CHAPTER 7

Invariant and Relatively Invariant Measures on Locally Compact Groups and Spaces

The various definitions and properties of measures are handled in this chapter in terms of continuous functions with compact support, in the spirit of the Bourbaki integration theory. The results hold then also for the integrable functions. This will be understood tacitly and will generally not be mentioned further.

In Sections 7.1 and 7.2 frequent reference will be made to Nachbin (1976). For short this will often be abbreviated "N."

7.1. Haar measure on locally compact groups. Let G be a l.c. group and $\mathcal{K}(G)$ the family of real valued continuous functions on G with compact support. Since G is l.c., the theory of Chapter 6 applies and any measure on G will be understood to be in the sense of Section 6.3. Consider the left and the right action of G on itself and recall the definition of the left g-translate gf and the right translate fg of a function f on G by an element $g \in G$ (Section 2.1).

7.1.1. DEFINITION. *A measure μ on G is called* **left invariant,** *or a* **left Haar measure,** *if*

$$\mu(gf) = \mu(f), \quad g \in G,\ f \in \mathcal{K}(G). \tag{7.1.1}$$

One also speaks of μ being a **left invariant integral** and (7.1.1) can be written $\int gf\, d\mu = \int f\, d\mu$. There is, of course, a similar definition for **right invariant** (or **right Haar**) measure, by replacing gf by fg. It is obvious from (7.1.1) that if μ is left invariant, then so is $c\mu$ for any $c > 0$.

Intuitively, it is a little easier to think in terms of left g-translates, gA, of sets $A \subset G$ (Section 2.1). Equation (7.1.1) implies the same equation for $f \in \mathcal{L}$, in particular for integrable sets A:

(7.1.2) $$\mu(gA) = \mu(A), \quad g \in G, \quad \text{integrable } A \subset G.$$

The existence of such a measure was first shown by Haar (1933) under the additional restriction that G be second countable. The idea is rather simple and will be sketched here briefly, following Nachbin (1976, Section II.2). It suffices to achieve (7.1.2) for compact sets. Let U be a neighborhood of the identity $e \in G$ and A an arbitrary nonempty compact set. Each $g \in A$ has a neighborhood gU, and by compactness A can be covered by a finite number of these left translates of U. Let $(A : U)$ be the smallest such number. This number is a rough measure of the size of A, but it is not a measure in the technical sense, and, besides, it depends on the choice of U. Instead, choose another fixed compact set B with nonempty interior and try to compare A with B. Put $\mu_U(A) = (A : U)/(B : U)$. Note that $\mu_U(gA) = \mu_U(A)$ since $(gA : U) = (A : U)$, for every $g \in G$. Also note $\mu_U(B) = 1$. This is not a measure either but something close to it. One can think of measuring A in units B and for that purpose there is available a measuring device whose smallest unit of measurement is U. Then both A and B are compared with U. Making U smaller increases the measurement precision. This suggests considering the neighborhood system of e partially ordered by inclusion. It turns out that as U becomes smaller and smaller μ_U has a limit μ in a certain sense, and μ is a measure (the details are far from trivial). Since $\mu_U(gA) = \mu_U(A)$, the same is true for μ so that μ satisfies (7.1.2). Furthermore, $\mu(B) = 1$. Another choice of B could have produced another left Haar measure, say μ'. However, it turns out that the relation between μ and μ' is very simple: $\mu' = c\mu$ for some $c > 0$.

7.1.2. THEOREM. *Let G be a l.c. group, then G admits a not identically zero left invariant measure that is unique up to multiplication by positive reals.*

PROOF. There are several proofs in the literature; here is a selection: Halmos (1950, Chap. XI; Bourbaki (1963), VII, §1.2; Nachbin (1976), Chap. II, Sect. 8 (proof of A. Weil), Sect. 9 (proof of H. Cartan). □

There is of course an analogous theorem for right invariant measure. It is also easy to construct a right Haar measure, say ν, from a given left Haar measure μ. For $f \in \mathcal{K}(G)$ define $\nu(f) \equiv \int f d\nu$ by $\nu(f) = \int f(x^{-1})\mu(dx)$ (notice that $g \to f(g^{-1})$ is also in $\mathcal{K}(G)$). That ν is right invariant follows from the following computation. First put $h(x) = f(x^{-1})$, $x \in G$. Then compute, for fixed $g \in G$, $\nu(fg) = \int (fg)(x^{-1})\mu(dx) = \int f(x^{-1}g^{-1})\mu(dx) = \int h(gx)\mu(dx) = \mu(g^{-1}h) = \mu(h) = \nu(f)$. These computations are sometimes easier conceived of and carried out if we consider sets rather than functions. Thus, the measure ν has the property that for compact (or even integrable) set A, $\nu(A) = \mu(A^{-1})$, where $A^{-1} = \{g \in G : g^{-1} \in A\}$. It is now immediate that for $g \in G$, $\nu(Ag) = \mu(g^{-1}A^{-1}) = \mu(A^{-1}) = \nu(A)$. For any right Haar measure ν we have the equivalent of (7.1.2):

$$\nu(Ag) = \nu(A), \quad g \in G, \text{ integrable } A \subset G. \tag{7.1.3}$$

If G is abelian, then, of course, a left Haar measure is also right invariant. But there are groups, e.g., $GL(n)$ and $O(n)$ (for notation see Section 5.1), that are not abelian, but where nevertheless left and right invariant measures coincide. For $O(n)$ this follows from compactness (Corollary 7.1.7); for $GL(n)$ see subsection 7.7.2.

Examples. All l.c. groups in this monograph are Lie groups. A very simple example is $G = R$ = reals under addition, with μ = Lebesgue measure (or a positive multiple thereof). Then for $f \in \mathcal{K}(G)$ we have for a fixed $g \in R$, $\mu(gf) = \int f(x-g)dx = \int f(x)dx = \mu(f)$ by a change of variable. It is also obvious that for a set $A \subset R$, for which the general notation gA now becomes $A + g$, the Lebesgue measures of A and $A+g$ are equal. I.e., Lebesgue measure is invariant under translation. Here G is abelian, and μ is both left and right invariant. The example can of course be extended immediately to $G = R^n$ under translations. Another simple example is $G = R_+^* =$ positive reals under multiplication. This is also abelian so that left and right Haar measures coincide. The integral $\mu(f) = \int_0^\infty f(x)x^{-1}\,dx$, for $f \in \mathcal{K}(G)$, is left (and right) invariant as the following computation shows: $\mu(gf) = \int_0^\infty f(g^{-1}x)x^{-1}\,dx = \int_0^\infty f(x)x^{-1}\,dx = \mu(f)$, by a change of variable. The measure μ can be defined informally as $(1/x)dx$ on R_+. One can also verify (7.1.2) for A an interval (a,b) : $\int_{ga}^{gb} x^{-1}dx = \int_a^b x^{-1}dx$. More examples will appear in Section 7.7.

Right- and left-hand moduli. Let μ be left invariant and for $g \in G$ fixed define a new measure μ' by $\mu'(f) = \mu(fg)$, $f \in \mathcal{K}(G)$. For $g_1 \in G$ we have $(g_1 f)g = g_1(fg)$ and therefore $\mu'(g_1 f) = \mu(g_1(fg)) = \mu(fg)$ (since μ is left invariant) $= \mu'(f)$. Thus, μ' is also left invariant and it follows from Theorem 7.1.2 that $\mu' = c\mu$ for some $c > 0$. Here c may depend on g. It is denoted $\Delta_r(g)$ or $\Delta_r^G(g)$ if the group G has to be specified. The function $g \to \Delta_r(g)$ is called the **right-hand modulus** of G.

7.1.3. PROPOSITION. *The right-hand modulus Δ_r defined by*

$$\mu(fg) = \Delta_r(g)\mu(f) \tag{7.1.4}$$

for $g \in G$, $f \in \mathcal{K}(G)$ and μ any left Haar measure on G, is a continuous homomorphism $G \to R_+^*$. The same conclusion can be drawn for the **left-hand modulus** *Δ_ℓ defined by*

$$\nu(gf) = \Delta_\ell(g)\nu(f) \tag{7.1.5}$$

where ν is a right Haar measure on G.

PROOF. The fact that Δ_r is a homomorphism follows from the simple computation that on one hand $\mu(fg_1g_2) = \mu(f(g_1g_2)) = \Delta_r(g_1g_2)\mu(f)$ and on the other $\mu(fg_1g_2) = \mu((fg_1)g_2) = \Delta_r(g_1)\Delta_r(g_2)$

$\mu(f)$, so that $\Delta_r(g_1 g_2) = \Delta_r(g_1)\Delta_r(g_2)$. In order to show Δ_r continuous rewrite (7.1.4):

$$\int f(xg^{-1})\mu(dx) = \Delta_r(g) \int f(x)\mu(dx). \tag{7.1.6}$$

Choose a fixed continuous f with support contained in the compact set K such that $\int f(x)\mu(dx) \neq 0$. It suffices then to show that the function

$$g \to \int f(xg^{-1})\mu(dx) \tag{7.1.7}$$

is continuous on G. Let $g_0 \in G$ be arbitrary and choose a compact neighborhood L of g_0. Put $h(x,g) = f(xg^{-1})$, then h is continuous on $G \times G$ since it is the composition of the continuous functions $(x,g) \to (xg^{-1})$ and f. If g is restricted to L, then $h(x,g) = 0$ unless $x \in KL = M$, say. By the compactness of M, the convergence $h(\cdot,g) \to h(\cdot,g_0)$ is uniform, as in the proof of Lemma 6.5.6(i). I.e., $\|h(\cdot,g) - h(\cdot,g_0)\| = \sup_{x \in M} |h(x,g) - h(x,g_0)| \to 0$ as $g \to g_0$. By the continuity of the linear functional μ on the Banach space $\mathcal{K}(G,M)$ (Section 6.3), $\mu(h(\cdot,g)) \to \mu(h(\cdot,g_0))$ which shows the continuity of (7.1.7) at g_0 and therefore the continuity of Δ_r on G. The proof for Δ_ℓ is similar. □

The following formula is useful in converting g^{-1} to g in the argument of a function that is being integrated with respect to an invariant measure on G. It also establishes a precise link between a left and right Haar measure.

7.1.4. PROPOSITION (N, II, Prop.8). *Let μ be left Haar measure on the l.c. group G and Δ_r its right-hand modulus. Then for $f \in \mathcal{K}(G)$,*

$$\int f(x^{-1})\mu(dx) = \int f(x)\Delta_r(x^{-1})\mu(dx), \tag{7.1.8}$$

and both sides define a right invariant integral.

PROOF (Sketch). The left-hand side was shown earlier to be a right invariant integral. A similar, straightforward computation establishes this also for the right-hand side. By the essential uniqueness of right Haar measure (Theorem 7.1.2 with "left" replaced by "right") the two sides must be equal except, possibly, for a factor $c > 0$. In order to show $c = 1$ observe that $\Delta_r(e) = 1$ (because Δ_r is a homomorphism) and that Δ_r in (7.1.8) can be kept arbitrarily close to 1 by restricting x to a small enough neighborhood U of e (use the continuity of Δ_r, Proposition 7.1.3). Then choose $f \geq 0$ in such a way that $f(x) = f(x^{-1})$, $\operatorname{supp} f \subset U$, and $f > 0$ on a neighborhood of e. □

According to Proposition 7.1.4 the following is a right Haar measure ν:

$$\nu(dx) = \Delta_r(x^{-1})\mu(dx), \tag{7.1.9}$$

or, more formally,

$$\int f(x)\nu(dx) = \int f(x)\Delta_r(x^{-1})\mu(dx), \quad f \in \mathcal{K}(G). \tag{7.1.10}$$

7.1.5. PROPOSITION. $\Delta_r\Delta_\ell = 1$ *on* G.

PROOF. Take the right Haar measure ν defined by (7.1.10), and take $g \in G$ fixed. Take $f \in \mathcal{K}(G)$ such that $\nu(f) \neq 0$. Put $h(x) = f(x)\Delta_r(x^{-1})$ so that $\nu(f) = \mu(h)$, and compute $(gh)(x) = h(g^{-1}x) = \Delta_r(g)f(g^{-1}x)\Delta_r(x^{-1})$. Integrate both sides with respect to μ, then on the left-hand side we have $\mu(gh) = \mu(h) = \nu(f)$, and on the right-hand side we get $\Delta_r(g)\nu(gf) = \Delta_r(g)\Delta_\ell(g)\nu(f)$ by the definition of Δ_ℓ. □

A l.c. group is called **unimodular** if $\Delta_r = 1$ on G. Then also $\Delta_\ell = 1$, by the previous proposition. Obviously, a group is unimodular if and only if a left Haar measure is also right invariant, and vice versa.

The following is a generalization of N, II, Prop. 13.

7.1.6. PROPOSITION. *Let G be compact and δ a continuous homomorphism $G \to R_+^*$. Then $\delta = 1$ on G.*

PROOF (Essentially 2nd proof of N, II, Prop. 13). The image of G under δ is compact since δ is continuous, and is a group since δ is a homomorphism. But R_+^* does not have any compact subgroup other than the trivial group $\{1\}$. (By taking log an equivalent statement is that the additive group R does not have any compact subgroup except $\{0\}$.) □

7.1.7. COROLLARY. *A compact group is unimodular.*

PROOF. In Proposition 7.1.6 take $\delta = \Delta_r$. There is also a very easy direct proof (1st proof of N, II, Prop. 13). Since G is compact, the function $f \equiv 1$ is in $\mathcal{K}(G)$ and $\mu(f) > 0$. Then for any $g \in G$, $fg = f$, so that $\mu(f) = \mu(fg) = \Delta_r(g)\mu(f)$. □

7.1.8. COROLLARY. *Let H be a compact subgroup of a l.c. group G and δ a continuous homomorphism $G \to R_+^*$. Then $\delta = 1$ on H. In particular, $\Delta_r^G(h) = \Delta_\ell^G(h) = 1$ for $h \in H$.*

PROOF. Restrict δ to H and use Proposition 7.1.6 with G replaced by H. □

The implication compact $\Rightarrow$ unimodular by Corollary 7.1.7 does not go in the opposite direction. For instance, $G = R$ and $G = R_+^*$ are not compact but they are unimodular because they are abelian. An example of a noncompact, nonabelian group that is unimodular is $GL(n)$ (see subsection 7.7.2).

Since a compact group is unimodular, left and right Haar measures coincide and one may drop the left-right designation. It will be recalled from Section 6.3 that if μ is a measure on a l.c. space X and K is a compact subset of X, then $\mu(K) < \infty$. In particular, if G is a compact group and μ Haar measure, then $\mu(G) < \infty$. This also follows from taking $f \equiv 1$, then $f \in \mathcal{K}(G)$ and $\mu(f) = \mu(G)$. If Haar measure is taken so that $\mu(G) = 1$, then one says that μ is **normalized**. It can also be shown that a l.c. group whose left or right Haar measure is finite must be compact (N, II, Prop. 4).

An **automorphism** of a group G is an isomorphism of G with itself, i.e., a bijection $a : G \to G$ that preserves group multiplication: $a(g_1 g_2) = a(g_1)a(g_2)$. The inverse function a^{-1} is also an automorphism. (NOTE: $a^{-1}(g)$ is not to be confused with $(a(g))^{-1}$.) If G is topological and the bijection a a homeomorphism, then a is called a **topological automorphism**. An example is an **inner automorphism** $g \to g_1^{-1} g g_1$ for $g_1 \in G$ fixed.

7.1.9. PROPOSITION N, II, Prop. 16). *Let a be a topological automorphism of the l.c. group G and let μ, ν be any left and right Haar measures, respectively. Then there exists a unique positive number $\delta(a)$ such that for $f \in \mathcal{K}(G)$,*

$$\int f(a^{-1}(x))\mu(dx) = \delta(a) \int f(x)\mu(dx), \tag{7.1.11}$$

$$\int f(a^{-1}(x))\nu(dx) = \delta(a) \int f(x)\nu(dx). \tag{7.1.12}$$

If G is compact, then $\delta(a) = 1$.

PROOF. If (7.1.11) is valid for one choice of left invariant μ, then it is valid for any other choice since that amounts to multiplying both sides of (7.1.11) by some $c > 0$. Similarly (7.1.12) and the choice of ν. In order to prove (7.1.11) put the left-hand side equal to $\mu'(f)$, then μ' is a not identically zero linear functional on $\mathcal{K}(G)$ which is nonnegative on $\mathcal{K}_+(G)$. We show now that it is left invariant. Put $h(x) = f(a^{-1}(x))$ so that $h \in \mathcal{K}(G)$, then $\mu'(f) = \mu(h)$. Fix $g \in G$ and put $a(g^{-1}) = g_1^{-1}$. Compute $(gf)(a^{-1}(x)) = f(g^{-1}a^{-1}(x)) = f(a^{-1}(g_1^{-1})a^{-1}(x)) = f(a^{-1}(g_1^{-1}x)) = h(g_1^{-1}x) = (g_1 h)(x)$. Therefore, $\mu'(gf) = \mu(g_1 h) = \mu(h)$ (since μ is left Haar) $= \mu'(f)$ so that μ' is left invariant. By Theorem 7.1.2 $\mu'(f) = c\mu(f)$ for some $c > 0$, which is (7.1.11) with c denoted $\delta(a)$. Now replace f in (7.1.11) by f^*, where $f^*(x) = f(x^{-1})$, and observe that $\int f^* d\mu$ is a right invariant integral by the left-hand side of (7.1.8): take $\int f d\nu \equiv \nu(f) = \int f^* d\mu$. Then (7.1.11) (with f^*) becomes (7.1.12). If G is compact, then we take $f \equiv 1$ in (7.1.11) and it follows that $\delta(a) = 1$. □

The number $\delta(a)$ is called the **modulus** of the topological automorphism a.

7.1.10. PROPOSITION. *If in Proposition 7.1.9, the automorphism is taken to be the inner automorphism $a = a_g$, defined for any $g \in G$ by $a_g(x) = g^{-1}xg$, $x \in G$, then $\delta(a_g) = \Delta_r(g)$.*

PROOF. The inverse function is $a_g^{-1}(x) = gxg^{-1}$. Then the left-hand side of (7.1.11) is $\mu(g^{-1}fg) = \mu(fg) = \Delta_r(g)\mu(f)$. □

Product of groups. Let G_1 and G_2 be two l.c. compact groups, then so is $G_1 \times G_2$ (Section 2.2) with the coordinatewise multiplication $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2)$. Let μ_i be left Haar measure on G_i, $i = 1, 2$, then the product measure $\mu = \mu_1 \otimes \mu_2$ is left Haar measure on $G_1 \times G_2$. By Section 6.5 it is sufficient to check this for f of the form $f_1 \otimes f_2$, $f_i \in \mathcal{K}(G_i)$. We have, for $g_i \in G_i$, $(g_1, g_2)(f_1 \otimes f_2) = (g_1f_1) \otimes (g_2f_2)$ so that $\mu((g_1, g_2)(f_1 \otimes f_2)) = \mu((g_1f_1) \otimes (g_2f_2)) =$ (formula 6.5.1) $\mu(g_1f_1)\mu_2(g_2f_2) = \mu_1(f_1)\mu_2(f_2) = \mu(f_1 \otimes f_2)$ ((6.5.1) again). Similarly, $\nu = \nu_1 \otimes \nu_2$ is right invariant if ν_i is right Haar measure on G_i, $i = 1, 2$.

7.2. Relatively invariant measures on groups.

Recall from Section 7.1 that left Haar measure of an integrable set A is not changed if the set is left translated: $A \to gA$, but is changed under a right translation $A \to Ag$ by a factor $\Delta_r(g)$. Similarly, right Haar measure of A is unchanged under $A \to Ag$, but changes by a factor $\Delta_\ell(g)$ under $A \to gA$. More generally, consider measures λ for which

$$\lambda(gA) = \chi_\ell(g)\lambda(A), \quad \lambda(Ag) = \chi_r(g)\lambda(A), \tag{7.2.1}$$

with positive functions χ_ℓ and χ_r on G. That is, under left translation the measure of a set may change, but only by a factor depending on the translation, not on the set, and similarly for a right translation. In terms of functions we have

7.2.1. DEFINITION. *A measure λ on a l.c. group G is said to be* **relatively invariant** *with* **left multiplier** χ_ℓ *and* **right multiplier**

χ_r *if for* $f \in \mathcal{K}(G)$

$$\lambda(gf) = \chi_\ell(g)\lambda(f) \tag{7.2.2}$$

$$\lambda(fg) = \chi_r(g)\lambda(f). \tag{7.2.3}$$

Thus, a left Haar measure is relatively invariant with multiplier $\chi_\ell \equiv 1$, $\chi_r = \Delta_r$, and a right Haar measure is relatively invariant with $\chi_\ell = \Delta_\ell$, $\chi_r \equiv 1$.

7.2.2. PROPOSITION. *The multipliers* χ_ℓ *and* χ_r *are continuous homomorphisms* $G \to R_+^*$.

PROOF. Similar to the proof of Proposition 7.1.3. □

The reader's attention is drawn to the fact that in Definition 7.2.1 for a relatively invariant measure λ both (7.2.2) and (7.2.3) are to be satisfied. Instead we could consider (as is done in Nachbin, 1976) measures λ that satisfy (7.2.2) but not necessarily (7.2.3) and call these "left relatively invariant;" similarly λ of (7.2.3) "right relatively invariant." A priori there is no reason to believe that those two classes of measures coincide, in the same way that left Haar measure is not necessarily a right Haar measure. However, it will be shown in Proposition 7.2.5 that either of the equations (7.2.2) and (7.2.3) implies the other. Therefore, the left and right relatively invariant measures are the same and in anticipation of that result it will not be necessary to make the distinction between the two. It should be kept in mind, however, that in general the left and right multipliers of a relatively invariant measure are not equal. The following proposition presents a useful representation of a relatively invariant measure in terms of left and right Haar measure.

7.2.3. PROPOSITION (N, II, Prop. 26). *Let G be a l.c. group with left Haar measure* μ *and let* χ_ℓ *be a continuous homomorphism* $G \to R_+^*$. *Then the measure* λ *defined by*

$$\lambda(f) = \int f\chi_\ell \, d\mu, \quad f \in \mathcal{K}(G), \tag{7.2.4}$$

satisfies (7.2.2). Conversely, if λ is a measure satisfying (7.2.2), then λ can be expressed as (7.2.4) with a unique choice for the left Haar measure μ. The theorem remains true for λ satisfying (7.2.3) and being represented by

$$\lambda(f) = \int f\chi_r d\nu, \quad f \in \mathcal{K}(G), \tag{7.2.5}$$

in which ν is right Haar measure.

PROOF. Clearly λ of (7.2.4) is a linear functional on $\mathcal{K}(G)$ and nonnegative on $\mathcal{K}_+(G)$. In order to show that λ satisfies (7.2.2) fix $g \in G$ and for simplicity of notation write χ for χ_ℓ. Observe that $(g\chi)(x) = \chi(g^{-1}x) = \chi(g^{-1})\chi(x) = (\chi(g))^{-1}\chi(x)$ for $x \in G$, so that $\chi = \chi(g)g\chi$ on G. Then compute $\lambda(gf) = \mu(\chi gf) = \mu(\chi(g)(g\chi)(gf)) = \chi(g)\mu(g(\chi f)) = \chi(g)\mu(\chi f) = \chi(g)\lambda(f)$, which is (7.2.2). Conversely, suppose that λ satisfies (7.2.2) (again, drop subscript ℓ on χ). Define $\mu'(f) = \lambda(\chi^{-1}f)$, then μ' is a linear functional on $\mathcal{K}(G)$ and nonnegative on $\mathcal{K}_+(G)$. Fix $g \in G$ and compute $\mu'(gf) = \lambda(\chi^{-1}gf) = \lambda(\chi(g^{-1})(g\chi)^{-1}gf) = \chi(g^{-1})\lambda(g(\chi^{-1}f)) =$ (use (7.2.2)) $\chi(g^{-1})\chi(g)\lambda(\chi^{-1}f) = \mu'(f)$. Since μ' is left invariant it must equal μ for some version of left Haar measure μ, by Theorem 7.1.2. That is, $\lambda(f) = \mu(f\chi) = \mu(f\chi_\ell)$, which is (7.2.4). By choosing f so that the left-hand side of (7.2.4) is $\neq 0$ it is obvious that μ is unique. The proof of the statement of the theorem pertaining to (7.2.5) is similar. □

7.2.4. COROLLARY. *A measure λ on the l.c. group G satisfying (7.2.2) with given left multiplier χ_ℓ is unique up to a positive multiplicative constant. The same is true for λ satisfying (7.2.3).*

PROOF. The representations (7.2.4) and (7.2.5) are unique. □

7.2.5. PROPOSITION (N, II, Prop. 27). *On a l.c. group G every measure λ satisfying (7.2.2) with a given left multiplier χ_ℓ also satisfies (7.2.3) with some right multiplier χ_r, and vice versa. The left and right multipliers are related by*

$$\chi_r = \chi_\ell \Delta_r^G \tag{7.2.6}$$

in which Δ_r^G is the right-hand modulus of G.

PROOF. Let λ satisfy (7.2.2) then it is expressible, by (7.2.4), as $\lambda(f) = \mu(f\chi)$ (again we have dropped subscript ℓ on χ for simplicity). Observe that for $g \in G$, $\chi = \chi(g)(\chi g)$, and compute $\lambda(fg) = \mu((fg)\chi) = \mu((fg)\chi(g)(\chi g)) = \chi(g)\mu((f\chi)g) = \chi(g)\Delta_r^G(g)\mu(f\chi) = \chi(g)\Delta_r^G(g)\lambda(f) = \chi_\ell(g)\Delta_r^G(g)\lambda(f)$, so that λ satisfies (7.2.3) with χ_r given by (7.2.6). The proof in the other direction is similar. □

The representations (7.2.4) and (7.2.5) can be written in the more convenient forms

$$\lambda(dg) = \chi_\ell(g)\mu(dg) = \chi_r(g)\nu(dg), \tag{7.2.7}$$

which may be abbreviated $\lambda = \chi_\ell\mu = \chi_r\nu$. This also suggests a method of obtaining the Haar measures μ and ν for a given group. Often there is an obvious measure λ to try for relative invariance; in Lie groups this is usually Lebesgue measure. Suppose this attempt is successful and that the computations have produced the left and right multipliers χ_ℓ and χ_r. Then we have

7.2.6. PROPOSITION. *Let G be a l.c. group and λ a relatively invariant measure on G with left and right multipliers χ_ℓ, χ_r, respectively. Then left Haar measure μ_G and right Haar measure ν_G on G may be taken as*

$$\mu_G = \chi_\ell^{-1}\lambda, \quad \nu_G = \chi_r^{-1}\lambda. \tag{7.2.8}$$

Furthermore, the right-hand and left-hand moduli are, respectively,

$$\Delta_r^G = \chi_r/\chi_\ell, \quad \Delta_\ell^G = \chi_\ell/\chi_r. \tag{7.2.9}$$

PROOF. Equations (7.2.8) follow from (7.2.7). The first of (7.2.9) is (7.2.6), and the second follows by an interchange of the roles of right and left. □

From (7.2.9) it is seen that G is unimodular if λ is a relatively invariant measure with $\chi_\ell = \chi_r$. By multiplying the two expressions in (7.2.9), Proposition 7.1.5 is recovered. Proposition 7.2.6 will be applied in subsection 7.7.2 to $LT(n)$ and $UT(n)$.

7.3. Invariant and relatively invariant measures on a space on which a group acts properly. Let X be a l.c. space and H a l.c. group that acts *properly* (Section 2.3) to the *right* of X. The reason for writing H here rather than G, and for wanting the action to the right, is that its most important application will be in Section 7.4 to the case $X = G$, where G is a l.c. group and H a closed subgroup. Then the orbit space X/H is the space of left cosets G/H (Section 2.1). It will be shown in Section 7.4 that G/H admits essentially unique left invariant and relatively invariant measures, and this will play an important role in the sequel. The most essential elements in the construction and proof are just as easy to establish for a more general space X rather than a group. It also has the additional benefit of showing the existence of a so-called quotient measure on X/H, which will have application in Section 13.3.

The material in this section is based on Bourbaki (1963), VII, §2, nos. 1 and 2. Left Haar measure on H will be denoted β. A measure μ on X is called **relatively invariant** with multiplier χ if

$$\mu(fh) = \chi(h)\mu(f), \quad f \in \mathcal{K}(X),\ h \in H, \tag{7.3.1}$$

and **invariant** if $\chi \equiv 1$. Note that the multiplier is necessarily a *right* multiplier since there is only one action to the right of X, unlike the case of a group in Section 7.2 where there is both right and left action. A relatively invariant measure with multiplier χ will sometimes be called a χ-relatively invariant measure. As in Section 7.2 χ is a continuous homomorphism $H \to R_+^*$.

The aim is to start from a given relatively invariant measure on X and create in a natural way a measure on X/H. Since the action of H is assumed proper, X/H is l.c. by Theorem 2.3.13(a) so that the integration theory of Chapter 6 applies. Any relation between measures on X and measures on X/H is provided by a relation between functions in $\mathcal{K}(X/H)$ and functions in $\mathcal{K}(X)$. There is a natural way of creating a member of $\mathcal{K}(X/H)$ from a member of $\mathcal{K}(X)$. Let $f \in \mathcal{K}(X)$ and define

$$f^1(x) = \int f(xy)\beta(dy). \tag{7.3.2}$$

The right-hand side is well-defined since $f(xy)$ as a function of y is in $\mathcal{K}(H)$. It is checked immediately that $f^1(xh) = f^1(x)$ for every $h \in H$ since β is left invariant. That is, f^1 is constant on each orbit and defines therefore a function, say $f^\flat$, on X/H:

$$f^1 = f^\flat \circ \pi, \tag{7.3.3}$$

in which π is the orbit projection $X \to X/H$. The function f^1 can be shown to be continuous, as in the proof of Proposition 7.1.3, so that $f^\flat$ is continuous on X/H since π is an open map (Section 2.3). Also, if $\operatorname{supp} f \subset K$ compact $\subset X$, then $\operatorname{supp} f^\flat \subset \pi(K)$ compact $\subset X/H$. Hence $f \in \mathcal{K}(X) \Rightarrow f^\flat \in \mathcal{K}(X/H)$. If in (7.3.2) f is replaced by fh, $h \in H$, the right-hand side of (7.3.2) becomes $\int f(xyh^{-1})\beta(dy) = \Delta_r^H(h) \int f(xy)\beta(dy)$, where Δ_r^H is the right-hand modulus of H. That is, $(fh)^1 = \Delta_r^H(h) f^1$, and therefore

$$(fh)^\flat = \Delta_r^H(h) f^\flat, \quad h \in H,\ f \in \mathcal{K}(X). \tag{7.3.4}$$

We have now a function $\Phi : \mathcal{K}(X) \to \mathcal{K}(X/H)$ defined by

$$\Phi(f) = f^\flat. \tag{7.3.5}$$

The function Φ is obviously linear, and $\Phi(f) \geq 0$ if $f \geq 0$. For a given measure ν on X/H there is a natural measure μ on X defined by

$$\mu(f) = \nu(f^\flat), \quad f \in \mathcal{K}(X) \tag{7.3.6}$$

(this is really the dual of the linear mapping Φ). With help of (7.3.4) we see that

$$\mu(fh) = \Delta_r^H(h)\mu(f), \quad f \in \mathcal{K}(X),\ h \in H. \tag{7.3.7}$$

Thus (see (7.3.1)) μ is relatively invraiant with multiplier $\chi = \Delta_r^H$.

So far we have started from a measure on X/H and created a measure on X, by (7.3.6). But what we really want is to go in the other direction; i.e., given a measure μ on X and $g \in \mathcal{K}(X/H)$ define ν on X/H by $\nu(g) = \mu(f)$ for f such that $f^\flat = g$. We know from (7.3.7) that a necessary condition for this to be possible is that μ be relatively invariant with multiplier Δ_r^H. But there are two more missing ingredients: (1) for $g \in \mathcal{K}(X/H)$ it has to be shown that there exists $f \in \mathcal{K}(X)$ such that $f^\flat = g$, also with $\mathcal{K}$ replaced by $\mathcal{K}_+$; and (2) if $f_1^\flat = f_2^\flat = g$, then $\mu(f_1) = \mu(f_2)$, so that $\nu(g)$ can be defined unambiguously. These two questions will be settled first in the following two lemmas.

7.3.1. LEMMA. *The function Φ defined by (7.3.5) maps $\mathcal{K}(X)$ onto $\mathcal{K}(X/H)$ and $\mathcal{K}_+(X)$ onto $\mathcal{K}_+(X/H)$.*

PROOF. It suffices to prove the second part of the statement. Let $g \in \mathcal{K}_+(X/H)$, with compact support $K' \subset X/H$. We have to show the existence of $f \in \mathcal{K}_+(X)$ such that $f^1 = g \circ \pi$. By Proposition 2.3.3 there is compact $K \subset X$ such that $\pi(K) = K'$. By Lemma 6.3.2 (with g in that lemma replaced by u) there is a function $u \in \mathcal{K}_+(X)$ with $u = 1$ on K (the constant value 1 is unimportant; any positive nonconstant value would do just as well). Now u^1 defined by (7.3.2) with f replaced by u is positive on the saturation KH of K, so in particular it is positive on K. Hence, u^1 has a positive lower bound on K, and therefore on KH since u^1 is constant on orbits. It follows that $(g \circ \pi)/u^1$ is nonnegative, is continuous on KH, and vanishes outside KH. Moreover, it is constant on orbits. Then $f = u(g \circ \pi)/u^1$ is in $\mathcal{K}_+(X)$ and $f^1 = g \circ \pi$. □

7.3.2. LEMMA. *If a measure μ on X satisfies (7.3.7), then $f^\flat = 0$ implies $\mu(f) = 0$ for $f \in \mathcal{K}(X)$.*

PROOF. First we show that for $f, g \in \mathcal{K}(X)$,

$$\mu(fg^1) = \mu(f^1 g). \tag{7.3.8}$$

In order to treat the left-hand side of (7.3.8) we need to verify first that $h(x,y) \equiv f(x)g(xy)$ as a function of (x,y) on the product space $X \times H$ is in $\mathcal{K}(X \times H)$. The continuity of h follows from the continuity of f and g and the continuity of the action $(x,y) \to xy$. In order to check that h has compact support let A, B be the compact supports of f, g, respectively. Then $h(x,y) = 0$ unless $x \in A$, $xy \in B$; i.e., unless $y \in ((A,B))$ (see (2.3.2) with slight change of notation) and the latter has compact closure, by Proposition 2.3.8 (with G replaced by H). The left-hand side of (7.3.8) is an iterated integral which can be written as an integral of the above function h with respect to the product measure $\mu \otimes \beta$, and then the order of integration may be reversed (Fubini, see (6.5.13) and (6.5.14)). We get

$$\begin{aligned}\mu(fg^1) &= \int f(x)\mu(dx) \int g(xy)\beta(dy) \\ &= \int \beta(dy) \int f(x)g(xy)\mu(dx) \\ &= \int \beta(dy)\Delta_r^H(y^{-1}) \int f(xy^{-1})g(x)\mu(dx) \quad \text{by (7.3.7)} \\ &= \int g(x)\mu(dx) \int f(xy^{-1})\Delta_r^H(y^{-1})\beta(dy) \\ &= \int g(x)\mu(dx) \int f(xy)\beta(dy) \quad \text{by (7.1.8)} \\ &= \mu(gf^1),\end{aligned}$$

so that (7.3.8) has been shown. Equation (7.3.8) may also be written $\mu(f(g^\flat \circ \pi)) = \mu(g(f^\flat \circ \pi))$. Therefore, if $f^\flat = 0$, then $\mu(f(g^\flat \circ \pi)) = 0$ for every $g \in \mathcal{K}(X)$. Let $\operatorname{supp} f = K$, compact, and take $g_0 \in \mathcal{K}(X/H)$ in such a way that $g_0 = 1$ on $\pi(K)$; this can be done by Lemma 6.3.2. By Lemma 7.3.1 there exists $g \in \mathcal{K}(X)$ such that $g^\flat = g_0$ so that $g^\flat = 1$ on $\pi(K)$ and hence $g^\flat \circ \pi = 1$ on K. Therefore, $f(g^\flat \circ \pi) = f$ so that $\mu(f) = \mu(f(g^\flat \circ \pi)) = 0$. □

Given a measure μ on X satisfying (7.3.7) define a measure $\mu^\flat$ on X/H by

$$\mu^\flat(g) = \mu(f), \quad g \in \mathcal{K}(X/H) \tag{7.3.9}$$

in which f is any function in $\mathcal{K}(X)$ such that $f^\flat = g$. Lemma 7.3.1 shows that there is such a function and Lemma 7.3.2 shows that $\mu(f)$ is unique (even if f is not). Together with the notation $\mu^\flat$ of (7.3.9) it is also convenient to have the notation $\nu^\sharp$ for the measure defined by (7.3.6). That is,

$$\nu^\sharp(f) = \nu(f^\flat), \quad f \in \mathcal{K}(X). \tag{7.3.10}$$

With help of (7.3.9) and (7.3.10) a simple calculation shows $\mu^{\flat\sharp} = \mu$ and $\nu^{\sharp\flat} = \nu$. That is, the operations $^\flat$ and $^\sharp$ are inverses of each other.

7.3.3. THEOREM. *Let the l.c. group H act properly on the right of a l.c. space X and let β be a given left Haar measure on H. Then given a relatively invariant measure μ on X with multiplier χ there exists a unique measure ν that has the property (7.3.6) if and only if $\chi = \Delta_r^H$. If this condition is satisfied, then ν is given by $\mu^\flat$ of (7.3.9).*

PROOF. That the condition $\chi = \Delta_r^H$ is necessary was already established in (7.3.7). Suppose the condition holds, then $\mu^\flat$ of (7.3.9) satisfies the equation (7.3.6) for ν. That $\mu^\flat$ is the unique such measure follows from the fact that the equation (7.3.6) for ν can also be expressed as $\nu^\sharp = \mu$ with $\nu^\sharp$ defined in (7.3.10). Then $\nu = \nu^{\sharp\flat} = \mu^\flat$. □

7.3.4. DEFINITION. *The unique measure $\mu^\flat$ in Theorem 7.3.3 is called the* **quotient** *of μ and β, and is denoted μ/β.*

The correspondence between the measures on X and those on X/H in the case $\chi = \Delta_r^H$ can also be put in the following way. For $f_1, f_2 \in \mathcal{K}(X)$ write $f_1 \sim f_2$ if $f_1^\flat = f_2^\flat$. Then $\sim$ is an equivalence relation. Lemma 7.3.2 shows that all functions in the same equivalence class have the same value of μ. Since there is now a 1-1 correspondence between the equivalence classes of $\mathcal{K}(X)$ and the functions in $\mathcal{K}(X/H)$ we can define $\mu^\flat$ by equating its value at a function in $\mathcal{K}(X/H)$ to the value of μ at the corresponding equivalence class of $\mathcal{K}(X)$.

7.3.5. PROPOSITION. *Let the compact group H act continuously on the right of a l.c. space X and let β be normalized Haar measure on H, i.e., $\beta(H) = 1$. If the measure μ on X is invariant, then the quotient measure μ/β coincides with the induced measure $\pi(\mu)$, where π is the orbit projection $X \to X/H$.*

PROOF. Since H is compact, its action is proper (Corollary 2.3.10) and π is proper (by essentially the same proof as the one of Proposition 2.3.5). Therefore, $\pi(\mu)$ is a measure on X/H, defined by (6.3.4) with h replaced by π, Y by X/H:

$$\pi(\mu)(g) = \mu(g \circ \pi), \quad g \in \mathcal{K}(X/H). \tag{7.3.11}$$

Now μ is given to be relatively invariant with multiplier $\chi \equiv 1$. Therefore, $\chi = \Delta_r^H$ since H is unimodular. Hence, Theorem 7.3.3 applies. For $g \in \mathcal{K}(X/H)$ the function $g \circ \pi$ on X has compact support since by Lemma 7.3.1 there exists $f \in \mathcal{K}(X)$ with support K such that $f^\flat = g$, and then $g \circ \pi = f^1$ vanishes outside the compact set KH. Now for given $g \in \mathcal{K}(X/H)$ we may take $f = g \circ \pi \in \mathcal{K}(X)$ and then $f = f^1$ since f is constant on orbits and $\beta(H) = 1$. The equation $f^1 = g \circ \pi$ then shows, by (7.3.3), that $f^\flat = g$. The measure $\mu/\beta = \mu^\flat$ of Theorem 7.3.3 is defined by (7.3.9), which now reads $\mu^\flat(g) = \mu(g \circ \pi)$, $g \in \mathcal{K}(X/H)$. Comparing this with (7.3.11) finishes the proof. □

7.4. Invariant and relatively invariant measures on homogeneous spaces. Let G be a l.c. group acting properly and transitively on the left of a l.c. space X. Take $x \in X$ arbitrarily, then $X = Gx$ and coincides with G/G_x as a point set since G is transitive. Apply Theorem 2.3.13, then by part (c) G_x is compact, and by part (e) the bijection $\psi_x : G/G_x \to X$ is a homeomorphism. It follows that the mapping α_x of part (b) is an open mapping, because for U open $\subset G$, UG_x is also open and may be regarded as an open subset of G/G_x; then observe that $\alpha_x(U) = Ux = UG_x x = \psi_x(UG_x)$ which is open in X since ψ_x is open. Thus, X is what Nachbin (1976, III, 3) calls a **topological left homogeneous space**.

In order to study invariant and relatively invariant measures on X we may without loss of generality do this on the coset space G/G_x. Replacing G_x by an arbitrary *closed* (but for the time being not necessarily compact) subgroup H of G, we consider the homogeneous space G/H of left cosets of G mod H. The closedness of H ensures that H is l.c. (Section 2.2). An application of Proposition 2.3.8 with

X and G there replaced by G and H shows that H acts properly on G to the right. Thus, we have here a special case of Section 7.3 with X there replaced by G. However, there is more structure here since not only do we have the group H acting on the space G, but also G as a group acts on itself and acts to the left on the coset space G/H (see (2.1.3)). Thus, we can now also consider relatively invariant measures on G/H.

7.4.1. THEOREM (Weil, 1951, II, §9; Nachbin, 1976, III.3, Thm.1; Bourbaki, 1963, VII, §2.6, Thm. 3). *Let H be a closed subgroup of a l.c. group G and G/H the space of left cosets* mod$H$. *Let μ_G and β be left Haar measures on G, H, respectively and let χ be a continuous homomorphism $G \to R_+^*$. Then in order that there exist on G/H a relatively invariant measure with multiplier χ it is necessary and sufficient that*

$$\Delta_r^H(h) = \chi(h)\Delta_r^G(h) \text{ for every } h \in H, \tag{7.4.1}$$

in which Δ_r^H and Δ_r^G are the right-hand moduli of H, G, respectively. If (7.4.1) is satisfied, then every χ-relatively invariant measure on G/H is of the form $\mu^\flat = \mu/\beta$ (Definition 7.3.4) in which $\mu = c\chi\mu_G$ for some $c > 0$.

PROOF. The left action of G on G/H induces a left action of G on functions on G/H. Consider in particular $f^\flat$ of (7.3.3) if $f \in \mathcal{K}(G)$. It is easy to check that $(gf)^\flat = gf^\flat$. Now suppose a measure ν on G/H is χ-relatively invariant. Then $\nu(gf^\flat) = \chi(g)\nu(f^\flat)$ for $f \in \mathcal{K}(G)$ so that $\nu((gf)^\flat) = \chi(g)\nu(f^\flat)$. By (7.3.10), $\nu^\sharp(gf) = \chi(g)\nu^\sharp(f)$ so that $\nu^\sharp$ is a relatively invariant measure on G with left multiplier $\chi_\ell = \chi$. By Proposition 7.2.3 $\nu^\sharp$ must be of the form $\mu = c\chi\mu_G$ for some $c > 0$. By Proposition 7.2.5 the right multiplier of μ is

$$\chi_r(g) = \chi(g)\Delta_r^G(g). \tag{7.4.2}$$

On the other hand, we know from (7.3.6) and (7.3.7) that under right translation with an element $h \in H$, μ is relatively invariant with multiplier $\Delta_r^H(h)$. Equating this to the right-hand side of (7.4.2) with $g = h$ gives (7.4.1). If this condition is satisfied, then $\mu = c\chi\mu_G$ satisfies Theorem 7.3.3 so that $\nu = \nu^{\sharp\flat} = \mu^\flat$. □

7.4.2. COROLLARY. *On a topological left homogeneous space any χ-relatively invariant measure is unique up to a positive multiplicative constant.*

7.4.3. COROLLARY. *If H is a closed normal subgroup of a l.c. group, then $\Delta_r^H(h) = \Delta_r^G(h)$ for every $h \in H$.*

PROOF. Since G/H now is a group it admits a left invariant measure. The conclusion follows from (7.4.1) after applying Theorem 7.4.1 with $\chi \equiv 1$. □

For us the greatest interest of Theorem 7.4.1 lies in the case where H is compact.

7.4.4. COROLLARY. *If H is a compact subgroup of a l.c. group G, then for every continuous homomorphism $\chi : G \to R_+^*$ there is a χ-relatively invariant measure ν on the space G/H of left cosets under the left action of G. This measure is necessarily of the form*

$$\nu = \pi(\chi\mu_G) \tag{7.4.3}$$

for any choice of left Haar measure μ_G on G, where $\pi : G \to G/H$ is the coset projection. Thus, ν is unique except for a positive multiplicative constant.

PROOF. Take Haar measure β on H normalized. Equation (7.4.1) is satisfied because each of the three homomorphisms, χ, Δ_r^G, and Δ_r^H, are identically equal to 1 on H, by Corollary 7.1.8. By Theorem 7.4.1 there is on G/H a χ-relatively invariant measure ν which is necessarily of the form $\mu^\flat$, with $\mu = \chi\mu_G$. Compactness of H ensures that μ is invariant under right translations with elements of H. Then Proposition 7.3.5 applies so that $\nu = \mu^\flat = \pi(\mu) = \pi(\chi\mu_G)$. □

The measure ν of (7.4.3) may be expressed in different ways. Put $Y = G/H$ and observe that χ is constant on every coset $\pi^{-1}(y)$, $y \in Y$, since $\chi(gh) = \chi(g)\chi(h) = \chi(g)$ for $h \in H$ since H is compact (Corollary 7.1.8). Therefore, χ is a function of g only through $y = [g]$. With some abuse of notation we shall write $\chi(y)$ when we want to consider χ as a function on Y. For any version μ_G of left Haar measure on G define the measure on Y

$$\mu_Y = \pi(\mu_G), \quad Y = G/H. \tag{7.4.4}$$

This is ν of (7.4.3) with $\chi = 1$. Then (7.4.3) can be written

$$\nu(dy) = \chi(y)\mu_Y(dy). \tag{7.4.5}$$

Furthermore, for $f \in \mathcal{K}(Y)$, (7.4.3) can be written in either of the two following forms:

$$\nu(f) = \int f(\pi(g))\chi(g)\mu_G(dg), \tag{7.4.6}$$

$$\nu(f) = \int f(y)\chi(y)\mu_Y(dy), \tag{7.4.7}$$

and these equations hold of course for any $f : Y \to R$ such that $f\chi$ is μ_Y-integrable.

7.5. Relatively invariant measure on a product space on which a group acts. Later applications will often involve the following situation: there is a group G that acts continuously and transitively on the left of a l.c. space X_1 and trivially on a l.c. space X_2, i.e., $gx_2 = x_2$ for every $g \in G$ and $x_2 \in X_2$. The left action of G on the l.c. product space $X_1 \times X_2$ is defined by $g(x_1, x_2) = (gx_1, x_2)$. Suppose that on X_i there is given a measure μ_i, $i = 1, 2$, where μ_1 is relatively invariant with multiplier χ. Consider the product measure $\mu_1 \otimes \mu_2$ (Section 6.5). It is easy to verify that $\mu_1 \otimes \mu_2$ is χ-relatively invariant under the action of G on $X_1 \times X_2$. By the theory of Section 6.5 it is sufficient to consider only $f \in \mathcal{K}(X_1 \times X_2)$ of the form $f = f_1 \otimes f_2$, where $f_i \in \mathcal{K}(X_i)$, $i = 1, 2$. Then $gf = (gf_1) \otimes f_2$, and one computes, using (6.5.12), $(\mu_1 \otimes \mu_2)(gf) = \mu_1(gf_1)\mu_2(f_2) = \chi(g)\mu_1(f_1)\mu_2(f_2) = \chi(g)(\mu_1 \otimes \mu_2)(f)$. The real problem we shall be faced with is in a sense the converse of the situation described above: there is given χ-relatively invariant measures μ_1 and μ on X_1 and $X_1 \times X_2$, respectively, and we want to show the existence of μ_2 on X_2 such that $\mu = \mu_1 \otimes \mu_2$. This will be the subject of the theorem below, which may be regarded as a Bourbaki version of Theorem 3.4.1 in Farrell (1985). A somewhat abbreviated proof can also be found in Andersson, Brøns, and Jensen (1983), Lemma 3. A proof of a version of the theorem along classical measure theory also appears in Bondar (1976), Theorem 2. The principal application of the theorem will be in the case that G acts properly on X_1. Then X_1 is a topological homogeneous space (first paragraph of Section 7.4) so that Theorem 7.4.1 applies. In particular, any χ-relatively invariant measure on X_1 will then be essentially unique.

7.5.1. THEOREM. *Let X_1 and X_2 be l.c. spaces, G a l.c. group acting continuously on X_1 to the left and trivially on X_2. Suppose on X_1 is given a nonzero measure μ_1 that is relatively invariant with multiplier χ and suppose that an arbitrary nonzero χ-relatively invariant measure on X_1 is necessarily of the form $c\mu_1$, with $c > 0$. Then if μ is a χ-relatively invariant measure on $X_1 \times X_2$, there exists a measure μ_2 on X_2 such that $\mu = \mu_1 \otimes \mu_2$. In particular, this conclusion holds if G acts transitively and properly on X_1, so that X_1 is a topological left homogeneous space under G.*

PROOF. We may assume in the proof that μ is nonzero, otherwise we can simply take $\mu_2 = 0$. Let $f_i \in \mathcal{K}(X_i)$, $i = 1, 2$, so that $f_1 \otimes f_2 \in \mathcal{K}(X_1 \times X_2)$. Temporarily fix $f_2 \geq 0$ and not $\equiv 0$, then $\mu(f_1 \otimes f_2)$ as a function of f_1 is a not identically zero linear functional on $\mathcal{K}(X_1)$ that is nonnegative on $\mathcal{K}_+(X_1)$ so that it is a nonzero measure on X_1. It is χ-relatively invariant by the computation $\mu((gf_1) \otimes f_2) = \mu(g(f_1 \otimes f_2)) = \chi(g)\mu(f_1 \otimes f_2)$. By hypothesis there exists a constant $c(f_2) > 0$ such that $\mu(f_1 \otimes f_2) = c(f_2)\mu_1(f_1)$. Now relax the temporary restriction $f_2 \geq 0$ and write $f_2 = f_2^+ - f_2^-$, with $f_2^+, f_2^- \geq 0$. Then by the linearity of μ we have $\mu(f_1 \otimes f_2) = [c(f_2^+) - c(f_2^-)]\mu_1(f_1) = c(f_2)\mu_1(f_1)$, say, with $c(f_2) \in R$. Now fix $f_1 \in \mathcal{K}_+(X_1)$ with $\mu_1(f_1) > 0$ and observe that $\mu(f_1 \otimes f_2)$ as a function of f_2 is a nonzero linear functional on $\mathcal{K}(X_2)$ that is nonnegative on $\mathcal{K}_+(X_2)$. Then so is $\mu(f_1 \otimes f_2)/\mu_1(f_1) = c(f_2)$ so that there is a measure μ_2 on X_2 such that $c(f_2) = \mu_2(f_2)$. Thus, $\mu(f_1 \otimes f_2) = \mu_1(f_1)\mu_2(f_2)$ and it follows from Theorem 6.5.1 that $\mu = \mu_1 \otimes \mu_2$. If X_1 is a topological left homogeneous space under G, then by Corollary 7.4.2 an arbitrary nonzero χ-relatively invariant measure on X_1 must be of the form $c\mu_1$, $c > 0$, so that the hypotheses of the theorem are satisfied. □

7.6. Haar measure on a group spanned by two subgroups. Suppose K is a l.c. group with the same structure as in Section 5.9; i.e., K has closed subgroups G and H such that

$$K = GH. \tag{7.6.1}$$

(In applications K will be Lie, but that aspect will not be used here.) Since G and H are also l.c. we have left Haar measures on all three groups: μ_K, μ_G, and μ_H, say. It is of great practical value to derive an expression for μ_K in terms of μ_G and μ_H.

A trivial example of (7.6.1) is the case of a product group, say $K' = G \times H$. Then define a group K with elements gh ($g \in G$, $h \in H$) and multiplication $(g_1h_1)(g_2h_2) = (g_1g_2)(h_1h_2)$ (then G and H commute as subgroups of K). Thus, there is an isomorphism between K' and K, with correspondence $(g, h) \leftrightarrow gh$. Left Haar measure on K is then the image under this isomorphism of the left Haar measure $\mu_G \otimes \mu_H$ on $G \times H$.

A much less trivial example is the case of a **semi-direct product** (see, e.g., Bourbaki, 1966b, III, §2.10). Then one of the groups G, H is normal in K and $G \cap H = \{e\}$ (so that the decomposition $k = gh$ is unique). Suppose H is normal in K so that $g^{-1}Hg = H$ for every $g \in G$. Then for each $g \in G$, $h \to \sigma_g(h) \equiv g^{-1}hg \in H$ is an automorphism σ_g of H. We have $hg = g\sigma_g(h)$, which shows that the elements of G and H "semi-commute," i.e., hg can be interchanged after replacing h by $\sigma_g(h)$ while g stays the same. One establishes easily that $\sigma_{g_1}\sigma_{g_2} = \sigma_{g_2g_1}$ so that $\{\sigma_g : g \in G\}$ is a group of automorphisms of H. The elements of K can be written as (g, h) in a unique way, but the group multiplication is in general not the same as in a product group (unless G and H commute in which case the semi-direct product is a direct product and each σ_g is trivial). From $h_1 g_2 = g_2 \sigma_{g_2}(h_1)$ it follows that the group multiplication is $(g_1, h_1)(g_2, h_2) = (g_1 g_2, \sigma_{g_2}(h_1) h_2)$.

A simple example of such a group is the affine group on R that arises from combining transformations $R \to R$ of the form $x \to ax$, $a > 0$, with $x \to x + b$, $b \in R$. Here $G = R_+^*$, $H = R$, and if g corresponds to a, h to b, then ghx corresponds to $x \to x+b \to a(x+b)$. One computes $\sigma_a(b) = b/a$ and if the elements of K are written (g, h), represented by (a, b), then $(a_1, b_1)(a_2, b_2) = (a_1 a_1, a_2^{-1} b_1 + b_2)$. (The same group K can also be written HG, corresponding to $x \to ax + b$.) An immediate generalization is to the affine group of transformations of $R^n : x \to A(x + b)$, $x, b \in R^n$, $A : n \times n$ nonsingular. Here $G = GL(n)$ (notation: Section 5.1), $H = R^n$, and H is normal in GH. Another important example of (7.6.1) is $K = UT(n)$ or $LT(n)$, $G =$ all $\mathrm{diag}(a_1, \ldots, a_n)$, $a_i > 0$ for all i, and $H =$ all members of K with diagonal elements equal to 1. Here also H is normal in K.

An important example of (7.6.1) where neither G nor H is normal in K is $K = GL(n)$, $G = UT(n)$ or $LT(n)$, and $H = O(n)$.

The construction of a left Haar measure μ_K on K from the left Haar measures μ_G and μ_H uses the same device of Bourbaki that was used in Section 5.9. That is, we consider K as a l.c. space on which the product group $G \times H$ acts to the left, continuously and transitively, according to (5.9.4). This sets up the 1-1 correspondence (5.9.5) and we shall again assume that it is a homeomorphism. Any measure on $(G \times H)/F^*$ can then be transferred to K, and vice versa. Furthermore, we shall now need $F = G \cap H$ to be compact. This certainly will be the case in the common situation where $F = \{e\}$.

7.6.1. PROPOSITION. *Let G and H be closed subgroups of the l.c. group K such that (7.6.1) is satisfied. Furthermore, assume that $F = G \cap H$ is compact and that Assumption 5.9.2 is satisfied. Let μ_G and μ_H be left Haar measures on G, H, respectively and let Λ be the right-hand modulus Δ_r^K of K restricted to H. Then*

$$\mu_K = \pi(\mu_G \otimes \Lambda^{-1} \mu_H) \tag{7.6.2}$$

is a left Haar measure on K, where π is the coset projection $G \times H \to (G \times H)/F^*$.

PROOF. Let μ_K be a version of left Haar measure on K and consider it to be a measure on the topological left homogeneous space $(G \times H)/F^* = Y$, say. We investigate the relative invariance of μ_K under the action (5.9.4) of $G \times H$ on Y. Let $f \in \mathcal{K}(Y)$ and let $g \in G$, $h \in H$ be fixed. While identifying Y and K compute $((g,h)f)(k) = f((g,h)^{-1}k) = f(g^{-1}kh) = (gfh^{-1})(k)$; i.e., $(g,h)f = gfh^{-1}$ (note that since $Y = K$ is not only a homogeneous space, but also a group, points and therefore functions can be acted on to the right as well as to the left). Next, compute $\mu_K((g,h)f) = \mu_K(gfh^{-1}) = \Delta_r^K(h^{-1})\mu_K(f)$ since μ_K is left invariant on K. This shows that under the action of $G \times H$, μ_K is a relatively invariant measure on Y with multiplier $\chi(g,h) = \Delta_r^K(h^{-1}) = \Lambda^{-1}(h)$. Apply Corollary 7.4.4 with G and H there replaced by $G \times H$ and F^*, respectively, and conclude $\mu_K = \pi(\chi(\mu_G \otimes \mu_H))$ for some versions of μ_G and μ_H. Then observe that $\chi(\mu_G \otimes \mu_H) = \mu_G \otimes \Lambda^{-1}\mu_H$. □

In the above proof the step showing relative invariance can be done simpler using integrable sets rather than functions in $\mathcal{K}(Y)$: let $A \subset K$ be μ_K-integrable, then $\mu_K(gAh^{-1}) = \Delta_r^K(h^{-1})\mu_K(A)$.

The result (7.6.2) can be put in the informal form

$$\mu_K(dk) = \Delta_r^K(h^{-1})\mu_G(dg)\mu_H(dh), \quad k = gh^{-1}, \tag{7.6.3}$$

or, more precisely, in integral form

$$\int f(k)\mu_K(dk) = \iint f(gh^{-1})\Delta_r^K(h^{-1})\mu_G(dg)\mu_H(dh), \quad f \in \mathcal{K}(K), \tag{7.6.4}$$

and by the usual extension (7.6.4) holds for all μ_K-integrable functions. With help of (7.1.8), (7.6.4) can also be put in the form

$$\int f(k)\mu_K(dk) = \int f(gh)\Delta_r^K(h)\Delta_r^H(h^{-1})\mu_G(dg)\mu_H(dh). \tag{7.6.5}$$

It should be kept in mind that (7.6.3)–(7.6.5) are valid for suitable versions of the left Haar measures μ_G, μ_H, μ_K. Choice of the versions of two of these fixes the third one. If it is desired to leave all three versions undetermined, then there should be an arbitrary multiplicative positive constant on the right-hand sides of (7.6.3)–(7.6.5).

7.6.2. COROLLARY. *In addition to the hypotheses of Proposition 7.6.1 suppose that H is normal in K. Then*

$$\int f(k)\mu_K(dk) = \iint f(gh)\mu_G(dg)\mu_H(dh) \tag{7.6.6}$$

for μ_K-integrable f.

PROOF. This follows from (7.6.5) and Corollary 7.4.3 with G in the latter replaced by K. □

The same formula (7.6.6) holds of course if G and H commute, which is a special case of Corollary 7.6.2. But whereas in the commuting case on the right-hand side of (7.6.6) we may equally well write $f(hg)$ instead of $f(gh)$, this is not true in the noncommuting case. We remark without proof that (7.6.6) remains valid if $f(gh)$ is replaced by $f(hg)$ and at the same time all left Haar measures are replaced by the corresponding right Haar measures.

7.6.3. COROLLARY. *In addition to the hypotheses of Proposition 7.6.1 suppose that G is normal in K. Then*

$$\int f(k)\mu_K(dk) = \int f(hg)\mu_G(dg)\mu_H(dh) \tag{7.6.7}$$

and

$$\int f(k)\mu_K(dk) = \iint f(gh)\delta(h)\mu_G(dg)\mu_H(dh) \tag{7.6.8}$$

in which $\delta(h)$ is the modulus of the topological automorphism $g \to h^{-1}gh \equiv a_h(g)$ defined in Proposition 7.1.9. If G and H commute, then $\delta \equiv 1$.

PROOF. Equation (7.6.7) is of course (7.6.6) with g and h interchanged. Then write $hg = a_h^{-1}(g)h$ and apply (7.1.11) with $\mu = \mu_G$. $\square$

The modulus $\delta(h) = \delta(a_h)$ can also be represented by $\mu(h^{-1}Ah)/\mu(A)$ for any integrable (e.g., compact) set $A \subset G$. This can be seen by writing (7.1.11) in the form $\int f(hgh^{-1})\mu_G(dg) = \delta(h) \int f(g)\mu_G(dg)$ and taking f to be the indicator of A. Informally one may write $\delta(h) = \mu(d(h^{-1}gh))/\mu(dg)$ and use this to pass from (7.6.7) to (7.6.8) by a change of variable. Replace in (7.6.7) g by g' and make the change of variable $g' = h^{-1}gh$, then $\mu_G(dg') = \delta(h)\mu_G(dg)$ so that $\int f(hg')\mu_G(dg') = \int f(gh)\delta(h)\mu_G(dg)$.

7.7. Invariant measures on Lie groups and its cosets by means of invariant differential forms. *Notation.* In addition to $GL(n)$, $LT(n)$, $UT(n)$, and $O(n)$, introduced in Sections 2.1 and 5.1, we shall also use the notation $M(m,n)$ for the group of all real $m \times n$ matrices under addition, $PD(n)$ for all $n \times n$ positive definite matrices, and $AS(n)$ for all $n \times n$ skew symmetric (= "anti-symmetric") matrices. If C is a real $n \times n$ matrix, then $|C|$ will stand for abs det(C). In order to shorten the notation for wedge products, following Muirhead (1982), we shall let (dC) stand for the appropriate wedge product of all or some of the elements dc_{ij} of dC (the order in the product is immaterial since any negative sign is discarded in forming a measure). To be more precise, if $B \in M(m,n)$, then $(dB) = \wedge_{ij} db_{ij}$ ($i = 1, \ldots, m$, $j = 1, \ldots, n$); if $C \in GL(n)$, then $(dC) = \wedge_{ij} dc_{ij}$ ($i, j = 1, \ldots, n$); if $T \in LT(n)$, then $(dT) = \wedge_{ij} dt_{ij}$ with $1 \le j \le i \le n$, and if $T \in UT(n)$ similarly with $i \le j$; if $S \in PD(n)$, $(dS) = \wedge_{ij} ds_{ij}$ with $i \ge j$ (or $i \le j$); and if $\Gamma \in O(n)$, then at $\Gamma = I_n$, $d\Gamma$ is skew symmetric and we take $(d\Gamma) = \wedge_{ij} d\gamma_{ij}$ where the wedge product is over $1 \le j < i \le n$ or $1 \le i < j \le n$.

Invariant differential forms on Lie groups, as defined in Section 5.3, can be used via Section 6.6 to obtain Haar measures.

7.7.1. PROPOSITION. *If ω is a left invariant d-form on the d-dimensional Lie group G, then it defines a left Haar measure on G.*

The same holds with "left" replaced by "right."

PROOF. In Proposition 6.6.3 take $M = N = G$ and take ϕ to be the left translation L_g, for any $g \in G$. Then $f \circ \phi$ on the right-hand side of (6.6.5) can also be written $g^{-1}f$. If ω is left invariant, then $\delta L_g(\omega) = \omega$. Hence (6.6.5) reads $\int f\omega = \int (g^{-1}f)\omega$. This being true for every $g \in G$ shows that $\int f\omega$ is a left invariant integral. The proof for right invariant ω is similar, with R_g replacing L_g. □

We shall usually write μ_G for left, ν_G for right Haar measure on G. It should be kept in mind that in converting ω to a measure we are using only $|\omega|$ rather than ω itself. Thus, for instance, even though the left invariant d-form (5.3.9) is not necessarily right invariant on all of G since sign change may occur, it does represent a right Haar measure (in agreement with Corollary 7.1.7).

7.7.2. EXAMPLES. **Haar measures on $GL(n)$, $LT(n)$, $UT(n)$, and $O(n)$.** Write elements of $GL(n)$ generically as $C : n \times n$, of $LT(n)$ and $UT(n)$ as $T : n \times n$, and of $O(n)$ as $\Gamma : n \times n$. On $GL(n)$ (5.3.4) defines a d-form that is both left and right invariant. Thus, on $GL(n)$ the measure

$$\mu_{GL(n)}(dC) = |C|^{-n}(dC) \tag{7.7.1}$$

is both left and right Haar measure, and it follows that $GL(n)$ is unimodular. From Examples 5.3.2 and 5.3.3 we obtain left and right Haar measures on $LT(n)$ and $UT(n)$:

$$\mu_{LT(n)}(dT) = t_{11}^{-1}t_{22}^{-2}\dots t_{nn}^{-n}(dT), \tag{7.7.2}$$

$$\nu_{LT(n)}(dT) = t_{11}^{-n}t_{22}^{-n+1}\dots t_{nn}^{-1}(dT), \tag{7.7.3}$$

whereas $\mu_{UT(n)}$ and $\nu_{UT(n)}$ are given by the right-hand sides of (7.7.3) and (7.7.2), respectively.

Proposition 7.2.6 can also be used to construct Haar measure in these cases, as well as right- and left-hand moduli. This is the method employed by Eaton (1983), Section 6.2, Examples 6.11, 6.12. On $GL(n)$ take $d\lambda = (dC)$ (Lebesgue measure), then the Jacobian computation in Example 5.3.1 produces $\chi_\ell(C) = \chi_r(C) = |C|^n$. Via (7.2.8) this reproduces (7.7.1). On $LT(n)$ take $d\lambda = (dT)$, then a Jacobian computation for a transformation of the type $y = gx$ as in Example 5.3.2, and a similar one for $y = xg$, produces the multipliers

$$\chi_\ell^{LT(n)}(T) = t_{11}t_{22}^2 \dots t_{nn}^n, \tag{7.7.4}$$

$$\chi_r^{LT(n)}(T) = t_{11}^n t_{22}^{n-1} \dots t_{nn}. \tag{7.7.5}$$

Then (7.7.4), (7.7.5), and (7.2.8) reproduce (7.7.2) and (7.7.3). By (7.2.9), the right- and left-hand moduli are the ratios of (7.7.4) and (7.7.5). Thus,

$$\Delta_r^{LT(n)} = t_{11}^{n-1} t_{22}^{n-3} \dots t_{nn}^{-n+1}, \tag{7.7.6}$$

and $\Delta_\ell^{LT(n)}$ is the reciprocal of (7.7.6). For $UT(n)$ the results may be obtained from those for $LT(n)$ by an interchange of left and right.

On $O(n)$ (5.3.9) is a formula for Haar measure. We shall not need this explicit form other than its expression at $\Gamma = I_n$:

$$\mu_{O(n)}(d\Gamma) = (d\Gamma) \text{ at } \Gamma = I_n. \tag{7.7.7}$$

This choice of $\mu_{O(n)}$ at the identity determines $\mu_{O(n)}$ uniquely on all of $O(n)$.

7.7.3. **Total Haar measure of $O(n)$.** The version of the Haar measure defined by (7.7.7) is not normalized, but it is for our purpose more convenient to leave it that way. However, we shall need the value

$$c_n = \mu_{O(n)}(O(n)), \tag{7.7.8}$$

i.e., the integral of the left-hand side of (7.7.7) over $O(n)$. Without proof we state the result:

$$c_n = \prod_{i=1}^{n} A_i, \quad A_i = 2\pi^{i/2}/\Gamma(i/2), \quad n = 1, 2, \dots, \tag{7.7.9}$$

and refer for its derivation to James (1954), equations (5.9) and (5.16), or Muirhead (1982), Corollary 2.1.16. In (7.7.9) A_i is the area of the unit $(i-1)$-sphere in R^i. It is also convenient to set $c_0 = 1$ by convention.

7.7.4. **Factorization of Haar measure on $GL(n)$.** Put $K = GL(n)$, $G = LT(n)$, or $UT(n)$, $H = O(n)$, then G and H are closed Lie subgroups of K with $G \cap H = \{e\}$ trivially compact. Hence, Proposition 7.6.1 applies. We shall pursue this for $G = LT(n)$. Choose the form (7.6.5) of the conclusion and observe that both K and H are unimodular, then (7.6.5) can be written

$$\mu_{GL(n)}(dC) = \mu_{LT(n)}(dT)\mu_{O(n)}(d\Gamma), \quad C = T\Gamma, \tag{7.7.10}$$

with $\mu_{GL(n)}$, $\mu_{LT(n)}$, and $\mu_{O(n)}$ of (7.7.1), (7.7.2), and (7.7.7), respectively. That the multiplicative constant on the right-hand side of (7.7.10) should be 1 follows by comparing the two sides of (7.7.10) in terms of their differential forms at $g = e$. This necessitates the following simple computation, which is typical in the use of differential forms. Take the equation $C = T\Gamma$ (see (7.7.10)) and take differentials on both sides: $dC = Td\Gamma + (dT)\Gamma$; evaluate at $e : dC = d\Gamma + dT$, at $C = I_n$. Since dT is lower triangular and $d\Gamma$ skew symmetric, we have $dc_{ii} = dt_{ii}$, and for $i > j$, $dc_{ij} = dt_{ij} + d\gamma_{ij}$, $dc_{ji} = d\gamma_{ji} = -d\gamma_{ij}$, so $dc_{ij} \wedge dc_{ji} = -dt_{ij} \wedge d\gamma_{ij} - d\gamma_{ij} \wedge d\gamma_{ij} =$ (use (4.1.10)) $-dt_{ij} \wedge d\gamma_j$. The minus sign may be discarded. Take the wedge product over all dc_{ij} with the result $(dC) = (dT)(d\Gamma)$, which, by (7.7.1), (7.7.2), and (7.7.7), is (7.7.10) evaluated at $g = e$.

7.7.5. **Left invariant measure on a coset space.** Let G be a d-dimensional Lie group, H a compact Lie subgroup, and $Y = G/H$ the space of left cosets. Any given left Haar measure μ_G on G induces via (7.4.4) a left invariant measure μ_Y. For application in subsequent sections it will be necessary to express in an explicit way the relation between μ_G and μ_Y.

7.7.6. PROPOSITION. *Let G be a d-dimensional Lie group and H a compact $(d-m)$-dimensional Lie subgroup, $0 < m < d$. Let*

$u_1, \dots, u_d$ be the coordinates of a canonical chart at e, of which $u_1, \dots, u_m$ are the coordinates of a chart on $Y = G/H$ at $[e]$ and $u_{m+1}, \dots, u_d$ the coordinates of a chart on H at e (Section 5.8). Let μ_G be the unique left Haar measure on G corresponding to the left invariant d-form ω^G whose value at e is

$$\omega_e^G = \bigwedge_{i=1}^{d} du_i. \tag{7.7.11}$$

Similarly μ_H on H and

$$\omega_e^H = \bigwedge_{i=m+1}^{d} du_i. \tag{7.7.12}$$

Then the unique induced left invariant measure μ_Y given by (7.4.4) corresponds to an m-form whose value at $[e]$ is

$$\omega_{[e]}^Y = c_H \bigwedge_{i=1}^{m} du_i \tag{7.7.13}$$

in which

$$c_H = \mu_H(H). \tag{7.7.14}$$

The statement remains true if $\dim H = 0$ and c_H equals the (finite) number of elements of H.

PROOF. Assume first $m < d$ so that $\dim H > 0$. Let μ_1 be the left invariant measure on Y corresponding to an m-form whose value at $[e]$ is $\bigwedge_1^m du_i$. It remains to be shown that $\mu_Y = c_H\mu_1$. Since in any case μ_Y and μ_1 are equal except for a positive multiplicative factor, put $\mu_Y = cc_H\mu_1$; then it will be shown that $c = 1$. Take the local cross section W in the proof of Theorem 5.8.1, parametrized by $u_1, \dots, u_m$, and consider the product space $W \times H$. On this space H acts to the right by acting to the right on H and trivially on W (since under right action the first m coordinates of a point don't change). On

$W \times H$ the measure μ_G is invariant under the right action of H since H is compact (so that Δ^G restricted to H is $\equiv 1$, Corollary 7.1.8). By Theorem 7.5.1 there is a measure μ_2 on W such that $\mu_G = \mu_2 \otimes \mu_H$ on $W \times H$. It follows that μ_G induces on W the measure $c_H \mu_2$. Then $c_H \mu_2$ is μ_Y restricted to W since μ_G induces μ_Y on the whole of Y. We had put $\mu_Y = c c_H \mu_1$ so that μ_2 is $c\mu_1$ restrcited to W. Then $\mu_G = c\mu_1 \otimes \mu_H$ on W. But at e both μ_G and $\mu_1 \otimes \mu_H$ equal $\bigwedge_1^d du_i$; therefore, $c = 1$.

The proof remains the same in the case $\dim H = 0$ if we take μ_H to be counting measure on H. □

7.7.7. EXAMPLE. Let G and H be as in Example 5.8.2: $G = O(n)$ and H is isomorphic to $O(s)$, $0 < s < n$. Partition $\Gamma = [\Gamma_1, \Gamma_2]$ with $\Gamma_2 : n \times s$, and define $(d\Gamma_1)$ as the wedge product of the $d\gamma_{ij}$ over all $(i,j) \in \Delta_1$ (notation Example 5.8.2). It was found in Example 5.8.2 that at $g = e$ the differentials of the canonical coordinates u_{ij} can be replaced by those of the γ_{ij}. Thus, the wedge product on the right-hand side of (7.7.13), which now reads $\bigwedge_{(i,j)\in\Delta_1} du_{ij}$, may be replaced by $(d\Gamma_1)$. Furthermore, the constant c_H in (7.7.13) equals c_s given by (7.7.9). With slight abuse of notation equate a measure to its defining differential form. Then (7.7.13) in this example reads

$$\mu_Y(dy) = c_s(d\Gamma_1) \text{ at } y = [e]. \tag{7.7.15}$$

7.7.8. EXAMPLE. Take the space $PD(n)$ of Example 5.3.7 with the action (5.3.10), but restrict G to $O(n)$. For some $\lambda_1 > \cdots > \lambda_n > 0$ take $S_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in PD(n)$, then at S_0 the isotropy subgroup H consists of all matrices $\operatorname{diag}(\pm 1, \ldots, \pm 1)$. This is a 0-dimensional finite group so that c_H of (7.7.14) equals 2^n. Let $\mu_G = \mu_{O(n)}$ be defined by (7.7.7), then since now in (7.7.13) $m = d$, the wedge product in (7.7.13) equals the right-hand side of (7.7.7). We get therefore

$$\mu_Y(dy) = 2^n(d\Gamma) \text{ at } y = [e]. \tag{7.7.16}$$