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Introduction to Algebraic Analysis on Complex Manifolds

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This talk is meant to be an introduction for algebraic geometers, who are familiar with local cohomology formalism and derived categories but not much with hard analysis, to the so-called "*algebraic analysis*" on complex manifolds. This is also meant to be an introduction to Kashiwara's talk later in this symposium [K5], where he will explain to us the relevance to algebraic geometry of holonomic systems with regular singularities (the twenty-first problem of Hilbert).

During the past ten years, there has been tremendous progress made in this field by Sato, Kashiwara, Kawai, Malgrange, Ramis and Mebkhout among others. The formulation, modelled after the idea and formalism of Sato's hyperfunctions, is a rather familiar one to algebraic geometers once they get used to what are going on. Most of the basic results can be found in Kashiwara [Km], [Kb], [K1], [K2] and Kashiwara-Kawai [K3]. Hopefully the by now voluminous literature is made a little bit more accessible to algebraic geometers when they get familiar with the formalism and typical examples found in this talk. We touch neither on the results concerning the differential operators of infinite order nor on the "*micro-local*" part of the theory, which in reality play very powerful roles in the proof of the results mentioned here. As a complete newcomer myself in this field, I benefited a great deal in reading Björk [B] and Pham [P], which are good introductions also to the micro-local theory.

§ 1. What is algebraic analysis?

(1.1) Let X be an n-dimensional complex manifold. We denote by $\mathscr{D}_X$ the sheaf of germs of holomorphic linear partial differential operators of finite order. At a point $x \in X$ with local coordinates $(z_1, \cdots, z_n)$, the stalk of $\mathscr{D}_X$ at x consists of finite sums

$$P=\sum_{\alpha} f_{\alpha}(z)(\partial/\partial z)^{\alpha}$$

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where $\alpha=(\alpha_1, \cdots, \alpha_n)$ runs through a finite set of n-tuples of nonnegative integers, $(\partial/\partial z)^\alpha=(\partial/\partial z_1)^{\alpha_1}(\partial/\partial z_2)^{\alpha_2}\cdots(\partial/\partial z_n)^{\alpha_n}$ and $f_\alpha(z)$ are holomorphic functions. The first basic result is that $\mathscr{D}_X$ is a *coherent* sheaf of (non-commutative) rings with each stalk left and right Noetherian of left and right *global dimension n* (cf. [Km]). $\mathscr{D}_X$ contains the structure sheaf $\mathscr{O}_X$ as well as the holomorphic tangent sheaf $\Theta_X=\mathscr{D}er_C(\mathscr{O}_X)$. Locally, $\mathscr{D}_X$ is generated by Θ_X over $\mathscr{O}_X$.

(1.2) As Kashiwara already noted in [Km], a system of holomorphic linear homogeneous partial differential equations

$$(*)\quad \sum_{1\leqslant j\leqslant l} P_{ij}u_j=0 \qquad (i=1, \cdots, s)$$

with l unknowns $u_1, \cdots, u_l$ and with $P_{ij} \in H^0(X, \mathscr{D}_X)$ can be described intrinsically as a left $\mathscr{D}_X$-module M of finite presentation

$$0\longleftarrow M\longleftarrow \mathscr{D}_X^{\oplus l}\xleftarrow{P}\mathscr{D}_X^{\oplus s}$$

where P sends a germ $(Q_1, \cdots, Q_s)$ of $\mathscr{D}_X^{\oplus s}$ to the matrix product $(Q_1, \cdots, Q_s)(P_{ij})$ of $\mathscr{D}_X^{\oplus l}$. Then a global holomorphic solution $u=(u_1, \cdots, u_l)$ of $(*)$ is nothing but an element of the $\boldsymbol{C}$-vector space

$$\mathrm{Hom}_{\mathscr{D}_X}(M, \mathscr{O}_X)$$

of left $\mathscr{D}_X$-homomorphisms, where $\mathscr{O}_X$ is naturally a left $\mathscr{D}_X$-module by differentiation (cf. (2.1)).

This simple-minded observation leads us also to consider the $\boldsymbol{C}_X$-module ($\boldsymbol{C}_X$ being the constant sheaf with fibers $\boldsymbol{C}$ on X)

$$\mathscr{H}om_{\mathscr{D}_X}(M, \mathscr{O}_X)$$

of germs of local holomorphic solutions, or more generally the derived functors

$$\mathrm{Ext}^q_{\mathscr{D}_X}(M, M'), \qquad \mathscr{E}xt^q_{\mathscr{D}_X}(M, M')$$

for left $\mathscr{D}_X$-modules M, M', as well as

$$\mathscr{T}or_q^{\mathscr{D}_X}(N, M)$$

for left $\mathscr{D}_X$-modules M and right $\mathscr{D}_X$-modules N. They will play important roles in "algebraic analysis", i.e., the study of $\mathscr{D}_X$-coherent left (or right) modules.

§ 2. Examples of $\mathscr{D}_X$-modules

We now list typical $\mathscr{D}_X$-modules, to get an idea of how widespread they are.

(2.1) $\mathcal{O}_X$ is canonically a left $\mathscr{D}_X$-module; for germs $P=\sum f_\alpha(z)(\partial/\partial z)^\alpha \in \mathscr{D}_X$ and $f \in \mathcal{O}_X$, we let $Pf=\sum f_\alpha(z)(\partial/\partial z)^\alpha f$, where $(\partial/\partial z)^\alpha f$ is the partial derivative of f. We have the following $\mathscr{D}_X$-locally free resolution:

$$0 \longleftarrow \mathcal{O}_X \longleftarrow \mathscr{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X$$

which will play important roles later, where $\bigwedge^{\bullet} \Theta_X$ is the exterior algebra of Θ_X over $\mathcal{O}_X$ and the boundary map

$$\mathscr{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{p-1} \Theta_X \overset{\partial}{\longleftarrow} \mathscr{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{p} \Theta_X$$

is defined by

$$\partial(P\otimes(v_1\wedge\cdots\wedge v_p)) = \sum_{1\leq i\leq p} (-1)^{i-1}(Pv_i)\otimes(v_1\wedge\cdots\overset{i}{\vee}\cdots\wedge v_p) + \sum_{i<j}(-1)^{i+j}P\otimes([v_i, v_j]\wedge v_1\wedge\cdots\overset{i}{\vee}\cdots\overset{j}{\vee}\cdots\wedge v_p)$$

for germs $P \in \mathscr{D}_X$ and $v_1, \cdots, v_p \in \Theta_X$. Here $[v, v']=vv'-v'v$ is the Lie bracket.

(2.2) Since $\mathcal{O}_X$ is a subalgebra of $\mathscr{D}_X$, a left $\mathscr{D}_X$-module M is an $\mathcal{O}_X$-module. M is *$\mathcal{O}_X$-coherent* if and only if M is a *vector bundle* (i.e., $\mathcal{O}_X$-locally free of finite rank) *with a holomorphic integrable connection.* Among other characterizations, this means the following: Θ_X acts $\boldsymbol{C}_X$-linearly on M via a $\boldsymbol{C}_X$-homomorphism

$$\Theta_X \otimes_{\boldsymbol{C}_X} M \longrightarrow M$$

with the image of $v\otimes m$ written as vm for germs $v \in \Theta_X$ and $m \in M$ so that

(a) $(av)m=a(vm)$ for $a \in \mathcal{O}_X$, $v \in \Theta_X$, $m \in M$,

(b) $v(am)=(v(a))m+a(vm)$ for $a \in \mathcal{O}_X$, $v \in \Theta_X$, $m \in M$,

(c) $[v, v']m=v(v'm)-v'(vm)$ for $v, v' \in \Theta_X$, $m \in M$.

The usual *de Rham complex* $\mathscr{DR}_X(M)=\Omega^{\bullet}_X\otimes_{\mathcal{O}_X} M$ of the form

$$0 \longrightarrow M \overset{\nabla}{\longrightarrow} \Omega^1_X\otimes_{\mathcal{O}_X} M \overset{\nabla}{\longrightarrow} \Omega^2_X\otimes_{\mathcal{O}_X} M \overset{\nabla}{\longrightarrow} \cdots \overset{\nabla}{\longrightarrow} \Omega^n_X\otimes_{\mathcal{O}_X} M \longrightarrow 0$$

built out of the integrable connection ∇ can be interpreted as the complex of $\boldsymbol{C}_X$-modules

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X\otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X, M)$$

whose cohomology sheaves, i.e., the *de Rham cohomology sheaves* are nothing but

$$\mathscr{H}^q(\mathscr{D}\mathscr{R}_X(M))=\mathscr{E}xt^q_{\mathscr{D}_X}(\mathscr{O}_X, M).$$

More generally, a left $\mathscr{D}_X$-module M is nothing but an $\mathscr{O}_X$-module with a $\boldsymbol{C}_X$-linear action of Θ_X on M satisfying the above properties (a), (b), (c), since $\mathscr{D}_X$ is locally generated by Θ_X over $\mathscr{O}_X$ with the relations $va=v(a)+av$, $[v, v']=vv'-v'v$ for $a \in \mathscr{O}_X$ and $v, v' \in \Theta_X$. Here $v(a)$ means the action of v on $a \in \mathscr{O}_X$.

(2.3) Using the above simple observation (cf. [K2]) we see the difference between left and right $\mathscr{D}_X$-module structures as follows: Let N be a right $\mathscr{D}_X$-module. Then we have $\boldsymbol{C}_X$-homomorphisms

$$\mathscr{O}_X\otimes_{\boldsymbol{C}_X} N \longrightarrow N, \qquad \Theta_X\otimes_{\boldsymbol{C}_X} N \longrightarrow N$$

sending $a\otimes n$ to $a*n=na$ and $v\otimes n$ to $v*n=-nv$ for $a \in \mathscr{O}_X$, $v \in \Theta_X$ and $n \in N$, where the right hand sides are via the right $\mathscr{D}_X$-module action. Then we easily see that

(a*) $(a*v)*n=v*(a*n)$,

(b*) $v*(a*n)=v(a)*n+a*(v*n)$,

(c*) $[v, v']*n=v*v'*n-v'*v*n$.

Conversely, $\boldsymbol{C}_X$-linear actions of $\mathscr{O}_X$ and Θ_X on a $\boldsymbol{C}_X$-module N satisfying (a*), (b*), (c*) give rise to a right $\mathscr{D}_X$-module structure on N. Thus formally, the difference between left and right $\mathscr{D}_X$-module structures lies *only* in (a) and (a*).

In this way, we see that the *canonical invertible sheaf* $\omega_X=\Omega^n_X$ is naturally a *right* $\mathscr{D}_X$-module by letting $v*\phi$ with $v \in \Theta_X$ and $\phi \in \omega_X$ to be the *Lie derivative* of ϕ with respect to the vector field v, i.e.,

$$(v*\phi)(v_1, \cdots, v_n)=v(\phi(v_1, \cdots, v_n))-\sum_{1\leqslant i\leqslant n} \phi(v_1, \cdots, \overset{i}{\check{[v, v_i]}}, \cdots, v_n)$$

$$\text{for } v_1, \cdots, v_n \in \Theta_X.$$

(2.4) Using the above simple observations (2.2) and (2.3), we see the following for left $\mathscr{D}_X$-modules M, M' and right $\mathscr{D}_X$-modules N, N':

$M\otimes_{\mathscr{O}_X} M'$ is a left $\mathscr{D}_X$-module via $v(m\otimes m')=(vm)\otimes m'+m\otimes(vm')$,

$M\otimes_{\mathscr{O}_X} N$ is a right $\mathscr{D}_X$-module via $v*(m\otimes n)=(vm)\otimes n+m\otimes(v*n)$,

$\mathscr{H}om_{\mathscr{O}_X}(M, M')$ is a left $\mathscr{D}_X$-module via $(v\psi)(m)=v(\psi(m))-\psi(vm)$,

$\mathscr{H}om_{\mathscr{O}_X}(N, N')$ is a left $\mathscr{D}_X$-module via $(v\psi)(n)=v*(\psi(n))-\psi(v*n)$,

$\mathscr{H}om_{\mathscr{O}_X}(M, N)$ is a right $\mathscr{D}_X$-module via $(v*\psi)(m)=v*(\psi(m))-\psi(vm)$.

Here is an easy way to remember all these: When X is a compact Riemann surface of genus g, an $\mathcal{O}_X$-invertible sheaf L on X can be a left $\mathcal{D}_X$-module (resp. right $\mathcal{D}_X$-module) if and only if $\deg L=0$ (resp. $\deg L=2g-2$).

(2.5) For instance, for $q\geqslant 0$ and a left $\mathcal{D}_X$-module M, we have a natural right $\mathcal{D}_X$-module structure on $\mathcal{E}xt^q_{\mathcal{D}_X}(M, \mathcal{D}_X)$ induced by that of the second factor $\mathcal{D}_X$. Hence

$$\mathcal{E}xt^q_{\mathcal{D}_X}(M, \mathcal{D}_X)\otimes_{\mathcal{O}_X}\omega_X^{-1}=\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{E}xt^q_{\mathcal{D}_X}(M, \mathcal{D}_X))$$

is naturally a *left* $\mathcal{D}_X$-module (cf. (4.3)).

(2.6) *Algebraic local cohomology sheaves.* Let us start with the simplest nontrivial case. Let X be a neighborhood of 0 in $\boldsymbol{C}$ with the coordinate z. The stalks of $\mathcal{O}_X$ and $\mathcal{D}_X$ at $z=0$ are the convergent power series ring $\mathcal{O}=\boldsymbol{C}\{z\}$ and $\mathcal{D}=\bigoplus_{m\geqslant 0}\mathcal{O}(d/dz)^m$. Then $\mathcal{O}$ as well as the field $\mathcal{O}[z^{-1}]$ of finite Laurent series are left $\mathcal{D}$-modules. Thus the quotient $\mathcal{O}[z^{-1}]/\mathcal{O}$ is also a left $\mathcal{D}$-module supported at $z=0$, with the $\boldsymbol{C}$-basis consisting of the residue classes $\delta^{(m)}$ of $(-1)^m m!z^{-(m+1)}$ mod $\mathcal{O}$ for $m=0, 1, \cdots$. Then $(d/dz)\delta^{(m)}=\delta^{(m+1)}$ and $z\delta^{(m)}=-m\delta^{(m-1)}$. Thus $\delta^{(m)}$ is the m-th derivative of ($2\pi i$ times) Dirac's delta function $\delta^{(0)}$.

These are particular cases of the following more general *algebraic local cohomology sheaves*: For a closed analytic subspace Y of a complex manifold X defined by an $\mathcal{O}_X$-ideal I, we define the q-th algebraic local cohomology sheaf of an $\mathcal{O}_X$-module M by the inductive limit

$$\mathcal{H}^q_{[Y]}(M)=\operatorname*{ind\,lim}_{\nu\geq 0}\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_X/I^\nu, M).$$

Compare this with the *transcendental* local cohomology sheaves $\mathcal{H}^q_Y(M)$, which are the usual derived functors of the subsheaf functor $\mathcal{H}^0_Y(M)$ of sections with support in Y. We also consider

$$\mathcal{H}^q_{[X|Y]}(M)=\operatorname*{ind\,lim}_{\nu\geqslant 0}\mathcal{E}xt^q_{\mathcal{O}_X}(I^\nu, M),$$

hence have a long exact sequence

$$0\longrightarrow\mathcal{H}^0_{[Y]}(M)\longrightarrow M\longrightarrow\mathcal{H}^0_{[X|Y]}(M)\longrightarrow\mathcal{H}^1_{[Y]}(M)\longrightarrow 0$$

and isomorphisms

$$\mathcal{H}^q_{[X|Y]}(M)\xrightarrow{\sim}\mathcal{H}^{q+1}_{[Y]}(M)\qquad q\geqslant 1.$$

When M is a *left* $\mathcal{D}_X$-*module*, we see that $\mathcal{H}^q_{[Y]}(M)$ and $\mathcal{H}^q_{[X|Y]}(M)$ are again naturally *left $\mathscr{D}_X$-modules*, since $\Theta_X I^{\nu} \subset I^{\nu-1}$. The above exact sequence and isomorphisms are those of left $\mathscr{D}_X$-modules. In contrast, the transcendental local cohomology sheaves $\mathscr{H}^q_Y(M)$ are naturally left $\mathscr{D}^{\infty}_X$-modules, where $\mathscr{D}^{\infty}_X$ is the sheaf of germs of holomorphic linear differential operators of infinite order.

(2.7) *Inverse images.* Let $f\colon Y \to X$ be a holomorphic map of complex manifolds. For an $\mathcal{O}_X$-module M, let $f^*M = \mathcal{O}_Y \otimes_{\mathcal{O}_X} M = \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1}(M)$ be the usual inverse image as ringed spaces. If M is a left $\mathscr{D}_X$-module, then f^*M is naturally a left $\mathscr{D}_Y$-module by the chain rule: For local coordinates $(w_1, \cdots, w_m)$ at a point $y \in Y$ and those $(z_1, \cdots, z_n)$ for $f(y) \in X$, with $z_j = f_j(w)$, the canonical homomorphism $\Theta_Y \longrightarrow f^*\Theta_X$ sends $\partial/\partial w_i$ to $\sum_{1 \leqslant j \leqslant n} (\partial f_j/\partial w_i)(f(w))(\partial/\partial z_j)$. Even if M is $\mathscr{D}_X$-coherent, however, f^*M need *not* be $\mathscr{D}_Y$-coherent. (cf. [Km] and (4.7.5)).

We mention here the following observation in [Km] concerning the *Cauchy problem* (the initial value problem) for linear partial differential equations: Given a coherent left $\mathscr{D}_X$-module M and an embedding $f\colon Y \to X$, we ask if the canonical homomorphism of $\boldsymbol{C}_Y$-modules

$$f^{-1}\,\mathscr{H}om_{\mathscr{D}_X}(M, \mathcal{O}_X) \longrightarrow \mathscr{H}om_{\mathscr{D}_Y}(f^*M, \mathcal{O}_Y),$$

sending a local holomorphic solution of the corresponding system of partial differential equations (cf. (1.2)) to its "initial value at Y", is injective or not, where f^{-1} on the left hand side is the inverse image of the $\boldsymbol{C}_X$-module. Kashiwara [Km, Chapter II] shows that it is injective and f^*M is $\mathscr{D}_Y$-coherent if f is "non-characteristic" with respect to M.

(2.8) *Direct images.* Compared with the inverse images in (2.7), the direct images are more difficult to define but are richer in geometric significance ("integration along fibers"). Let $f\colon Y \longrightarrow X$ be a holomorphic map of complex manifolds. We first define a sheaf $\mathscr{D}_{X \leftarrow Y}$ on Y which is both a right $\mathscr{D}_Y$-module and a left $f^{-1}(\mathscr{D}_X)$-module. Here and elsewhere, f^{-1} is the inverse image as a sheaf. Then the whole *sequence* of direct images of a left $\mathscr{D}_Y$-module N are defined to be the *hyperdirect image left $\mathscr{D}_X$-modules* (cf. [Kb, § 4])

$$\int_f^q N := \boldsymbol{R}^q f_* \Big(\mathscr{D}_{X \leftarrow Y} \overset{L}{\underset{\mathscr{D}_Y}{\otimes}} N \Big) \qquad q \in \boldsymbol{Z},$$

where $\otimes^L_{\mathscr{D}_Y}$ means the left derived functor of the tensor product over $\mathscr{D}_Y$ (or, more precisely, the tensor product in the derived category of $\mathscr{D}_Y$-modules).

The sheaf $\mathscr{D}_{X \leftarrow Y}$ above is defined (cf. [K2, § 4] and [Kb, § 4]) as

$$\mathscr{D}_{X\leftarrow Y}=f^{-1}(\mathscr{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{f^{-1}(\mathcal{O}_X)}\omega_Y=f^*(\mathscr{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1})\otimes_{\mathcal{O}_Y}\omega_Y,$$

where ω_X and ω_Y are the canonical invertible sheaves of X and Y, respectively, and f^* is the inverse image as ringed spaces. In the first description, $\mathscr{D}_X$ is a right $\mathscr{D}_X$-module by right multiplication, hence $\mathscr{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$ is a left $\mathscr{D}_X$-module by (2.4). Thus $f^{-1}(\mathscr{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1})$ is a left $f^{-1}(\mathscr{D}_X)$-module. On the other hand, $\mathscr{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}$ is a left $\mathscr{D}_X$-module by left multiplication, hence $f^*(\mathscr{D}_X\otimes_{\mathcal{O}_X}\omega_X^{-1})$ is a left $\mathscr{D}_Y$-module as an inverse image by (2.7). In view of (2.3) and (2.4), we conclude that the second description of $\mathscr{D}_{X\leftarrow Y}$ above is a right $\mathscr{D}_Y$-module.

Remark. Do not confuse the above $\mathscr{D}_{X\leftarrow Y}$ with

$$\mathscr{D}_{Y\to X}=f^*(\mathscr{D}_X)$$

which is a left $\mathscr{D}_Y$-module and a right $f^{-1}(\mathscr{D}_X)$-module and which appears in the literature.

Let us now describe rather complicated $\mathscr{D}_{X\leftarrow Y}$ and the direct images in more detail when f is smooth. The study in the general case is reduced to the composite of this case and the case of an embedding, since (i) a holomorphic map $f\colon Y\to X$ is a composite of an embedding

$$\gamma\colon Y\longrightarrow Y\times X,\quad \gamma(y)=(y,f(y))\quad \text{for } y\in Y$$

and the second projection $p_2\colon Y\times X\to X$ which is smooth, and since (ii) for another holomorphic map $g\colon Z\to Y$ we have (cf. [Kb, § 4])

$$\mathscr{D}_{X\leftarrow Z}=g^{-1}(\mathscr{D}_{X\leftarrow Y})\otimes_{g^{-1}(\mathscr{D}_Y)}\mathscr{D}_{Y\leftarrow Z}.$$

Sometimes convenient is the following description (cf. [Kb, § 4]):

$$\mathscr{D}_{X\leftarrow Y}=\gamma^{-1}(\mathscr{H}^{\dim X}_{[\gamma(Y)]}(p_1^*\omega_Y)).$$

When $f\colon Y\to X$ is *smooth* of relative dimension r, we have an exact sequence

$$0\longrightarrow\Theta_{Y/X}\longrightarrow\Theta_Y\longrightarrow f^*\Theta_X\longrightarrow 0,$$

hence, in particular, the subring $\mathscr{D}_{Y/X}$ of $\mathscr{D}_Y$ generated by the relative tangent sheaf $\Theta_{Y/X}=\mathscr{H}om_{\mathcal{O}_Y}(\Omega^1_{Y/X},\mathcal{O}_Y)$, and a surjective ring homomorphism $\mathscr{D}_Y\to f^*\mathscr{D}_X\to 0$. Then we see (cf. [P] and [B]) that $\mathscr{D}_{X\leftarrow Y}$ has the following right $\mathscr{D}_Y$-*locally free resolution*:

$$\mathscr{D}\mathscr{R}_{Y/X}(\mathscr{D}_Y)[r]\longrightarrow\mathscr{D}_{X\leftarrow Y}\longrightarrow 0.$$

Here $[r]$ means the usual dimension shift to the left by r of the relative de Rham complex for $\mathscr{D}_Y$ defined by

$$\mathscr{D}\mathscr{R}_{Y/X}(\mathscr{D}_Y)=\Omega^{\cdot}_{Y/X}\otimes_{\mathcal{O}_Y}\mathscr{D}_Y,$$

where $\Omega^{\cdot}_{Y/X}$ is the usual complex of relative holomorphic forms with respect to the relative exterior differentiation $d_{Y/X}$. The map

$$d\colon \Omega^p_{Y/X}\otimes_{\mathcal{O}_Y}\mathscr{D}_Y\longrightarrow\Omega^{p+1}_{Y/X}\otimes_{\mathcal{O}_Y}\mathscr{D}_Y$$

for various p's are defined uniquely in such a way that

(i) $d(\alpha\wedge\phi)=(d_{Y/X}\alpha)\wedge\phi+(-1)^p\alpha\wedge d\phi$ for $\alpha\in\Omega^p_{Y/X}$, $\phi\in\Omega^q_{Y/X}\otimes_{\mathcal{O}_Y}\mathscr{D}_Y$

(ii) d sends $1\in\mathscr{D}_Y=\Omega^0_{Y/X}\otimes_{\mathcal{O}_Y}\mathscr{D}_Y$ to the canonical element of $\Omega^1_{Y/X}\otimes_{\mathcal{O}_Y}\Theta_{Y/X}\subset\Omega^1_{Y/X}\otimes_{\mathcal{O}_Y}\mathscr{D}_Y$ corresponding to the identity element of $\mathscr{E}nd_{\mathcal{O}_Y}(\Theta_{Y/X})=\Omega^1_{Y/X}\otimes_{\mathcal{O}_Y}\Theta_{Y/X}$.

We obtain the augmentation homomorphism

$$\Omega^r_{Y/X}\otimes_{\mathcal{O}_Y}\mathscr{D}_Y\longrightarrow\mathscr{D}_{X\leftarrow Y}\longrightarrow 0$$

by noticing that $\Omega^r_{Y/X}=(f^*\omega_X)^{-1}\otimes_{\mathcal{O}_Y}\omega_Y$ is a right $\mathscr{D}_{Y/X}$-module as a relativized version of (2.3). Hence the direct image $\boldsymbol{R}^q f_*(\mathscr{D}_{X\leftarrow Y}\otimes^L_{\mathscr{D}_Y}N)$ of a left $\mathscr{D}_Y$-module N is seen to coincide with the hyperdirect image left $\mathscr{D}_X$-module

$$\boldsymbol{R}^{r+q}f_*(\mathscr{D}\mathscr{R}_{Y/X}(N))\qquad q\in\boldsymbol{Z}$$

of the relative de Rham complex for N given by

$$\mathscr{D}\mathscr{R}_{Y/X}(N)=\mathscr{D}\mathscr{R}_{Y/X}(\mathscr{D}_Y)\otimes_{\mathscr{D}_Y}N=\Omega^{\cdot}_{Y/X}\otimes_{\mathcal{O}_Y}N.$$

When N is a vector bundle on Y with a holomorphic integrable connection,we see (cf. (2.2)) that $\mathscr{D}\mathscr{R}_{Y/X}(N)$ coincides with the usual relative de Rham complex for the induced relative connection. Hence we see that the hyperdirect image left $\mathscr{D}_X$-module $\boldsymbol{R}^{r+q}f_*(\mathscr{D}\mathscr{R}_{Y/X}(N))$ coincides with the usual relative de Rham cohomology $\mathcal{O}_X$-module $\mathscr{H}^{r+q}_{DR}(Y/X;N)$ with the *Gauss-Manin connection* (cf., for instance, [D]).

§ 3. Structure of coherent $\mathscr{D}_X$-modules

Now that we have familiarized ourselves with left $\mathscr{D}_X$-modules, we explain briefly how their structure is studied.

(3.1) *Filtration.* $\mathscr{D}_X$ has the following increasing filtration by $\mathcal{O}_X$-submodules:

$$\mathcal{O}_X=\mathscr{D}_X^{(0)}\subset\mathscr{D}_X^{(1)}\subset\mathscr{D}_X^{(2)}\subset\cdots\subset\mathscr{D}_X^{(m-1)}\subset\mathscr{D}_X^{(m)}\subset\cdots\subset\mathscr{D}_X,$$

where the stalks of $\mathscr{D}_X^{(m)}$ consist of $P=\sum_\alpha f_\alpha(z)(\partial/\partial z)^\alpha$ with $f_\alpha(z)\neq 0$ only for $|\alpha|=\alpha_1+\cdots+\alpha_n\leqslant m$. We say P to be of order m if $P\in\mathscr{D}_X^{(m)}$ but $P\notin\mathscr{D}_X^{(m-1)}$. In this case, the residue class $\sigma_m(P)$ of P mod $\mathscr{D}_X^{(m-1)}$ is, by tradition, called the *principal symbol* of P.

As in commutative algebra (where we deal with the dual decreasing filtrations), we consider the *associated graded sheaf of rings*

$$\mathrm{gr}\,(\mathscr{D}_X)=\bigoplus_{m\geqslant 0}\mathscr{D}_X^{(m)}/\mathscr{D}_X^{(m-1)}.$$

We have a canonical isomorphism

$$\mathrm{gr}\,(\mathscr{D}_X)=\mathrm{Symm}_{\mathcal{O}_X}(\Theta_X),$$

the symmetric algebra over $\mathcal{O}_X$ of the holomorphic tangent sheaf Θ_X. Hence in particular it is commutative. Thus we are naturally led to consider the *cotangent bundle* (as a complex manifold)

$$X\xleftarrow{\check{\pi}}T^*X=\mathrm{Specan}\,(\mathrm{gr}\,(\mathscr{D}_X))=V(\Theta_X)$$

(this $\check{\pi}$ is sometimes also denoted π) or sometimes the *projectivized cotangent bundle*

$$X\xleftarrow{\pi}\boldsymbol{P}^*X=\mathrm{Projan}\,(\mathrm{gr}\,(\mathscr{D}_X))=\boldsymbol{P}(\Theta_X)$$

as well. Here we follow the notations of Grothendieck's EGA.

(3.2) Important analytic subspaces of T^*X are the *conormal* bundles $T_Y^*X\subset T^*X$ of analytic subspaces Y of X defined as follows: First of all, when Y is an r-codimensional closed *submanifold* of X defined by an $\mathcal{O}_X$-ideal I, we have the normal sheaf $\mathscr{N}_{Y/X}=\mathscr{H}om_{\mathcal{O}_Y}(I/I^2,\mathcal{O}_Y)$ and an exact sequence

$$0\longrightarrow\Theta_Y\longrightarrow\mathcal{O}_Y\otimes_{\mathcal{O}_X}\Theta_X\longrightarrow\mathscr{N}_{Y/X}\longrightarrow 0.$$

Hence we have a submanifold $T_Y^*X=V(\mathscr{N}_{Y/X})$ of $T^*X=V(\Theta_X)$, which is a rank r subbundle of the restriction $(T^*X)\times_X Y$ to Y of the cotangent bundle of X. More generally, if Y is an open subset of a closed analytic subspace possibly with singularities, we define its conormal bundle T_Y^*X to be the *closure* of that of the smooth part of Y. In particular, the conormal bundle T_X^*X of X in X is nothing but the *zero section* of T^*X.

(3.3) *The characteristic variety.* To a coherent left $\mathscr{D}_X$-module M, we associate a closed analytic subspace $V_M\subset T^*X$, called the characteristic variety of M, in the following way: Locally M is a quotient of a free left $\mathscr{D}_X$-module L of finite rank. Hence we have *locally* an increasing filtration Γ

$$0=M^{(0)}\subset M^{(1)}\subset\cdots\subset M^{(m-1)}\subset M^{(m)}\subset\cdots\subset M$$

induced by the obvious filtration of L. This filtration has the following properties:

(i) $\bigcup_{m\geqslant 0} M^{(m)}=M$.

(ii) Each $M^{(m)}$ is $\mathcal{O}_X$-coherent.

(iii) $\mathscr{D}_X^{(l)}M^{(m)}\subset M^{(l+m)}$ for all l, m, and the associated graded gr $(\mathscr{D}_X)$-module

$$\mathrm{gr}_\Gamma (M)=\bigoplus_{m\geqslant 0} M^{(m)}/M^{(m-1)}$$

is locally finitely generated.

A filtration Γ of M satisfying (i), (ii) , (iii) is called a *good filtration.*

We can then consider *locally* the annihilator ideal I in gr $(\mathscr{D}_X)$ of $\mathrm{gr}_\Gamma (M)$. One of the basic results says that its *radical* rad (I) does not depend on the good filtration Γ chosen. (cf. [Km], [B], [P]). We thus have a *globally defined* homogeneous radical ideal sheaf $J_M\subset$ gr $(\mathscr{D}_X)$, hence a closed *reduced* analytic subspace

$$V_M=\mathrm{Specan}\,(\mathrm{gr}\,(\mathscr{D}_X)/J_M)\subset T^*X,$$

which, by tradition, is called the *characteristic variety* of M. Sometimes V_M is also denoted $\check{\mathrm{SS}}\,(M)$ and is called the *singular spectrum* of M.

Remark. In the above, we had to paste together locally obtained data to define the characteristic variety. In "micro-local analysis" where we introduce a sheaf $\check{\mathscr{E}}_X$ on T^*X of micro-differential operators, V_M can be defined simply and more directly as the *support* of the "pulled-back" $\check{\mathscr{E}}_X$-module

$$\check{\mathscr{E}}_X \underset{\check{\pi}^{-1}(\mathscr{D}_X)}{\otimes} \check{\pi}^{-1}(M).$$

Besides being a fiber system of "conic subvarieties" (i.e., stable under the fiberwise scalar multiplication, since J_M is homogeneous), V_M is known to be *involutive*. This means the following: Locally, M has a good filtration Γ. Let P and P' be germs of $\mathscr{D}_X$ of orders m and m', respectively, such that their principal symbols $\sigma_m(P)$ and $\sigma_{m'}(P')$ belong to J_M, i.e., some powers of them annihilate $\mathrm{gr}_\Gamma (M)$. Then the principal symbol $\sigma_{m+m'-1}([P, P'])$ of $[P, P']=PP'-P'P$ also belongs to J_M. This

$\sigma_{m+m'-1}([P, P'])$ is known to depend only on $\sigma_m(P)$ and $\sigma_{m'}(P')$ and is called the *Poisson bracket* of $\sigma_m(p)$ and $\sigma_{m'}(P')$. The involutiveness is proved micro-locally. See, for instance, [SKK, p. 453, Theorem 5.3.2] and [B, Chap. 4, Theorem 9.1 and Chap. 5, Section 5]. See also Gabber [G] for a purely algebraic proof, where he considers a very big graded module $\oplus_{m\in Z} M^{(m)}$ over a very big ring $\oplus_{m\in Z}\mathscr{D}_X^{(m)}$.

Geometrically, the involutiveness means the following: First of all, the cotangent bundle T^*X is a so-called *symplectic manifold* globally equipped with the *canonical* 1-*form* θ and its exterior derivative $\omega=d\theta$, called the *fundamental* 2-*form*. In local coordinates $(z_1, \cdots, z_n)$ of X and the principal symbols $\zeta_i=\sigma_1(\partial/\partial z_i)$, they can be written as

$$\theta=\sum_{1\leqslant i\leqslant n}\zeta_i dz_i, \qquad \omega=\sum_{1\leqslant i\leqslant n} d\zeta_i\wedge dz_i.$$

Then a closed reduced analytic subspace $V\subset T^*X$ is *involutive* if and only if for any *smooth* point v of V, the tangent space T_vV in $T_v(T^*X)$ satisfies $(T_vV)^{\perp}\subset T_vV$, where the left hand side is the perpendicular with respect to the fundamental 2-form ω. As a consequence, *each* irreducible component of the characteristic variety V_M is of dimension $\geqslant n=\dim X$ (*Bernstein's inequality*). This is also a consequence of the fact that the stalks of $\mathscr{D}_X$ are of left global dimension n (cf. [B], [Km]).

§ 4. Holonomic $\mathscr{D}_X$-modules with regular singularities

A left $\mathscr{D}_X$-module M is said to be *holonomic* if M is $\mathscr{D}_X$-coherent and, moreover, if the dimension of its characteristic variety V_M is equal to $n=\dim X$, i.e., the smallest possible. In this sense, M is also called "maximally overdetermined".

If M is a holonomic $\mathscr{D}_X$-module, then M satisfies various properties:

(4.1) For each $x\in X$, the stalk M_x is a monogenic $\mathscr{D}_{X,x}$-module of finite length. (cf., for instance, [B, Chap. 1, Theorem 8.18 and Chap. 2, Theorem 7.13]).

(4.2) For each irreducible component V of the characteristic variety $V_M\subset T^*X$, the image $\check{\pi}(V)\subset X$ is an irreducible closed analytic subspace and V coincides with the conormal bundle $T^*_{\check{\pi}(V)}X$ in the sense of (3.2). (cf., for instance, [P, pp. 92–94]).

(4.3) $\mathscr{E}xt^q_{\mathscr{D}_X}(M, \mathscr{D}_X)=0$ for $q\neq n$ and the *adjoint* left $\mathscr{D}_X$-module (cf. (2.5)) defined by

$$M^*=\mathscr{E}xt^n_{\mathscr{D}_X}(M, \mathscr{D}_X)\otimes_{\mathcal{O}_X}\omega_X^{-1}$$

is also holonomic with $V_{M^*}=V_M$ and $M^{**}=M$. (cf. [Km, 3.1.5, 3.1.2, 3.1.8] and [Kb, Theorem 2.3]).

(4.4) By far the most important result is the *finiteness theorem* [K2, Theorem 4.8]: If M and M' are holonomic left $\mathscr{D}_X$-modules, then the $\boldsymbol{C}_X$-modules $\mathscr{E}xt^q_{\mathscr{D}_X}(M, M')$ are $\boldsymbol{C}_X$-constructible for all q. Recall that a $\boldsymbol{C}_X$-module F is said to be *$\boldsymbol{C}_X$-constructible* if there exists a finite sequence of closed analytic subspaces

$$\emptyset=X_0\subset X_1\subset\cdots\subset X_{j-1}\subset X_j\subset\cdots\subset X$$

such that the restriction of F to each $X_j\setminus X_{j-1}$ is a locally constant C_X-module of finite rank.

Actually, Kashiwara showed that there exists a *Whitney stratification* of X such that the restriction of $\mathscr{E}xt^q_{\mathscr{D}_X}(M, M')$ to each stratum is a locally constant $\boldsymbol{C}_X$-module. Of particular importance are the following special cases: The "solution sheaves"

$$\mathscr{E}xt^q_{\mathscr{D}_X}(M, \mathscr{O}_X)$$

and the "de Rham sheaves"

$$\mathscr{E}xt^q_{\mathscr{D}_X}(\mathscr{O}_X, M)$$

are mutually dual constructible $\boldsymbol{C}_X$-modules (cf. (1.2) and (2.2)). Note that $\mathscr{O}_X$ is holonomic, since its characteristic variety is easily seen to be the zero section T^*_XX.

(4.5) Among holonomic $\mathscr{D}_X$-modules, the following are of utmost importance: A holonomic $\mathscr{D}_X$-module M is said to have *regular singularities* if it satisfies the following equivalent conditions.

(i) Locally at each point, there exists a coherent $\mathscr{O}_X$-submodule F of M with $M=\mathscr{D}_XF$ such that if the principal symbol $\sigma_m(P)$ of $P\in\mathscr{D}_X^{(m)}$ vanishes on the characteristic variety V_M, then $PF\subset\mathscr{D}_X^{(m-1)}F$. In another word, in a neighborhood of each point of X, there exists locally a good filtration Γ

$$0=M^{(0)}\subset M^{(1)}\subset\cdots\subset M^{(l-1)}\subset M^{(l)}\subset\cdots\subset M$$

of M such that if $\sigma_m(P)$ for $P\in\mathscr{D}_X^{(m)}$ vanishes on V_M, then $\sigma_m(P)$ annihilates $\mathrm{gr}_\Gamma(M)$.

(ii) [K3, Corollary 5.1.11] There exists a *global* good filtration Γ of M such that if $\sigma_m(P)$ for $P\in\mathscr{D}_X^{(m)}$ vanishes on V_M, then $\sigma_m(P)$ annihilates $\mathrm{gr}_\Gamma(M)$.

(iii) [K3, Theorems 6.3.1 and 6.4.1] The "formal solutions" coincide with the "holomorphic solutions", i.e., for each q and for each point $x \in X$, the canonical homomorphism

$$\mathscr{E}xt^q_{\mathscr{D}_X}(M, \mathscr{O}_X)_x \xrightarrow{\sim} \mathrm{Ext}^q_{\mathscr{D}_{X,x}}(M_x, \hat{\mathscr{O}}_{X,x})$$

is an isomorphism, where the suffices x mean the stalks at x and $\hat{\mathscr{O}}_{X,x}$ is the usual completion of the local ring $\mathscr{O}_{X,x}$. The right hand side is the usual extension module.

(iv) [K3, Theorem 6.4.7] For each point $x \in X$, we have the equality

$$\sum_q (-1)^q \dim_C \mathscr{E}xt^q_{\mathscr{D}_X}(M, \mathscr{O}_X)_x = \sum_q (-1)^q \dim_C \mathrm{Ext}^q_{\mathscr{D}_{X,x}}(M_x, \hat{\mathscr{O}}_{X,x}).$$

Note that by the finiteness theorem in (4.4), the left hand side is well defined.

These seemingly unrelated conditions are shown to be equivalent only through other micro-local characterizations (i.e., in terms of $\mathscr{E}_X$-modules and, indeed, $\mathscr{E}^\infty_X$-modules, where $\mathscr{E}^\infty_X$ is the sheaf on T^*X of germs of micro-differential operators of infinite order) and through the reduction to the important special cases introduced by Deligne [D] (i.e., holonomic $\mathscr{D}_X$-modules of Deligne-type along hypersurfaces of X.) The whole paper [K3] is devoted to the task.

(4.6) The above definition of holonomicity with regular singularities is a successful generalization to higher dimensions of the classical notion of ordinary differential equations with regular singularities in the one-dimensional case. To see this, let us go back to the situation in (2.6), namely, X is a neighborhood of 0 in C, $\mathscr{O} = C\{z\}$ and $\mathscr{D} = \oplus_{m \geqslant 0} \mathscr{O}(d/dz)^m$. Consider a left $\mathscr{D}$-module $M = \mathscr{D}/\mathscr{D}P$ with

$$P = a_m(d/dz)^m + a_{m-1}(d/dz)^{m-1} + \cdots + a_p(d/dz)^p + \cdots + a_0$$

such that $a_p \in \mathscr{O}$ and $a_m \neq 0$. M is obviously a holonomic $\mathscr{D}$-module. Let us denote by $\mathrm{ord}(a_p)$ the usual order of zero at $z=0$ of the holomorphic function a_p. Then P is classically said to have regular singularity at $z=0$ if the coefficients satisfy

$$\mathrm{ord}\,(a_p/a_m) \geqslant -(m-p) \qquad \text{for all } p.$$

It was Malgrange [Ma] (and Deligne, cf. [K]), who first formulated this condition in such a way that (iv) of (4.5) is a natural generalization to higher dimensions. We have a $\mathscr{D}$-free resolution

$$0 \longleftarrow M \longleftarrow \mathscr{D} \xleftarrow{P} \mathscr{D} \longleftarrow 0,$$

where P is the right multiplication by $P \in \mathscr{D}$. Hence we have

$$\mathscr{E}xt^q_{\mathscr{D}}(M, \mathcal{O}) = \begin{cases} \ker (\mathcal{O} \xrightarrow{P} \mathcal{O}) & q=0 \\ \operatorname{coker} (\mathcal{O} \xrightarrow{P} \mathcal{O}) & q=1 \\ 0 & q \neq 0, 1. \end{cases}$$

Malgrange showed that the *index* defined by

$$\operatorname{index} (\mathcal{O} \xrightarrow{P} \mathcal{O}) = \dim_C \ker (\mathcal{O} \xrightarrow{P} \mathcal{O}) - \dim_C \operatorname{coker} (\mathcal{O} \xrightarrow{P} \mathcal{O})$$

is equal to $m - \operatorname{ord}(a_m)$, while if the completion $\hat{\mathcal{O}} = C[[z]]$ is taken instead of $\mathcal{O}$, then

$$\operatorname{index} (\hat{\mathcal{O}} \xrightarrow{P} \hat{\mathcal{O}}) = \sup_p (p - \operatorname{ord}(a_p)).$$

Thus $\operatorname{index} (\mathcal{O} \xrightarrow{P} \mathcal{O}) = \operatorname{index} (\hat{\mathcal{O}} \xrightarrow{P} \hat{\mathcal{O}})$ if and only if P has regular singularity at $z=0$.

Here are simple examples to give the feeling for the equivalence of (i) through (iv) in (4.5).

(4.6.1) $\mathcal{O} \cong \mathscr{D}/\mathscr{D}(d/dz)$ is holonomic with regular singularities. Take the one-step filtration $0 = M^{(0)} \subset M^{(1)} = M^{(2)} = \cdots = \mathcal{O}$. The characteristic variety is the zero section $T^*_X X = \{\zeta = 0\}$ with $\zeta = \sigma_1(d/dz)$.

(4.6.2) $M = \mathcal{O}[z^{-1}] \cong \mathscr{D}/\mathscr{D}(zd/dz+1)$ is holonomic with regular singularities. Take the filtration $M^{(j)} = z^{-j}\mathcal{O}$. The characteristic variety is $\check{\pi}^{-1}(0) \cup T^*_X X = \{z\zeta = 0\}$.

(4.6.3) $M = \mathcal{O}[z^{-1}]/\mathcal{O} = \oplus_{m \geqslant 0} C\delta^{(m)} \cong \mathscr{D}/\mathscr{D}z$ is holonomic with regular singularities. Take the filtration $M^{(j)} = \oplus_{0 \leqslant m \leqslant j} C\delta^{(m)}$. The characteristic variety is $\check{\pi}^{-1}(0) = \{z=0\}$.

(4.6.4) $M = \mathcal{O}[z^{-1}] \exp (z^{-1}) = \mathscr{D}/\mathscr{D}(z^2 d/dz + 1)$ does not have regular singularities, but the characteristic variety is $\{z\zeta = 0\}$.

(4.6.5) For $\lambda \in C$ with $\lambda \neq 0, 1, 2, \cdots$ and for a nonnegative integer l, the left $\mathscr{D}$-module

$$M_{\lambda, l} = \mathscr{D}/\mathscr{D}(zd/dz - \lambda)^l \cong \bigoplus_{0 \leqslant j < l} \mathcal{O}[z^{-1}]z^\lambda (\log z)^j$$

is holonomic with regular singularities. Take the filtration

$$M^{(i)}_{\lambda,l}=\bigoplus_{0\leqslant j<l} z^{-i}\mathcal{O}z^{\lambda}(\log z)^{j}.$$

The characteristic variety is again $\{z\zeta=0\}$.

(4.6.6) (cf. [P, p. 102—p. 106]) A coherent left $\mathscr{D}$-module M is holonomic with regular singularities if and only if $M\neq 0$ and if there exists an $\mathcal{O}$-coherent submodule $L\subset M$ with $\mathscr{D}L=M$ and $(zd/dz)L\subset L$. In fact this is the case if and only if the kernel and the cokernel of the "localization $\mathscr{D}$-homomorphism"

$$M\longrightarrow\mathcal{O}[z^{-1}]\otimes_{\mathcal{O}} M$$

are isomorphic to direct sums of $\mathcal{O}[z^{-1}]/\mathcal{O}$ and if $\mathcal{O}[z^{-1}]\otimes_{\mathcal{O}} M$, which is nothing but the algebraic local cohomology sheaf $\mathscr{H}^{0}_{[X|Y]}(M)$ with $Y=\{0\}$, is a direct sum of $M_{\lambda,l}$'s with λ 's as well as the differences among them not being integers.

(4.7) Coming back to a general complex manifold X, here are some of the important general results concerning holonomic $\mathscr{D}_X$-modules and those with regular singularities.

(4.7.1) $\mathcal{O}_X$ or, more generally, $\mathcal{O}_X$-coherent $\mathscr{D}_X$-modules $M\neq 0$ are holonomic with regular singularities (cf. (2.1) and (2.2)).

(4.7.2) For an exact sequence $0\to M'\to M\to M''\to 0$ of left $\mathscr{D}_X$-modules, M is holonomic (resp. holonomic with regular singularities) if and only if so are M' and M''. (cf. [BK, Proposition 1.3, (1) and (3)]).

(4.7.3) For an r-codimensional closed *submanifold* $Y\subset X$, we see that the algebraic local cohomology sheaves $\mathscr{H}^{q}_{[Y]}(\mathcal{O}_X)$ vanish for $q\neq r$ and that $\mathscr{H}^{r}_{[Y]}(\mathcal{O}_X)$ is $\mathscr{D}_X$-holonomic with regular singularities. The characteristic variety is the conormal bundle $T^{*}_{Y}X$.

(4.7.4) For a closed analytic subspace $Y\subset X$ and a left $\mathscr{D}_X$-module M which is holonomic (resp. holonomic with regular singularities), the algebraic local cohomology sheaves $\mathscr{H}^{q}_{[Y]}(M)$ and $\mathscr{H}^{q}_{[X|Y]}(M)$ are $\mathscr{D}_X$-holonomic (resp. $\mathscr{D}_X$-holonomic with regular singularities). (cf. [K2, Theorem 1.3], resp. [K3, Theorem 5.4.1]). By the Mayer-Vietoris exact sequence, the proof is reduced to the case of hypersurfaces, and then the formalism used to study the Bernstein-Sato polynomials is employed.

More generally, [BK, Proposition 1.3, (7)] shows the following: For closed analytic subspaces $Y\supset Z$ of X, the algebraic local cohomology sheaves $\mathscr{H}^{q}_{[Y\setminus Z]}(M)$ are defined to be the derived functors of $\mathscr{H}^{0}_{[Y]}\mathscr{H}^{0}_{[X|Z]}(M)$. If M is $\mathscr{D}_X$-holonomic with regular singularities, then so are $\mathscr{H}^{q}_{[Y\setminus Z]}(M)$.

(4.7.5) Let f: $Y \to X$ be a holomorphic map. For a left $\mathscr{D}_X$-module M which is holonomic (resp. holonomic with regular singularities), the inverse image (cf. (2.7)) f^*M is $\mathscr{D}_Y$-holonomic (resp. $\mathscr{D}_Y$-holonomic with regular singularities). (cf. [K2, Theorem 4.4], resp. [K3, Corollary 5.4.8]).

(4.7.6) For left $\mathscr{D}_X$-modules M, M' which are holonomic (resp. holonomic with regular singularities), $\mathscr{T}or_q^{\mathscr{O}_X}(M, M')$ is $\mathscr{D}_X$-holonomic (resp. $\mathscr{D}_X$-holonomic with regular singularities) for each q. (cf. [K2, Theorem 4.6], resp. [K3, Corollary 5.4.7]).

(4.7.7) For a *proper* holomorphic map f: $Y \to X$ and a holonomic $\mathscr{D}_Y$-module N with regular singularities, the direct images (cf. (2.8))

$$\int_f^q N := \boldsymbol{R}^q f_*(\mathscr{D}_{X \leftarrow Y} \otimes^L_{\mathscr{D}_Y} N)$$

are $\mathscr{D}_X$-holonomic with regular singularities. (cf. [K3, Theorem 6.2.1] for f projective. The proof in the general case will appear in [K6]).

Note that for a smooth (algebraic) morphism f: $Y \to X$ of (not necessarily compact) algebraic manifolds and an $\mathscr{O}_Y$-coherent left $\mathscr{D}_Y$-module N, we have a theorem of Griffiths-Katz-Deligne to the effect that the Gauss-Manin connection on the direct images is with regular singularities (cf. (2.8)).

(4.7.8) For a holonomic $\mathscr{D}_X$-module M with regular singularities, the adjoint (cf. (4.3))

$$M^* = \mathscr{E}xt^n_{\mathscr{D}_X}(M, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \omega_X^{-1}$$

is $\mathscr{D}_X$-holonomic with regular singularities.

§ 5. The twenty-first problem of Hilbert

An extremely lucid account of this problem as well as Deligne's contributions to its solution in the algebraic case can be found in Katz [K]. Recently, Kashiwara [K4] and also Mebkhout [Me] gave the following solution to the problem in higher dimensions. See also Kashiwara's account in [K5] as well as his [K6] now in preparation.

The *de Rham functor*

$$M^\cdot \longmapsto \mathscr{D}\mathscr{R}_X(M^\cdot) := \boldsymbol{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, M^\cdot)$$

(resp. the *solution functor*

$$M^\cdot \longmapsto \mathscr{S}ol_X(M^\cdot) := \boldsymbol{R}\,\mathscr{H}om_{\mathscr{D}_X}(M^\cdot, \mathscr{O}_X))$$

gives rise to an *equivalence* (resp. *anti-equivalence*) from the derived category $\boldsymbol{D}^b(\mathscr{D}_X)_{hr}$ of the bounded complexes of left $\mathscr{D}_X$-modules whose cohomology sheaves are $\mathscr{D}_X$-holonomic with regular singularities, to the derived category $\boldsymbol{D}^b(\boldsymbol{C}_X)_c$ of the bounded complexes of $\boldsymbol{C}_X$-modules with $\boldsymbol{C}_X$-constructible cohomology sheaves. Note that $\mathscr{D}\mathscr{R}_X$ and $\mathscr{S}ol_X$ are mutually dual in the sense that

$$\mathscr{S}ol_X(M^\cdot)=\boldsymbol{R}\,\mathscr{H}om_{\boldsymbol{C}_X}(\mathscr{D}\mathscr{R}_X(M^\cdot), \boldsymbol{C}_X)$$
$$\mathscr{D}\mathscr{R}_X(M^\cdot)=\boldsymbol{R}\,\mathscr{H}om_{\boldsymbol{C}_X}(\mathscr{S}ol_X(M^\cdot), \boldsymbol{C}_X)$$

for a bounded complex $M^\cdot$ of $\mathscr{D}_X$-modules with holonomic cohomology sheaves. (cf., [K3, Proposition 1.4.6]).

We have the following compatibility for $\mathscr{D}\mathscr{R}_X$ with respect to various operations on $\boldsymbol{D}^b(\mathscr{D}_X)_{hr}$ we considered in (4.7) and corresponding operations on $\boldsymbol{D}^b(\boldsymbol{C}_X)_c$.

(5.1) $\mathscr{O}_X$ corresponds to the usual de Rham complex $\mathscr{D}\mathscr{R}_X(\mathscr{O}_X)$ (cf. (2.2)), which is quasi-isomorphic to the constant sheaf $\boldsymbol{C}_X$.

(5.2) We can extend the notion of the adjoint in (4.7.8) to $\boldsymbol{D}^b(\mathscr{D}_X)_{hr}$ by

$$(M^\cdot)^*:=\boldsymbol{R}\,\mathscr{H}om_{\mathscr{D}_X}(M^\cdot, \mathscr{D}_X)\otimes_{\mathscr{O}_X}\omega_X^{-1}[n],$$

where $n=\dim X$. Then it corresponds to the $\boldsymbol{C}_X$-dual in $\boldsymbol{D}^b(\boldsymbol{C}_X)_c$, i.e.,

$$\mathscr{D}\mathscr{R}_X(M^{\cdot *})=\boldsymbol{R}\,\mathscr{H}om_{\boldsymbol{C}_X}(\mathscr{D}\mathscr{R}_X(M^\cdot), \boldsymbol{C}_X)$$

(cf. [BK, Proposition 1.1, (5)]).

(5.3) For $M^\cdot$ and $M'^\cdot$ in $\boldsymbol{D}^b(\mathscr{D}_X)_{hr}$, we have

$$\boldsymbol{R}\,\mathscr{H}om_{\mathscr{D}_X}(M^\cdot, M'^\cdot)=\boldsymbol{R}\,\mathscr{H}om_{\boldsymbol{C}_X}(\mathscr{D}\mathscr{R}_X(M^\cdot), \mathscr{D}\mathscr{R}_X(M'^\cdot))$$

(cf. [BK, Proposition 1.3, (4) and (9)]).

(5.4) For a holomorphic map $f\colon Y\to X$, the inverse image in (4.7.5) corresponds to the inverse image f^{-1} of $\boldsymbol{C}_X$-modules, i.e., for $M^\cdot$ in $\boldsymbol{D}^b(\mathscr{D}_X)_{hr}$ we have

$$\mathscr{S}ol_Y(f^*M^\cdot)=f^{-1}\,\mathscr{S}ol_X(M^\cdot).$$

(The proof will be published in [K6].)

(5.5) For a proper holomorphic map $f\colon Y\to X$, the direct image in

(4.7.4) corresponds, up to the degree shift, to the direct image of C_X modules, i.e., for $N^{\cdot}$ in $D^b(\mathscr{D}_Y)_{hr}$ we have

$$\mathscr{S}ol_X\left(\int_f^{\cdot} N^{\cdot}\right)[\dim X] = Rf_*(\mathscr{S}ol_Y(N^{\cdot}))[\dim Y].$$

(The proof will be published in [K6].)

(5.6) For an r-codimensional closed *submanifold* $Y \subset X$, we have seen in (4.7.3) that the algebraic local cohomology sheaf $\mathscr{H}^r_{[Y]}(\mathscr{O}_X)$ is $\mathscr{D}_X$-holonomic with regular singularities. It corresponds to $\mathscr{D}\mathscr{R}_X(\mathscr{H}^r_{[Y]}(\mathscr{O}_X))$, which is quasi-isomorphic to $j_*C_Y[-r]$ for the embedding $j\colon Y \to X$. It is the complex with the direct image j_*C_Y of the constant sheaf C_Y on Y being in degree r and zero elsewhere. For instance for a neighborhood X of $Y = \{0\}$ in C, we have $M := \mathscr{H}^1_{[Y]}(\mathscr{O}_X) = \mathscr{O}[z^{-1}]/\mathscr{O}$ in the notation of (4.6), and have an exact sequence

$$0 \longrightarrow M \longrightarrow \Omega^1 \otimes_{\mathscr{O}} M \longrightarrow C_Y \longrightarrow 0.$$

(5.7) More generally, for a closed analytic subspace $Y \subset X$, we have a functor $R\mathscr{H}^0_{[Y]}$ from $D^b(\mathscr{D}_X)_{hr}$ into itself sending $M^{\cdot}$ to the algebraic local cohomology $R\mathscr{H}^0_{[Y]}(M^{\cdot})$ by (4.7.4). This corresponds to the transcendental local cohomology $R\mathscr{H}^0_Y$ for C_X-modules, i.e.,

$$\mathscr{D}\mathscr{R}_X(R\mathscr{H}^0_{[Y]}(M^{\cdot})) = R\mathscr{H}^0_Y(\mathscr{D}\mathscr{R}_X(M^{\cdot}))$$

for $M^{\cdot}$ in $D^b(\mathscr{D}_X)_{hr}$ (cf. [BK, Proposition 1.4]).

(5.8) Let Y be a purely r-codimensional closed analytic subspace of X. Let Z be a closed analytic subspace of Y containing the singular locus of Y. Then we have the following generalization of (5.6) (cf. [BK, Proposition 8.5 and Theorem 8.6]): On the one hand, there exists a unique holonomic $\mathscr{D}_X$-module $\mathscr{L} = \mathscr{L}(Y, X)$ with regular singularities such that (cf. (4.7.4))

$$\mathscr{L}|_{X \setminus Z} = \mathscr{H}^r_{[Y \setminus Z]}(\mathscr{O}_{X \setminus Z}),$$

hence a natural extension of (5.6) for $Y \setminus Z \subset X \setminus Z$, and that

$$\mathscr{H}^0_Z(\mathscr{L}) = \mathscr{H}^0_Z(\mathscr{L}^*) = 0.$$

Moreover, $\mathscr{L}$ is self-adjoint, i.e., $\mathscr{L}^* = \mathscr{L}$. On the other hand, the corresponding de Rham complex $\mathscr{D}\mathscr{R}_X(\mathscr{L}(Y, X))$ is quasi-isomorphic to $\pi_Y[-r]$, where π_Y is the *intersection cohomology complex* of C_X-modules introduced by Deligne-Goresky-MacPherson [GM]. The restriction of π_Y

to the smooth locus of Y is known to be the constant sheaf $\boldsymbol{C}$ there, hence this side is also a natural extension of (5.6) for $Y\setminus Z\subset X\setminus Z$. The self-adjointness of $\mathscr{L}(Y, X)$ corresponds to the self-duality of $\pi_Y[-r]$ by (5.2), hence to the Poincaré duality for the intersection cohomology groups.

In the special case when Y is a Schubert variety in the flag manifold $X=G/B$, this result was a key in the proof of a conjecture of Kazhdan-Lusztig concerning the multiplicity of irreducible factors in the Verma modules. (cf. [KL1], [KL2], [BK], [BB] and [K5]).

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