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Holonomic Systems of Linear Differential Equations with Regular Singularities and Related Topics in Topology

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The purpose of this talk is to give applications of the theory of holonomic systems of linear differential equations with regular singularities (we shall call them in this note regular holonomic systems, for short). Regular holonomic systems appear, besides purely analytic applications, as tools to connect topological objects with geometric or algebraic objects. This comes from the fact that Hilbert's twenty-first problem holds for regular holonomic systems. In this note, we take two topics as examples: "intersection homology groups" and "vanishing cycle cohomologies".

§ 1. Hilbert's 21-st problem

1.1 We employ the same notations as in § 5 [0]. Then, Hilbert's 21-st problem for regular holonomic systems is stated as follows. (This was announced in [7].)

Theorem 1. The functor $\mathscr{DR}_X = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \quad)$: $D^b(\mathscr{D}_X)_{hr} \to D^b(\boldsymbol{C}_X)_c$ is an equivalence of the categories.

1.2 We shall discuss briefly how to construct the inverse functor Ψ: $D^b(\boldsymbol{C}_X)_c \to D^b(\mathscr{D}_X)_{hr}$ of $\mathscr{DR}_X$. Let $X_{\boldsymbol{R}}$ denote the underlying real analytic manifold of X, and let $\mathscr{D}b^{(0,p)}$ denote the sheaf of $(0,p)$-forms with distribution (in the sense of L. Schwartz) coefficients. A sheaf $\mathscr{F}$ of $\boldsymbol{C}$-vector spaces on $X_{\boldsymbol{R}}$ is called *$\boldsymbol{R}$-constructible* if there exists a stratification of $X_{\boldsymbol{R}}$ by subanalytic (see [2]) strata on which $\mathscr{F}$ is locally constant. For an $\boldsymbol{R}$-constructible sheaf $\mathscr{F}$, we define the subsheaf $\mathscr{T}\mathscr{H}(\mathscr{F})^{(0,p)}$ of $\mathscr{H}om_{\boldsymbol{C}_X}(\mathscr{F}, \mathscr{D}b^{(0,p)})$ as follows: for any open subset U of X, $\Gamma(U, \mathscr{T}\mathscr{H}(\mathscr{F})^{(0,p)}) = \{\varphi; \mathscr{F}|_U \to \mathscr{D}b^{(0,p)}|_U$; for any relatively compact open subanalytic subset V of U and for any $s \in \Gamma(V, \mathscr{F})$, there exists $u \in \Gamma(U, \mathscr{D}b^{(0,p)})$ such that $\varphi(s) = u|_V\}$. Then, $\mathscr{T}\mathscr{H}(\mathscr{F})^{(0,p)}$ is an *exact* contra-

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variant functor in $\mathscr{F}$. We can define the Dolbeault sequence

$$\mathscr{T}\mathscr{H}(\mathscr{F})^{(0,\cdot)}:\quad 0\longrightarrow \mathscr{T}\mathscr{H}(\mathscr{F})^{(0,0)}\xrightarrow{\bar\partial}\mathscr{T}\mathscr{H}(\mathscr{F})^{(0,1)}\xrightarrow{\bar\partial}\cdots \xrightarrow{\bar\partial}\mathscr{T}\mathscr{H}(\mathscr{F})^{(0,n)}\longrightarrow 0.$$

$(n=\dim X)$

This is naturally a complex of $\mathscr{D}_X$-Modules.

Now, let $\mathscr{F}^{\cdot}$ be an object of $D^b(C_X)_c$. Then there exists a bounded complex $\mathscr{G}^{\cdot}$ of $\boldsymbol{R}$-constructible sheaves, which is isomorphic to $\boldsymbol{R}\mathscr{H}om_{C_X}(\mathscr{F}^{\cdot}, C_X)$ in $D^b(C_X)_c$. Then $\Psi(\mathscr{F}^{\cdot})$ is defined as the simple complex associated with the double complex $\mathscr{T}\mathscr{H}(\mathscr{G}^{\cdot})^{(0,\cdot)}$.

1.3. The functor $\mathscr{D}\mathscr{R}_X$ enjoys the property

$$\mathscr{D}\mathscr{R}_X(\mathscr{M}^{\cdot *})=\mathscr{D}\mathscr{R}_X(\mathscr{M}^{\cdot})^{*} \qquad \text{for } \mathscr{M}^{\cdot}\in D^b(\mathscr{D}_X)_{hr} \tag{1.1}$$

where $\mathscr{M}^{\cdot *}=\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}^{\cdot}, \mathscr{D}_X)\otimes\Omega^{\otimes -1}[n]$ and $\mathscr{F}^{\cdot *}=\boldsymbol{R}\mathscr{H}om_{C_X}(\mathscr{F}^{\cdot}, C_X)$ for $\mathscr{F}^{\cdot}\in D^b(C_X)_c$. Here Ω denotes the sheaf of holomorphic n-forms on X. (See [6]).

The following theorem characterizes the vanishing of cohomology groups of $\mathscr{M}^{\cdot}$ in terms of $\mathscr{D}\mathscr{R}_X(\mathscr{M}^{\cdot})$.

Theorem 2. *For* $\mathscr{M}^{\cdot}\in D^b(\mathscr{D}_X)_{hr}$ *set* $\mathscr{F}^{\cdot}=\mathscr{D}\mathscr{R}_X(\mathscr{M}^{\cdot})$. *Then we have the following, where* $\mathscr{H}^{\cdot}$ *means the cohomology sheaves*:

(i) $\mathscr{H}^j(\mathscr{M}^{\cdot})=0$ *for any* $j>0$ *if and only if* codim Supp $\mathscr{H}^j(\mathscr{F}^{\cdot})\geqq j$ *for any* $j\geqq 0$

(ii) $\mathscr{H}^j(\mathscr{M}^{\cdot})=0$ *for any* $j<0$ *if and only if* codim Supp $\mathscr{H}^j(\mathscr{F}^{\cdot *})\geqq j$ *for any* $j\geqq 0$.

Corollary 3. $\mathscr{H}^j(\mathscr{M}^{\cdot})=0$ *for any* $j\neq 0$ *if and only if*

$$\text{codim Supp}\, \mathscr{H}^j(\mathscr{F}^{\cdot})\geqq j \quad \textit{and} \quad \text{codim Supp}\, \mathscr{H}^j(\mathscr{F}^{\cdot *})\geqq j \quad \textit{for any } j\geqq 0. \tag{1.2}$$

Therefore, the abelian category of regular holonomic systems is equivalent to the full subcategory of $D^b(C_X)_c$ consisting of $\mathscr{F}^{\cdot}$ satisfying (1.2). Thus, for an $\mathscr{F}^{\cdot}$ with (1.2), we can expect to study its property through the corresponding regular holonomic system. In subsequent sections, we discuss two examples of such $\mathscr{F}^{\cdot}$.

§ 2. Minimal extension and intersection homology groups

The first example is the "intersection homology group", introduced by Goresky-MacPherson [1].

2.1. Let Y be a closed analytic subset of a complex manifold X and j the inclusion $X-Y \hookrightarrow X$. Let $\mathcal{M}$ be a regular holonomic system defined on $X-Y$. We assume that $\mathcal{M}$ is extendable onto X. In another word, $j_!\mathcal{H}^k(\mathcal{DR}(\mathcal{M}))$ is constructible for any k, where $j_!$ is the direct image with compact support.

Proposition 4. *Under these conditions, there exists a regular holonomic system* $\tilde{\mathcal{M}}$ *defined on* X *with an isomorphism* $\mathcal{M} \xrightarrow{\sim} j^{-1}\tilde{\mathcal{M}}$, *which satisfies*

(2.1) $\mathcal{H}_Y^0(\tilde{\mathcal{M}})=\mathcal{H}_Y^0(\tilde{\mathcal{M}}^*)=0$, *i.e.* $\tilde{\mathcal{M}}$ *has neither coherent sub-Module supported in* Y *nor coherent quotient supported in* Y.

Moreover, such an $\tilde{\mathcal{M}}$ is unique up to isomorphism. (See [8], for the proof.)

This $\tilde{\mathcal{M}}$ is called the *minimal extension* of $\mathcal{M}$. The corresponding complex $\tilde{\mathcal{F}}^\cdot=\mathcal{DR}_X(\tilde{\mathcal{M}})$ is characterized as follows.

Proposition 5. $\tilde{\mathcal{F}}^\cdot$ *satisfies*

(2.2) $$j^{-1}\tilde{\mathcal{F}}^\cdot \cong \mathcal{DR}_X(\mathcal{M}),$$

(2.3) $$\operatorname{codim}_X Y \cap \operatorname{Supp} \mathcal{H}^j(\tilde{\mathcal{F}}^\cdot) > j \qquad \textit{for any } j,$$

(2.4) $$\operatorname{codim}_X Y \cap \operatorname{Supp} \mathcal{H}^j(\tilde{\mathcal{F}}^{\cdot *}) > j \qquad \textit{for any } j.$$

Moreover, such an $\tilde{\mathcal{F}}^\cdot \in D^b(\boldsymbol{C}_X)_c$ is uniquely determined by $\mathcal{DR}(\mathcal{M})$ up to isomorphism.

2.2. Now, let Z be a closed analytic subspace imbedded into a complex manifold X and let Z_{sing} denote the singular locus of Z. Let $\mathcal{M}$ be the minimal extension of the regular holonomic system $\mathcal{B}_{Z-Z_{\text{sing}}|X-Z_{\text{sing}}}$ defined on $X-Z_{\text{sing}}$. Here, for an r-codimensional closed submanifold W of a complex manifold V, we denote by $\mathcal{B}_{W|V}$ the algebraic local cohomology $\mathcal{D}_V$-module $\mathcal{H}_{[W]}^r(\mathcal{O}_V)$. Then $\pi_Z=\mathcal{DR}_X(\mathcal{M})[\dim X]$ is called the "intersection homology group" of Z. It belongs to $D^b(\boldsymbol{C}_X)_c$ and is supported on Z. By using Proposition 5, π_Z is characterized as follows:

(2.5) $$\pi_{Z|Z-Z_{\text{sing}}}=\boldsymbol{C}_{Z-Z_{\text{sing}}}[\dim Z].$$

(2.6) $$\dim Z_{\text{sing}} \cap \operatorname{Supp} \mathcal{H}^j(\pi_Z) < -j.$$

(2.7) $$\dim Z_{\text{sing}} \cap \operatorname{Supp} \mathcal{E}xt^j_{\boldsymbol{C}_Z}(\pi_Z, \omega_Z) < -j,$$

where ω_Z is the dualizing sheaf $\boldsymbol{R}\Gamma_Z(\boldsymbol{C}_X)[2\dim X]$. The uniqueness of the minimal extension implies $\mathcal{M} \cong \mathcal{M}^*$, which holds trivially on $X - Z_{\text{sing}}$. Therefore by (1.1) and the duality theorem, we have

$$R\,\mathcal{H}om_{C_Z}(\pi_Z, \omega_Z) = \pi_Z. \tag{2.8}$$

The Poincaré duality theorem (e.g. see [3]) implies

$$H^j(U, \pi_Z) = \mathrm{Hom}_C(H_c^{-j}(U, \pi_Z), C) \tag{2.9}$$

for any open subset U of Z.

§ 3. Vanishing cycle sheaves

3.1. Another example is the "vanishing cycle sheaf". Let $f: X \to D$ be a holomorphic map from a complex manifold X into the unit disk $D = \{z \in C; |z| < 1\}$. Let $\tilde{D}$ be the universal covering of $D - \{0\}$, p the projection $\tilde{D} \times_D X \to X$ and j the inclusion $X_0 = f^{-1}(0) \hookrightarrow X$. Then $\mathcal{F}^\cdot = j^{-1}Rp_*(C_{\tilde{D}\times_D X})$ is called the vanishing cycle sheaf (see [10]). For any $x \in X_0$, we have

(3.1) $\mathcal{H}^j(\mathcal{F}^\cdot)_x = H^j(U \cap f^{-1}(t), C)$, where U is the ball centered at x with radius ε and $0 < |t| \ll \varepsilon \ll 1$.

Then, $j_*\mathcal{F}^\cdot[-1]$ satisfies (1.2) and

$$R\,\mathcal{H}om_{C_{X_0}}(\mathcal{F}^\cdot, j^{-1}R\Gamma_{X_0}(C_X)) = \mathcal{F}^\cdot[-2]. \tag{3.2}$$

Hence

$$R\,\mathcal{H}om_{C_X}(j_*\mathcal{F}^\cdot, C_X) = j_*\mathcal{F}^\cdot[-2] \tag{3.2$'$}$$

by the duality theorem (3), since $\omega_{X_0} = R\Gamma_{X_0}(C_X)[2\dim X]$ is the dualizing sheaf for X_0. Therefore, to $j_*\mathcal{F}^\cdot[-1]$, there corresponds a regular holonomic system $\mathcal{V}$ which satisfies

$$\mathcal{DR}_X(\mathcal{V}) = j_*\mathcal{F}^\cdot[-1] \tag{3.3}$$

$$\mathcal{V} \cong \mathcal{V}^*. \tag{3.4}$$

The last property (3.4) follows from (3.2)′ and (1.1). Since the monodromy M for f acts on $\mathcal{F}^\cdot$, it acts also on $\mathcal{V}$. Hence $\mathcal{V}$ can be considered as a $\mathcal{D}_X[M, M^{-1}]$-Module.

3.2. The $\mathcal{D}_X[M, M^{-1}]$-module $\mathcal{V}$, which is a regular holonomic $\mathcal{D}_X$-module is easier to handle than $\mathcal{F}^\cdot$, since it can be described directly in terms of f as follows. Let $\mathcal{D}_X[s]$ denote the sheaf of rings $\mathcal{D}_X \otimes_C C[s]$ and set

$\mathcal{J}=\{P(s)\in\mathcal{D}_X[s];\ P(s)f^s=0 \text{ on } X-X_0\}$. Let $\mathcal{N}$ denote the $\mathcal{D}_X[s]$-Module $\mathcal{D}_X[s]/\mathcal{J}$, whose canonical generator is denoted by f^s. In [4], it is shown that $\mathcal{N}$ is a coherent $\mathcal{D}_X$-Module. Let t denote the $\mathcal{D}_X$-linear endomorphism of $\mathcal{N}$ given by $P(s)f^s \mapsto P(s+1)f^{s+1}=P(s+1)f\cdot f^s$. Then t and the multiplication by s do not commute but have the relation

$$[t, s]=t. \tag{3.5}$$

Let $C[t, s]$ denote the ring generated by variables t and s with the relation (3.5), and let $\mathcal{D}_X[s, t]$ denote $\mathcal{D}_X\otimes_C C[s, t]$. Then $\mathcal{N}$ has a structure of $\mathcal{D}_X[s, t]$-Module.

We know that for any $\mathcal{D}_X[s, t]$-sub-Module $\mathcal{N}'$ of $\mathcal{N}$ which is coherent over $\mathcal{D}_X$, $\mathcal{N}'/t\mathcal{N}'$ is a regular holonomic $\mathcal{D}_X$-Module. The minimal polynomial of s considered as an element of $\mathcal{E}nd\,(\mathcal{N}'/t\mathcal{N}')$ is denoted by $b(s, \mathcal{N}')$. We endow $\mathcal{N}'/t\mathcal{N}'$ with the structure of $\mathcal{D}_X[M, M^{-1}]$-Module by $M=\exp(2\pi\sqrt{-1}\,s)$. The function $b(s, \mathcal{N})$ is called the b-function of f and it is known that the roots of the b-function are negative rational numbers (see [4]).

Theorem 6. (i) *$\mathcal{N}/t\mathcal{N}$ and $\mathcal{N}'/t\mathcal{N}'$ give the same object in the Grothendieck group of the abelian category of $\mathcal{D}_X[M, M^{-1}]$-Modules supported in X_0 which are regular holonomic systems as $\mathcal{D}_X$-Modules.*

(ii) *There exists an $\mathcal{N}'$ such that no difference of two distinct roots of $b(s, \mathcal{N}')$ is an integer. For such an $\mathcal{N}'$, $\mathcal{N}'/t\mathcal{N}'$ is isomorphic to $\mathcal{V}$ as a $\mathcal{D}_X[M, M^{-1}]$-Module.*

We can prove this theorem by applying the theory of asymptotic expansions ([5]) to the regular holonomic system for $\delta(t-f(x))$. As its corollary, we have

Corollary 6. (i) *If λ is an eigenvalue of the monodromy on $\mathcal{H}^j(\mathcal{F}^\cdot)_x$, then there is a root s of $b(s)$ such that $\lambda=e^{2\pi\sqrt{-1}s}$.*

(ii) *If α is a root of the b-function, there exist j and x such that $e^{2\pi\sqrt{-1}\alpha}$ is an eigenvalue of $\mathcal{H}^j(\mathcal{F}^\cdot)_x$.*

This is known in the case of the isolated singularity. ([9]). I am informed by B. Malgrange that he and A. Beilinson and I. Bernstein obtained analogous results [11].

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