraic space of dimension 3. We shall show that X' has a conic bundle structure $f' : X' \to Y$, which makes X' projective as well.

Let A be a very ample divisor on Y and let $|D| = |f^*A|$. We claim that:

(1) $|\mu^*D - \mu_2^*E_1 - E_2|$ cuts on $\mu_2^*E_1 = \mu_2'E_1$ a pencil without base points and such that every member except one is a nonsingular rational curve, while one member consists of two irreducible components, where $\mu := \mu_1 \cdot \mu_2$;

(2) $(\mu^*D - \mu_2^*E_1 - E_2 \cdot s_2) = 0$.

In order to verify these assertions, write $(\mu_1^*D - E_1)|_{E_1} \sim \alpha l_1 + \beta s_1$, where l_1 is a fiber of E_1 and s_1 is the minimal section of E_1. Then $(\mu_1^*D - E_1 \cdot l_1) = -(E_1 \cdot l_1) = 1$, whence $\beta = 1$. Note that $\operatorname{Tr}_{E_1}|\mu_1^*D - E_1|$ has a unique base point $P := E_1 \cap L_2'$. On the other hand, we have

$$\begin{aligned} ((\mu_1^*D - E_1)^2 \cdot E_1) &= ((\mu_1^*D)^2 \cdot E_1) - 2(\mu_1^*D \cdot E_1^2) + (E_1^3) \\ &= (D^2 \cdot L_1) + 2(D \cdot L_1) - c_1(N_{L_1/X}) = 1. \end{aligned}$$

Hence $(s_1 + \alpha l_1)^2 = -1 + 2\alpha = 1$, i.e., $\alpha = 1$. Therefore, $\operatorname{Tr}_{E_1}|\mu_1^*D - E_1|$ is a pencil of rational curves, whose unique singular fiber is $s_1 + l_1$. Let $E_1' := \mu_2'E_1$. The blowing-up of X_1 with center at L_2' induces the blowing-up of E_1 with center at P. Thus $\mu_2'(\operatorname{Tr}_{E_1}|\mu_1^*D - E_1|)$ is a pencil of rational curves without base points, whose section is $E_2 \cap E_1'$ and whose unique singular fiber is $\mu_2'(s_1 + l_1)$. Note that

$$(\mu^*D - E_1' - E_2 \cdot s_2) = -(E_1' \cdot s_2) - (-s_2 - l_2 \cdot s_2)_{E_2} = -1 + 1 = 0.$$

Therefore the linear system $|\mu^*D - E_1' - E_2|$ defines a flat morphism $f' : X' \to Y'$ such that f' induces an isomorphism between $\mu_2^{-1}(P) = E_2 \cap E_1'$ and $\sigma^{-1}(y)$. Note that $\Delta_{f'} = \sigma'(\Delta_f)$.

We can show that f': $X' \to Y'$ is standard if so is f: $X \to Y$.

(III) Suppose that $y \in \mathrm{Sing}\, \Delta_f$. Then y is an ordinary double point of Δ_f. Let $L_1 := (X_y)_{\mathrm{red}} \cong \boldsymbol{P}^1_k$, let μ_1: $X_1 \to X$ be the blowing-up with center at L_1 and let $E_1 := \mu_1^{-1}(L_1)$. Let A be a very ample divisor on Y and let $D := f^*A$. Then the following assertions hold, which we state without proof.

(1) Let Z be a general member of $|D| - X_y$. Then Z has two ordinary double points lying on L_1, and is smooth elsewhere.

(2) The linear system $|\mu_1^* D - E_1|$ has a base curve s_1 on E_1.

(3) $N_{L_1/X} \cong \mathcal{O}(-2) \oplus \mathcal{O}(1)$ and $E_1 = \boldsymbol{P}(N^*_{L_1/X}) \cong F_3$, where the above s_1 is the minimal section.

(4) Let $Z' := \mu_1'(Z)$. Then Z' is a smooth ruled surface, and $Z' \cdot E_{1|Z'} = s_1 + l_1^{(1)} + l_1^{(2)}$, where $(s_1^2)_{Z'} = -1$ and $(l_1^{(i)})^2_{Z'} = -2$ for $i = 1, 2$. On the other hand, $Z' \cdot E_{1|E_1} = s_1 + l_1^{(1)} + l_1^{(2)}$ with $(s_1^2)_{E_1} = -3$ and $(l_1^{(i)})^2_{E_1} = 0$ for $i = 1, 2$.

(5) Let μ_2: $X_2 \to X_1$ be the blowing-up along $L_2 := s_1$, let $E_2 := \mu_2^{-1}(L_2)$ and let $E_1' := \mu_2'(E_1)$. Then $N_{L_2/X_1} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $E_2 = \boldsymbol{P}(N^*_{L_2/X}) \cong F_0$. Let l_2 and s_2 be a fiber and a minimal section of E_2, respectively.

(6) $E_1' \cdot E_{2|E_2} \sim s_2 + 2l_2$ and $E_1' \cdot E_{2|E_1'} \sim s_1$ if we identify E_1' with E_1. $E^2_{2|E_2} \sim -s_2 - l_2$ and E_2 is contractible analytically onto a fiber l_2.

(7) $\mathrm{Tr}_{E_1'} |\mu^* D - E_1' - 2E_2|$ is a pencil contained in $|2l_1|$ and $(\mu^* D - E_1' - 2E_2 \cdot s_2) = 0$.

(8) Let ν: $X_2 \to X'$ be the contraction of E_2 onto a fiber l_2. Then $\nu_* |\mu^* D - E_1' - 2E_2|$ defines a conic bundle structure f': $X' \to Y'$ such that

(i) $\sigma^{-1}(y) \subset \Delta_{f'}$

(ii) $X'_{y'}$ is a reducible conic if $y' \in \sigma^{-1}(y) - (\sigma^{-1}(y) \cap \sigma' \Delta_f)$ and $X'_{y'}$ is a double line if $y' \in \sigma^{-1}(y) \cap \sigma' \Delta_f$.

(9) The conic bundle f': $X' \to Y'$ is standard if so is f: $X \to Y$.

Remark. The notations and the assumptions being the same as above, if X_y is a reducible conic, there are two different β-processes with center at y. Namely, one is to blow up L_1 first and L_2' next, and the other is to blow up L_2 first and L_1' next.

We have the following result:

Corollary. *Let f': $X' \to Y'$ be the conic bundle obtained from a conic bundle f: $X \to Y$ by the β-process with center at $y \in Y$. Then $\Delta_{f'} = \sigma^* \Delta_f$ if $y \notin \Delta_f$ and $\Delta_{f'} = \sigma^*(\Delta_f) - \sigma^*(y)$ if $y \in \Delta_f$.*

Remark. Let f: $X \to Y$ be a conic bundle. Suppose that there exists an exceptional curve γ of the first kind on Y such that $\gamma \cap \Delta_f = \phi$. Let $\Gamma := f^{-1}(\gamma)$. Then Γ is a relatively minimal ruled surface with the $\boldsymbol{P}^1$-fibration induced by f. Let l be its fiber and s a minimal section. Then $(\Gamma \cdot s) = (\Gamma^2 \cdot s)_\Gamma = -(l \cdot s) = -1$. Hence if $\Gamma \cong F_0$, then Γ is contractible onto a fiber l. Namely, there exists a conic bundle $\bar{f}\colon \bar{X}\to\bar{Y}$ such that $\bar{Y}=\mathrm{cont}_{\gamma}\,Y$, $Q=\mathrm{cont}_{\gamma}\,\gamma$ and $\bar{X}=\mathrm{cont}_{\Gamma}\,X$, and that $f\colon X\to Y$ is obtained from $\bar{f}\colon \bar{X}\to\bar{Y}$ by the β-process with center at Q. However, Γ is not necessarily isomorphic to F_0. A counterexample can be constructed as follows: Start with a conic bundle $\bar{f}\colon \bar{X}\to\bar{Y}$ and an irreducible curve $\bar{C}$ on $\bar{Y}$ with an ordinary n-ple point Q, where $Q\notin \Delta_{\bar{f}}$. By applying β-processes to $\bar{f}\colon \bar{X}\to\bar{Y}$, we may assume that $\bar{C}\cap\Delta_{\bar{f}}=\phi$. Let $\sigma\colon Y\to\bar{Y}$ be the blowing-up of Q, let $\gamma:=\sigma^{-1}(Q)$, let $C:=\sigma'(\bar{C})$ and let $f_1\colon X_1\to Y$ be the base change of $\bar{f}\colon\bar{X}\to\bar{Y}$ relative to $\sigma\colon Y\to\bar{Y}$. Then $\Gamma_1:=f_1^{-1}(\gamma)$ is isomorphic to F_0. Let $C\cap\gamma=\{P_1,\cdots,P_n\}$, let s_1 be a minimal section of Γ_1 and let $\tilde{P}_i:=f_1^{-1}(P_i)\cap s_1$ for $1\leqq i\leqq n$. We assume that C is a nonsingular curve of genus g. Let $Z:=f_1^{-1}(C)$. Then Z is a relatively minimal ruled surface over C. Let l be a fiber of Z and let b be a section of Z. For $m\gg 0$, we know that $\dim|b+ml|_Z=2m-\alpha+2(1-g)$, where $\alpha=-(b^2)_Z$. Then a general member b_1 of $|b+ml|_Z-\sum_{i=1}^n\tilde{P}_i$ is an irreducible curve if $m\gg 0$. Let $f\colon X\to Y$ be the conic bundle obtained from $f_1\colon X_1\to Y$ by the φ-process with center at b_1. Let $\Gamma:=f^{-1}(\gamma)$. Then it is apparent that Γ is obtained from Γ_1 by means of elementary transformations at the points $\tilde{P}_1,\cdots,\tilde{P}_n$. Hence $\Gamma\cong F_n$. Since n can be an arbitrary nonnegative integer, this is one of the required counterexamples.

4.5. Lemma. *Let $f\colon X\to Y$ be a conic bundle. Then the following conditions are equivalent to each other*:

(1) *A general fiber X_y is an extremal rational curve.*

(2) $\rho(X)=\rho(Y)+1$.

(3) *For any irreducible curve C on Y, $f^{-1}(C)$ is irreducible.*

Proof. (1)⇒(3). Suppose that $f^{-1}(C)=D_1\cup D_2$ for an irreducible curve C. Then $C\subset\Delta_f$, and D_1 and D_2 are irreducible reduced surfaces. Let y be a general point of C and write $X_y=l_1+l_2$ with $l_i\subset D_i$ $(i=1,2)$. Then clearly $(l_1\cdot D_2)=(l_2\cdot D_1)=1$. Hence $(l_1\cdot D_1)=(l_2\cdot D_2)=-1$. However, since $[l_1]$, $[l_2]\in\boldsymbol{R}_+[X_{y'}]$ with a general point y' of Y, we have $[l_i]=\alpha_i[X_{y'}]$ and $(D_i\cdot X_{y'})=0$. Hence $(l_i\cdot D_i)=0$. This is a contradiction. Thus $f^{-1}(C)$ is irreducible for every irreducible curve C on Y.

(3)⇒(2). Let D be a divisor on X. Define D' by $D':=2D+(D\cdot X_y)K_X$. Then $(D'\cdot X_y)=0$ for every fiber X_y of f. We shall show that $D'\sim f^*N$ for some divisor N on Y. Since $\mathrm{Pic}\,(X_\eta)\cong\boldsymbol{Z}$ for the generic point η of Y, we may assume that every irreducible component of D' has the image of dimension one on Y. Thus, for our purpose, we may assume that D' is an irreducible reduced surface with $(D'\cdot X_y)=0$. Let $C=f(D')$, which is an irreducible reduced curve. By the hypothesis, $f^*(C)$ is an irreducible reduced surface such that $D'\leqq f^*(C)$. Hence $D'=f^*(C)$. Hence $\rho(X)=\rho(Y)+1$.

(2)$\Rightarrow$(1). Clearly, $(X_y \cdot c_1(X))=2>0$. We shall show that $R=\boldsymbol{R}_+[X_y]$ is an extremal ray. Suppose that $\xi+\eta=a[X_y]$ with $\xi, \eta \in \overline{NE}(X)$ and $a \in \boldsymbol{R}_+$. Then $f_*\xi+f_*\eta=af_*X_y=0$ on Y, where $f_*\xi$ and $f_*\eta$ are effective 1-cycles on Y. Hence $f_*\xi=f_*\eta=0$. This implies that both ξ and η are $\boldsymbol{R}_+$-linear combinations of irreducible components of fibers of f. However, since $\rho(X)=\rho(Y)+1$, we have $\overline{NE}(X)\cap(f^* \operatorname{Pic} Y\otimes_{\boldsymbol{Z}} \boldsymbol{R})^{\perp}=\boldsymbol{R}_+[X_y]$. Therefore $\xi, \eta \in \boldsymbol{R}_+[X_y]$. Namely, X_y is an extremal rational curve.

Q.E.D.

4.6. Lemma (cf. [28]). *Let f: $X\rightarrow Y$ be a conic bundle. Let C be an irreducible curve on Y such that $f^{-1}(C)$ is reducible and let $f^{-1}(C)=D_1\cup D_2$, where D_1 and D_2 are irreducible reduced surfaces. Then the following assertions hold*:

(1) $C\subseteq \Delta_f-\operatorname{Sing}\Delta_f$ *and C is smooth.*

(2) *For $i=1, 2$, there exist a conic bundle f_i: $X_i\rightarrow Y$ and the contraction θ_i: $X\rightarrow X_i$ of D_i to a nonsingular curve such that $f=f_i\cdot\theta_i$.*

Remark. The above lemmas imply that, given a conic bundle f: $X\rightarrow Y$, a composition of contractions of $\boldsymbol{P}^1$-bundles onto nonsingular curves applied to X provides us with a conic bundle $\bar{f}$: $\bar{X}\rightarrow Y$ with $\rho(\bar{X})=\rho(Y)+1$.

4.7. Recall that a conic bundle f: $X\rightarrow Y$ is said to be standard if $\operatorname{Pic} X\cong f^* \operatorname{Pic} Y\oplus \boldsymbol{Z}K_X$. The following result is an easy consequence of the definitions.

Lemma. *Let f:$X\rightarrow Y$ be a conic bundle with $\rho(X)=\rho(Y)+1$. Then the following conditions are equivalent to each other*:

(1) *f: $X\rightarrow Y$ is not a standard conic bundle*;

(2) *f: $X\rightarrow Y$ is a $\boldsymbol{P}^1$-bundle in the Zariski topology.*

Remark. (1) It is clear that if a conic bundle f: $X\rightarrow Y$ is standard, then we have $\rho(X)=\rho(Y)+1$. Thus the above lemma asserts that if one excludes $\boldsymbol{P}^1$-bundles in the Zariski topology, two conditions "standard" and "$\rho(X)=\rho(Y)+1$" are equivalent.

(2) If a conic bundle f: $X\rightarrow Y$ satisfies the condition $\rho(X)=\rho(Y)+1$ and $\Delta_f\neq\phi$, then the generic fiber X_η is not rational over $k(\eta)=k(Y)$.

4.8. As a generalization of the notion of conic bundle, we introduce the notion of conical fibration. Let V be an irreducible projective variety defined over k. A structure of *conical fibration over* an irreducible projective variety S is a dominant mapping π: $V\rightarrow S$, whose general fibers are rational curves. Two conical fibrations π: $V\rightarrow S$ and π': $V'\rightarrow S'$ are *equivalent* to each other if there exist birational mappings $\lambda\colon V\to V'$ and $\eta\colon S\to S'$ such that $\pi'\cdot\lambda=\eta\cdot\pi$.

We shall prove the following result due to Zagorskih. Since his original proof contains a gap, we propose another proof.

Theorem (cf. Zagorskih [42]). *Let $g\colon V\to S$ be a conical fibration over an irreducible projective surface S. Then there exists a conic bundle $f\colon X\to Y$ such that $\rho(X)=\rho(Y)+1$ and $f\colon X\to Y$ is equivalent to $g\colon V\to S$.*

Proof. The proof consists of three steps.

(I) Resolving the singularities on V and S and the indeterminacy of g by blowing-ups on V and S, we may assume that

(1) V and S are nonsingular;

(2) g is a morphism whose general fibers are isomorphic to $\boldsymbol{P}^1$, although g may not be flat.

If the generic fiber V_η is rational over $k(\eta)=k(S)$, the assertion holds clearly. In fact, $g\colon V\to S$ is equivalent to the trivial $\boldsymbol{P}^1$-bundle $p_1\colon S\times\boldsymbol{P}^1\to S$. Therefore we assume henceforth that V_η is not rational over $k(\eta)$.

There exists an open set U of S such that every closed fiber of $g_U\colon V_U:=g^{-1}(U)\to U$ is a nonsingular rational curve. Let $\mathscr{K}$ be the relative canonical sheaf $\Omega^1_{V_U/U}$ on V_U. Let $\mathscr{E}_U:=(g_U)_*\mathscr{K}^{-1}$. Then, as in Lemma 4.1, there exists a section q_U of $H^0(\boldsymbol{P}(\mathscr{E}_U), \mathscr{O}_{\boldsymbol{P}(\mathscr{E}_U)}(2)\otimes p^*\mathscr{M}^2)$ such that V_U, embedded into $\boldsymbol{P}(\mathscr{E}_U)$ as a U-scheme, is identified with the zero locus of q_U, where $p\colon \boldsymbol{P}(\mathscr{E}_U)\to U$ is the canonical projection and $\mathscr{M}$ is an invertible sheaf on U.

By shrinking U if necessary, we assume that $\mathscr{E}_U$ (resp. $\mathscr{M}$) is a trivial $\mathscr{O}_U$-bundle of rank 3 (resp. rank 1). Hence $\boldsymbol{P}(\mathscr{E}_U)\cong U\times\boldsymbol{P}^2_k$. Choose a system of homogeneous coordinates (X_0, X_1, X_2) on $\boldsymbol{P}^2_k$ and write q_U in the form:

$$q_U=\sum_{0\leqq i\leqq j\leqq 2} a_{ij}X_iX_j \quad \text{with} \quad a_{ij}\in\Gamma(U, \mathscr{O}_S).$$

By shrinking U again if necessary, we may assume that the matrix

$$\begin{pmatrix} a_{00} & \frac{1}{2}a_{01} & \frac{1}{2}a_{02} \\ \frac{1}{2}a_{10} & a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{20} & \frac{1}{2}a_{21} & a_{22} \end{pmatrix}$$

is a diagonal matrix, i.e., $a_{ij}=0$ for $i\neq j$; this is possible after a suitable change of homogeneous coordinates of $U\times\boldsymbol{P}^2_k$. Thus we have

$$q_U=a_0X_0^2+a_1X_1^2+a_2X_2^2 \quad \text{with} \quad a_i\in\Gamma(U, \mathscr{O}_S).$$

Identify $\boldsymbol{P}(\mathscr{E}_U)$ with an open set of $S\times_k\boldsymbol{P}^2_k=\boldsymbol{P}^2_S$ and let W be the closure in $\boldsymbol{P}_S^2$ of V_U: $q_U=0$. Let π: $W\to S$ be the canonical projection. This process is construed in the following way: Associating q_U with (a_0, a_1, a_2), identify q_U with a regular section of $\Gamma(U, \mathcal{O}_S^{\oplus 3})$ and extend it to a regular section $q\in\Gamma(S, \mathcal{O}_S^{\oplus 3}\otimes\mathcal{M})$ in such a way that the zero locus W of q is irreducible, where $\mathcal{M}$ is an invertible sheaf on S and is chosen so that q, extending q_U, has no common zero divisors.

Suppose that q (or (a_0, a_1, a_2)) has a common zero P_1. Let σ_1: $S_1\to S$ be the blowing-up of P_1 and let $E_1:=\sigma_1^{-1}(P_1)$. Let $\nu_1=\min\{\nu_{P_1}(a_i)\,|\,i=0, 1, 2\}$, where $\nu_{P_1}(a)$ is defined as the largest integer n such that $a\in\mathfrak{M}_{P_1}^n$ and $a\notin\mathfrak{M}_{P_1}^{n+1}$, $\mathfrak{M}_{P_1}$ being the maximal ideal of $\mathcal{O}_{P_1,S}$. Then considering the proper transform q_1 of q in $\Gamma(S_1, \sigma_1^*(\mathcal{O}_S^{\oplus 3}\otimes\mathcal{M})\otimes\mathcal{O}(-\nu_1E_1))$, we can reduce the order of zero of q at P_1 by ν_1. We apply this operation at all common zeroes (including infinitely near common zeroes) of a_0, a_1, a_2, and we may assume that the equation $q=a_0X_0^2+a_1X_1^2+a_2X_2^2$ satisfies the conditions:

(i) a_0, a_1, a_2 are regular sections of $\Gamma(S, \mathcal{M})$, where $\mathcal{M}$ is an inver tible sheaf on S;

(ii) a_0, a_1, a_2 have no common zeroes; in other words, the variety W defined by $q=0$ does not contain any fiber of $\boldsymbol{P}_S^2\to S$.

However, W might not be smooth. In the next step, we shall consider how to desingularize W.

(II) Let Δ_π be the discriminant locus of π: $W\to S$. Namely, Δ_π is the curve (or the effective divisor) on S defined by $a_0a_1a_2=0$. Let τ: $S'\to S$ be the shortest succession of blowing-ups with centers at singular points (including infinitely near singular points) of $(\Delta_\pi)_{\mathrm{red}}$ such that, for $\tau^*\Delta_\pi=\sum_i\alpha_iC_i$, every C_i is smooth and $(\tau^*\Delta_\pi)_{\mathrm{red}}$ has only normal crossings as singularities. Let $W':=W\times_S(S', \tau)$ and let π': $W'\to S'$ be the canonical projection. For the sake of simplicity, we may assume that π: $W\to S$ has already these properties. Let $P\in S$. If $P\notin\Delta_\pi$ then $\pi^{-1}(P)$ is a smooth conic and W is nonsingular along $\pi^{-1}(P)$. Suppose that $P\in\Delta_\pi$. Let $\hat{\mathcal{O}}_{P,S}=k[[u, v]]$, where we choose u, v in such a way that if C_1 is an irreducible component of Δ_π passing through P, then C_1 is defined by $u=0$ locally at P; if C_1 and C_2 are irreducible components of Δ_π passing through P, then C_1 and C_2 are defined by $u=0$ and $v=0$ locally at P, respectively. Then we may assume that W is defined over an open neighborhood of P by

$$(*)\qquad q=\begin{cases}X_0^2+u^\alpha X_1^2+u^\beta X_2^2 & \text{if } P\in\Delta_\pi-\mathrm{Sing}\,((\Delta_\pi)_{\mathrm{red}})\\ X_0^2+u^\alpha v^\gamma X_1^2+u^\beta v^\delta X_2^2 & \text{if } P\in\mathrm{Sing}\,((\Delta_\pi)_{\mathrm{red}}),\end{cases}$$

where α, β, γ, $\delta\geqq 0$. We may assume that $\beta\geqq\alpha$. Moreover, if P moves on C_1 toward one point of $C_1\cap C_2$, then q varies from the first equation to the second one. Let $\mu:=[\beta/2]$ and $\nu:=[\alpha/2]$. Replace Proj $(\mathcal{O}_U[X_0, X_1, X_2])$ by Proj $(\mathcal{O}_U[X'_0, X'_1, X'_2])$, where $X'_0=X_0/u^{\mu}$, $X'_1=X_1/u^{\mu-\nu}$ and $X'_2=X_2$, and where U is an open neighborhood of P. Then $W_U:=\pi^{-1}(U)$ is replaced by $W'_U\subset\text{Proj}\,(\mathcal{O}_U[X'_0, X'_1, X'_2])$ defined by

$$q'=\begin{cases} X_0'^2+u^{\alpha-2\nu}X_1'^2+u^{\beta-2\mu}X_2'^2 & \text{if } P\in\Delta_\pi-\text{Sing}\,((\Delta_\pi)_{\text{red}}) \\ X_0'^2+u^{\alpha-2\nu}v^{\gamma}X_1'^2+u^{\beta-2\mu}v^{\delta}X_2'^2 & \text{if } P\in\text{Sing}\,((\Delta_\pi)_{\text{red}}). \end{cases}$$

Suppose that $\delta\geqq\gamma$; we consider only this case because the other case $\gamma\geqq\delta$ can be treated similarly. Let $\lambda:=[\delta/2]$ and $\rho:=[\gamma/2]$. When $P\in\text{Sing}\,((\Delta_\pi)_{\text{red}})$, we replace Proj $(\mathcal{O}_U[X'_0, X'_1, X'_2])$ by Proj $(\mathcal{O}_U[X''_0, X''_1, X''_2])$, where

$$X''_0=X'_0/v^{\lambda},\quad X''_1=X'_1/v^{\lambda-\rho}\quad\text{and}\quad X''_2=X'_2.$$

Then W'_U is replaced by W''_U, which is defined by

$$q''=X_0''^2+u^{\alpha-2\nu}v^{\gamma-2\rho}X_1''^2+u^{\beta-2\mu}v^{\delta-2\lambda}X_2''^2.$$

Note that the second replacement is compatible and commutative with the first replacement near the point $P\in\text{Sing}\,((\Delta_\pi)_{\text{red}})$. After these replacements made at all points of Δ_π, we may assume that if W is defined by $q=0$ in Proj $(\mathcal{O}_U[X_0, X_1, X_2])$ over an open neighborhood U of P and if q is written in the form as in (*), then we have $0\leqq\alpha,\ \beta\leqq1$ when $P\in\Delta_\pi-\text{Sing}\,((\Delta_\pi)_{\text{red}})$ while $0\leqq\alpha,\ \beta,\ \gamma,\ \delta\leqq1$ if $P\in\text{Sing}\,((\Delta_\pi)_{\text{red}})$. Note that W is then not necessarily embedded into $\boldsymbol{P}^2_S$ and hence that (X_0, X_1, X_2) is nothing but a local system of homogeneous coordinates.

If $\alpha=\beta=1$ in the case $P\in\Delta_\pi-\text{Sing}\,((\Delta_\pi)_{\text{red}})$ as well as the case $P\in\text{Sing}\,((\Delta_\pi)_{\text{red}})$, we make, as a convention, the following change of coordinates:

$$X'_0=X_0/u,\quad X'_1=X_1\quad\text{and}\quad X'_2=X_2.$$

Then q is changed to the equation:

$$q'=\begin{cases} uX_0'^2+X_1'^2+X_2'^2 & \text{if } P\in\Delta_\pi-\text{Sing}\,((\Delta_\pi)_{\text{red}}) \\ uX_0'^2+v^{\gamma}X_1'^2+v^{\delta}X_2'^2 & \text{if } P\in\text{Sing}\,((\Delta_\pi)_{\text{red}})\quad\text{and}\quad(\gamma,\delta)\neq(1,1) \\ uvX_0'^2+X_1'^2+X_2'^2 & \text{if } P\in\text{Sing}\,((\Delta_\pi)_{\text{red}})\quad\text{and}\quad(\gamma,\delta)=(1,1). \end{cases}$$

It is readily ascertained that the case where we have to consider another type of desingularization is, after a permutation of coordinates (X_0, X_1, X_2), of the form:

$$q=X_0^2+X_1^2+uvX_2^2\quad\text{with}\quad P\in\text{Sing}\,((\Delta_\pi)_{\text{red}}).$$

Let σ: $S' \to S$ be the blowing-up with center at P and let π': $W' := W \times_S (S', \sigma) \to S'$ be the base change of π: $W \to S$ relative to σ. Let $E := \sigma^{-1}(P)$. Now introduce u' and v' by the relations $u = vu'$ and $v = uv'$. Then W' is defined by the following equation in an open neighborhood of E:

$$q = \begin{cases} X_0^2 + X_1^2 + u^2 v' X_2^2 & \text{if } v' \neq \infty \\ X_0^2 + X_1^2 + u' v^2 X_2^2 & \text{if } v' = \infty. \end{cases}$$

By virtue of a replacement of coordinates of the said type, q is transformed to

$$q' = \begin{cases} X_0'^2 + X_1'^2 + v' X_2'^2 & \text{if } v' \neq \infty \\ X_0'^2 + X_1'^2 + u' X_2'^2 & \text{if } v' \neq \infty. \end{cases}$$

Then the variety W obtained by the above procedure is desingularized.

(III) We found a conic bundle f': $X' \to Y$ and birational mappings σ: $Y \to S$ and ε': $X' \to V$ such that $g \cdot \varepsilon' = \sigma \cdot f'$. By virtue of Lemma 4.6, we find a conic bundle f: $X \to Y$ and a birational morphism η: $X' \to X$ such that

(1) $\rho(X) = \rho(Y) + 1$,

(2) η is a composition of contractions of the type considered in Lemma 4.6, and $f' = f \cdot \eta$.

Since $\sigma \cdot f = g \cdot \varepsilon$ with $\varepsilon := \varepsilon' \cdot \eta^{-1}$: $X \to V$, the conical fibration g: $V \to S$ is equivalent to a conic bundle f: $X \to Y$ with $\rho(X) = \rho(Y) + 1$. Q.E.D.

4.9. We shall recall from Artin-Mumford [1] several basic results on the Brauer group of a function field of two variables. Let S be a non-singular projective simply connected surface defined over k and let $K := k(S)$ be the function field of S over k. We use the following notations: μ_n denotes the group of n-th roots of unity, $\mu = \bigcup_n \mu_n$ and $\mu^{-1} = \bigcup_n \mu_n^{-1} = \bigcup_n \mathrm{Hom}(\mu_n, \boldsymbol{Q}/\boldsymbol{Z})$. Then μ and μ^{-1} are non-canonically isomorphic to $\boldsymbol{Q}/\boldsymbol{Z}$. Br (S) denotes the Brauer group of Azumaya algebras over S; then $\mathrm{Br}(S) \cong H^2_{\text{ét}}(S, G_m)$. Similarly, Br (K) denotes the Brauer group of Azumaya algebras over K. For an irreducible curve C on S, $H^1_{\text{ét}}(k(C), \boldsymbol{Q}/\boldsymbol{Z})$ denotes the group of cyclic extensions of the function field $k(C)$, or the group of cyclic ramified coverings of the normalization $\tilde{C}$ of C.

4.9.1. Lemma (Artin-Mumford [1; Th. 1, p. 84]). *There is a canonical exact sequence*

$$0 \longrightarrow \mathrm{Br}\, S \xrightarrow{\text{res.}} \mathrm{Br}\, K \xrightarrow{a} \bigoplus_{\substack{\text{curves}\\ C}} H^1_{\text{ét}}(k(C), \boldsymbol{Q}/\boldsymbol{Z}) \xrightarrow{r} \bigoplus_{\substack{\text{points}\\ P}} \mu^{-1}$$
$$\xrightarrow{s} \mu^{-1} \longrightarrow 0,$$

where the maps are explained below.

(1) *The map* res. *is the restriction to the generic point of S.*

(2) *For an irreducible curve C on S, the local ring* $\mathcal{O}_{C,S}$ *of S at the generic point of C is a discrete valuation ring. The map a associates a finite central simple algebra D with a collection of cyclic extensions L of* $k(C)$, *C moving over the set of all irreducible curves on S, where L is obtained from a maximal order A for D over* $\mathcal{O}_{C,S}$ *as* $A \otimes_{\mathcal{O}_{C,S}} k(C)/(\text{radical})$.

(3) *Given a cyclic extension of* $k(C)$, *one may measure its ramification at a point* $\tilde{P}$ *of* $\tilde{C}$. *This is canonically an element of* μ^{-1}. *The map r is defined as the sum of the ramification at all points of the various* $\tilde{C}$ *lying over P.*

(4) *The map s is the summation.*

4.9.2. Let S and K be the same as above. For a positive integer n, consider an exact sequence of étale sheaves on K,

$$0 \longrightarrow \mu_{n,K} \longrightarrow SL(n)_K \longrightarrow PGL(n)_K \longrightarrow 1,$$

from which we obtain an injection $H^1_{\text{ét}}(K, PGL(n)) \xhookrightarrow{\iota_n} H^2_{\text{ét}}(K, \mu_n)$. If $n \mid n'$, we have a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(K, PGL(n)) & \xhookrightarrow{\iota_n} & H^2_{\text{ét}}(K, \mu_n) \\ \Big\downarrow \alpha_{n'n} & & \Big\downarrow \beta_{n'n} \\ H^1_{\text{ét}}(K, PGL(n')) & \xhookrightarrow{\iota_n} & H^2_{\text{ét}}(K, \mu_{n'}), \end{array}$$

where injections $\alpha_{n'n}$ and $\beta_{n'n}$ are induced by

$$A \in GL(n) \longmapsto \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix} \in GL(n')$$

$$r = n'/n,\ A_1 = \cdots = A_r = A$$

and the canonical injection $\mu_n \hookrightarrow \mu_{n'}$. Then the injections $\{\iota_n\}$ fit into the following commutative diagram

$$\begin{array}{ccc} \bigcup\limits_{n>0} H^1_{\text{ét}}(K, PGL(n)) & \hookrightarrow & \bigcup\limits_{n>0} H^2_{\text{ét}}(K, \mu_n) \\ \wr\| & & \wr\| \\ \operatorname{Br}(K) & \xrightarrow{\sim} & H^2_{\text{ét}}(K, \mu) \end{array}$$

(cf. Serre [38; pp. 164～166]). This implies that $H^1_{\text{ét}}(K, PGL(n)) \cong H^2_{\text{ét}}(K, \mu_n) \cong \operatorname{Br}(K)_n$ (= the n-torsion part of $\operatorname{Br}(K)$). In particular, when

$n=2$ this means that the Brauer-Severi varieties of dimension 1 over K (or quaternion algebras over K) represent the 2-torsion points of $\mathrm{Br}(K)$ (cf. Grothendieck [10]).

Let η be the generic point of S. Let A_η be a quaternion algebra over K, which represents an element d of $\mathrm{Br}(K)$ of order 2. By virtue of Lemma 4.9.1, there is a finite number of irreducible curves $C_1, \cdots, C_n$ on S at which $a(d)$ is nonzero. The union $C=C_1 \cup \cdots \cup C_n$ is called the ramification curve of A_η and $S-C$ is the maximal Zariski open set U of S such that A_η extends to an Azumaya algebra over U.

An order in A_η over S is a coherent $\mathcal{O}_S$-algebra $\mathcal{R}$ such that $\mathcal{R}_\eta = A_\eta$. Let $\mathcal{A}$ be a *maximal order* in A_η over S. Then it is known (cf. [2]) that $\mathcal{A}$ is a locally free $\mathcal{O}_S$-Module of rank four. With the open set U of S as above, $\mathcal{A}|_U$ is an Azumaya algebra over U. If the ramification curve C of A_η is *nonsingular*, we have a more precise result:

Lemma [1; Prop. 2, p. 88]. *With the above notations and assumptions, a maximal order $\mathcal{A}$ may be represented at a point $P \in C$ as the $\mathcal{O}_S$-algebra generated by elements x, y with relations*

$$x^2=a,\ y^2=bt \quad \textit{and} \quad xy=-yx,$$

where $t=0$ is a local equation for C, and a, b are units in $\mathcal{O}_S$. Moreover, a is not congruent to a square (modulo t).

Conversely, when a is not congruent to a square, the algebra presented in this way is a maximal order in some (non-trivial) quaternion algebra over K.

4.9.3. Lemma [1; Th. 2, p. 90]. *There is a canonical one-one correspondence between*

(1) *maximal orders in quaternion algebras A_η over K, whose ramification curve $C=C_1 \cup \cdots \cup C_n$ is nonsingular*

(2) *conic bundles $\pi: V \to S$ with the discriminant locus $\Delta_\pi = C$ and* $\rho(V)=\rho(S)+1$, i.e., *for every irreducible component C_i, the two components of $\pi^{-1}(\xi_i)$ ($\xi_i \in C_i$, the generic point) are not rational over $k(C_i)$, but define a quadratic extension of $k(C_i)$.*

The correspondence is given by assigning to a maximal order $\mathcal{A}$ an S-scheme V which represents the functor

$$S' \in (\mathrm{Sch}/S) \longmapsto \left\{ \textit{left } \mathcal{O}_{S'}\textit{-ideals of } \mathcal{A} \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'} \right\}.$$

Moreover, the quadratic extensions thus defined are just those given by $a(A_\eta)$ in Lemma 4.9.1.

4.10. We consider the case where S is a nonsingular complete rational surface defined over k. Then it is known that $\mathrm{Br}(S)=(0)$ (cf. [12]). Then Lemma 4.9.1 gives us an exact sequence

$$(*) \qquad 0 \longrightarrow \mathrm{Br}(K) \xrightarrow{a} \bigoplus_{\substack{\text{curves}\\ C}} H^1_{\text{ét}}(k(C), Q/Z) \xrightarrow{r} \bigoplus_{\substack{\text{points}\\ P}} \mu^{-1}.$$

Let C be a nonsingular irreducible curve on S and let $\rho: \tilde{C} \to C$ be a nontrivial unramified double covering of C. Then there exists a quaternion algebra A_η such that $a([A_\eta])=[\rho]$, where $[A_\eta]$ is the element of $\mathrm{Br}(K)$ represented by A_η and $[\rho]$ is the element of the second term of $(*)$ represented by ρ. By virtue of Lemma 4.9.3, there exists a conic bundle $\pi: V \to S$ such that $\Delta_\pi = C$, that $\rho: \tilde{C} \to C$ is the double covering associated with π (cf. 4.2) and that $\rho(V)=\rho(S)+1$. After Sarkisov [37], we call the pair (C, ρ) *the local invariant* of the conic bundle $\pi: V \to S$, or we say that $\pi: V \to S$ is defined by the local invariant (C, ρ).

Lemma. *Let S be a nonsingular complete rational surface. Let π_i: $V_i \to S$ $(i=1, 2)$ be conic bundles defined by the local invariants (C_i, ρ_i), where C_i is an irreducible nonsingular curve on S and ρ_i: $\tilde{C}_i \to C_i$ is a nontrivial unramified double covering. Then there exists a birational mapping θ: $V_1 \to V_2$ such that $\pi_1 = \pi_2 \cdot \theta$ if and only if $C_1 = C_2$ and $[\rho_1] = [\rho_2]$ in $H^1_{\text{ét}}(C, Z/2Z)$.*

Proof. If there is a birational mapping θ: $V_1 \to V_2$ such that $\pi_1 = \pi_2 \cdot \theta$, then θ_η: $(V_1)_\eta \xrightarrow{\sim} (V_2)_\eta$. Hence $[(V_1)_\eta]=[(V_2)_\eta]$ in $\mathrm{Br}(K)$ and $(C_1, [\rho_1]) = a([(V_1)_\eta]) = a([(V_2)_\eta]) = (C_2, [\rho_2])$. Conversely, if $C_1 = C_2$ and $[\rho_1]=[\rho_2]$ then $[(V_1)_\eta]=[(V_2)_\eta]$ in $\mathrm{Br}(K)$. Hence there exists a K-isomorphism θ_η: $(V_1)_\eta \xrightarrow{\sim} (V_2)_\eta$, which extends to a birational mapping θ: $V_1 \to V_2$ such that $\pi_1 = \pi_2 \cdot \theta$. Q.E.D.

Remark. The above construction shows the abundance of conic bundles.

4.11. We shall now state the result of Sarkisov [37].

Theorem. *Let Y be a nonsingular projective surface with $\chi(\mathcal{O}_Y) \geqq 1$ and let f: $X \to Y$ be a conical fibration with a nonsingular projective threefold X. Suppose that there exists a standard conic bundle f': $X' \to Y'$ which is equivalent to f: $X \to Y$ and satisfies $|4K_{Y'} + \Delta_{f'}| \neq \phi$. Then any conical fibration on X is equivalent to the given fibration f: $X \to Y$.*

We shall only note the following result which explains why the linear system $|4K_{Y'} + \Delta_{f'}|$ comes in.

Let f: $X \to Y$ be a conic bundle. Then $f_*(K_X^2) \approx -(4K_Y + \Delta_f)$.

Indeed, let T be an irreducible curve on Y. Since $T \sim H_1 - H_2$ with very ample divisors H_1, H_2 on Y, we may assume that T is a nonsingular irreducible curve not contained in Δ_f and meeting Δ_f transversally. Let $F := f^{-1}(T)$. Then we have

$$(K_X^2 \cdot F) = (K_X|_F)_F^2 = (K_F - F^2)_F^2 = (K_F^2) + 4(T^2).$$

Since F is a smooth ruled surface over T with $(\Delta_f \cdot T)$ degenerate fibers (consisting of 2 components), we have

$$(K_F^2) = -4((K_Y \cdot T) + (T^2)) - (\Delta_f \cdot T),$$

whence $(K_X^2 \cdot F) = (f_*(K_X^2) \cdot T) = -(4K_Y + \Delta_f \cdot T)$.

Corollary. *Let f: $X \to Y$ be a standard conic bundle with $|4K_Y + \Delta_f| \neq \phi$. Then X is irrational.*

4.12. We shall consider several examples.

4.12.1. Example (cf. Sarkisov [37], Roth [6]). Let V be an irreducible hypersurface of degree m in $\boldsymbol{P}_k^4$ passing through a line $\tilde{l} \subset \boldsymbol{P}^4$ with multiplicity $m-2$ and nonsingular outside of $\tilde{l}$. Then V has a conical fibration over $Y := \boldsymbol{P}^2$. In order to see this, take a plane Y in $\boldsymbol{P}^4$ such that $\tilde{l} \cap Y = \phi$. Then, for each plane Π of $\boldsymbol{P}^4$ with $\tilde{l} \subset \Pi$, we have $\Pi \cdot V = C_\Pi + (m-2)\tilde{l}$, where C_Π is a conic. Hence the projection of V with center $\tilde{l}$ gives a structure of conical fibration φ: $V \to Y$. More precisely, let W be the blowing-up of $\boldsymbol{P}^4$ with center $\tilde{l}$. Then the mapping φ gives rise to a morphism ψ: $W \to Y$, with respect to which W is a $\boldsymbol{P}^2$-bundle over Y. Indeed, $W = \mathrm{Proj}\,(\mathscr{E})$ with $\mathscr{E} := \mathcal{O}_{\boldsymbol{P}^2}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^2}$. Let L be a tautological divisor of W. Then $L \sim$ the exceptional divisor $\sigma^{-1}(\tilde{l})$ on W, where σ: $W \to \boldsymbol{P}^4$ is the said blowing-up. Let X be the proper transform of V on W. Let H be a hyperplane on $\boldsymbol{P}^4$ and let $F := \psi^*(l)$ for a line l on Y. Since $\sigma^* H \sim L + F$ and $\sigma^* V \sim X + (m-2)L$, we know that $X \sim 2L + mF$. Let $f := \psi|_X$: $X \to Y$. Then f gives a standard conic bundle structure on X because $K_W \sim -3L - 5F$, $K_X \sim -(L + (5-m)F)|_X$ and $\mathrm{Pic}\,(X) \cong f^*\,\mathrm{Pic}\,(Y) \oplus \boldsymbol{Z}(L|_X)$. If one identifies L with $\sigma^{-1}(\tilde{l})$, then $X \cdot L$ is a rational 2-section because $L \cong \boldsymbol{P}^1 \times \boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times (t) \subset \psi^{-1}(\psi(t))$ for every $t \in \boldsymbol{P}^2$. Thus X is unirational by virtue of the easy Lemma 4.12.2. The discriminant locus Δ_f is linearly equivalent to $(3m-4)l$. This can be computed as follows:

$$f_* K_X^2 \sim \psi_*((L - (m-5)F)^2 \cdot (2L + mF)) \sim (16 - 3m)l$$

and

$$f_* K_X^2 \sim -(4K_Y + \Delta_f) \sim 12l - \Delta_f,$$

whence $\Delta_f \sim (3m-4)l$. If $m=3$, then deg $\Delta_f = 5$; if $m=4$, then deg $\Delta_f =$ 8.

4.12.2. Lemma (cf. Beauville [4; Prop. 4.1 and Cor. 4.4]). *Let f: $X \to Y$ be a conic bundle over a nonsingular projective rational surface. Then the following assertions hold*:

(1) *If f has a rational section, then X is rational.*

(2) *If there exists a rational surface $S \subset X$ such that $f|_S$: $S \to Y$ is a surjective morphism of degree d, then X is unirational. More precisely, there exist a rational variety $\tilde{X}$ and a generically finite surjective morphism $\tilde{X} \to X$ of degree d.*

4.12.3. The following example is a generalization of 4.12.1.

Example (Sarkisov [37]). Let $\mathscr{E} := \mathcal{O}_{P^2}(-n) \oplus \mathcal{O}_{P^2}(-n) \oplus \mathcal{O}_{P^2}$, let $W := \mathrm{Proj}\,(\mathscr{E})$ and let L be the tautological divisor of W, where n is a non-negative integer. Let X be a general member of $|2L+(2n+1)F|$, where F is defined in a fashion similar to that in 4.12.1. Then X is a nonsingular threefold endowed with conic bundle structure by the restriction f of the canonical projection ψ: $W \to P^2$, whose discriminant locus Δ_f is linearly equivalent to $(2n+3)l$. Moreover, since $X \cdot L$ is a rational 2-section of f, X is unirational. Of course, we can take X so that X is irrational; the condition $n \geqq 5$ suffices.

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