LIMIT THEOREMS FOR RANDOM CENTRAL ORDER STATISTICS

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Let $X_{n,1} \leq \ldots \leq X_{n,n}$ be the order statistics of a random sample $X_1, \ldots, X_n$ from a distribution function F. If $\{\nu_n\}$ is a sequence of integer-valued random variables such that $1 \leq \nu_n \leq n$ and $\nu_n/n \xrightarrow{P} p$, for some $p \in (0,1)$, the sequence $\{X_{n,\nu_n}\}$ is referred to as a sequence of random central order statistics (of limiting rank p) corresponding to the random central rank sequence $\{\nu_n\}$. In this paper, we establish the weak as well as strong consistency of X_{n,ν_n} in estimating the p-th quantile of F. We derive several central limit theorems for X_{n,ν_n} for regular as well as non-regular cases, and for each case, we provide remainder term estimates of the Berry-Esséen type.

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*Research supported by the Office of Naval Research, Contract N00014-85-K-0648.

AMS 1980 subject classifications. Primary 60F05,60F15, 62G30.

Key words and phrases. Random central order statistics, weak consistency, strong consistency, central limit theorem, Bahadur representation, Berry-Esséen bounds.

1. Introduction. Let X_i, $i \geq 1$ be a sequence of i.i.d. r.v.'s (independent and identically distributed random variables) with a cdf (cumulative distribution function) F, and let $X_{n,1} \leq X_{n,2} \leq \ldots \leq X_{n,n}$ be the order statistics of $X_1,\ldots,X_n$. In classical theory, one defines the central order statistics as sequences $\{X_{n,k_n}\}$ where $k_n \in \{1,\ldots,n\}$ (deterministic integers) and k_n/n has a limit $p \in (0,1)$ as $n \to \infty$. (The ratio k_n/n is generally called the rank of X_{n,k_n} and p is called the limiting rank). A typical example of central order statistics having a limiting rank $p \in (0,1)$ is provided by the sequence of the sample p-th quantile $\{\hat{\xi}_{np}\}$, where $\hat{\xi}_{np} = X_{n,np}$ if np is an integer, and $\hat{\xi}_{np} = X_{n,[np]+1}$ if np is not an integer ($[\cdot]$ denotes the integer part). Central order statistics serve to provide consistent estimators, tolerance limits and distribution free confidence intervals for "central" parameters, e.g., quantiles. They have, in general, an asymptotically normal distribution, and they converge strongly to appropriate limits (see Smirnov (1952) for a characterization, under suitable conditions, of the class of all possible limit distributions and the corresponding domains of attraction). An important feature of central order statistics is that they can be expressed asymptotically as sums of indpendent random variables, via the Bahadur (1966) representation.

In many problems dealing with central parameters of the underlying distribution function, there are in general several candidates based on sequences of central order statistics that can be used in deriving optimal statistical procedures. A relative efficiency comparison based on these procedures may still leave us with the difficult task of having to select the "best" central rank sequence $\{k_n\}$. Thus, it would be of interest to introduce a new class of statistics for which one allows random flexibility on the sequence of ranks and to study their asymptotic properties.

Our objective in this paper is to generalize the classical asymptotic theory for the so called random central order statistics. Specifically, given a sequence of i.i.d.r.v.'s $\{X_n\}$ with a common cdf F, let $X_{n,j}$ denote the j-th order statistic from $X_1,\ldots,X_n$ and let $\{\nu_n\}$ be a sequence of integer-valued random variables such that $1 \leq \nu_n \leq n$ for each n. Then, if for some $p \in (0,1)$, $\nu_n/n \xrightarrow{P} p$, $\{X_{n,\nu_n}\}$ is called a sequence of random central order statistics, and $\{\nu_n\}$ a random central rank sequence. In section 2, we establish, under quite general conditions, the weak and strong consistency of a sequence of random central order statistics $\{X_{n,\nu_n}\}$ of the limiting rank p in estimating the (unknown) p-th quantile $\xi_p = \inf\{x\colon\ F(x) \geqslant p\}$. Further, confining attention to the so called regular cases (those for which $F'(\xi_p)$ exists and is positive), if $\nu_n/n \xrightarrow{P} p$ sufficiently fast, we obtain, in section 3, a central limit theorem which shows that the random flexibility of the ranks (ν_n/n) does not disturb the form of the limiting distribution of the normalized sequence $\{X_{n,\nu_n}\}$, a feature which adds greatly to the usefulness of the theory. We conclude this section with the derivation of a weak Bahadur representation for random central order statistics, a result which generalizes Ghosh (1971). In section 4, a Berry-Esséen type theorem is established for the distribution of random central order statistics in regular cases. Our results generalize as well as extend those of Reiss (1974) and Serfling (1980), Theorem C, p. 81) who derived the bound $O(n^{-1/2})$ for the departure from normality of the distribution function of the sample quantile. In section 5, we study the limit law of X_{n,ν_n} in non-regular cases. In this context, we present a central limit theorem for a suitably normalized X_{n,ν_n} which generalizes a result of Chanda (1975). We also give a remainder term estimate of the Berry-Esséen type for the corresponding normal distribution approximation.

2. Consistency of X_{n,ν_n}. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s with a common cdf $F(x) = P[X_1 \leqslant x]$ and let $\{\nu_n\}$ be a random central rank sequence such that $\nu_n/n \xrightarrow{P} p \in (0,1)$. We begin with the simplest result of interest.

THEOREM 2.1. (Weak Consistency). If ξ_p is the unique solution y of $F(y-) \leqslant p \leqslant F(y)$, then

$$X_{n,\nu_n} \xrightarrow{P} \xi_p, \text{ as } n \to \infty. \tag{2.1}$$

Proof. Let $F_n(x)$ be the empirical distribution function corresponding to the sample $X_1,\ldots,X_n$, i.e. $nF_n(x)$ = number of $X_i \leq x$, $1 \leq i \leq n$. For each $\alpha > 0$, we have (by the uniqueness condition of the theorem),

$$F(\xi_p - \alpha) < p < F(\xi_p + \alpha) \tag{2.2}$$

Note that

$$\begin{aligned} P\{X_{n,\nu_n} > \xi_p + \alpha\} &= 1 - P\{F_n(\xi_p + \alpha) \geq \frac{\nu_n}{n}\} \\ &= P\{[F_n(\xi_p + \alpha) - F(\xi_p + \alpha)] + [F(\xi_p + \alpha) - p] < \frac{\nu_n}{n} - p\} \end{aligned} \tag{2.3}$$

and, since $F_n(\xi_p + \alpha) \xrightarrow{P} F(\xi_p + \alpha)$ and $\frac{\nu_n}{n} \xrightarrow{P} p$, we obtain on account of (2.2) and (2.3) that $P\{X_{n,\nu_n} > \xi_p + \alpha\} \to 0$ as $n \to \infty$.

A similar argument shows that $P\,\{X_{n,\nu_n} < \xi_p - \alpha - \alpha\} \to 0$ as $n \to \infty$. The proof follows.

To achieve strong consistency of X_{n,ν_n} for the estimation of ξ_p, we assume the following:

$$\sum_{n=1}^{\infty} P\{|\frac{\nu_n}{n} - p| > \varepsilon\} < \infty \quad \text{for every } \varepsilon > 0. \tag{2.4}$$

THEOREM 2.2. (Strong Consistency) If ξ_p is the unique solution y of $F(y-) \leq p \leq F(y)$ and, in addition, (2.4) holds, then, with probability one,

$$X_{n,\nu_n} \to \xi_p, \text{ as } n \to \infty. \tag{2.5}$$

Proof: For each $\alpha > 0$, observe that

$$P\{X_{n,\nu_n} > \xi_p + \alpha\} = P\{F_n(\xi_p + \alpha) < \frac{\nu_n}{n}\}$$

(2.6)

$$\leq P\{|\frac{\nu_n}{n} - p| \leq \varepsilon,\ F_n(\xi_p + \alpha) < \frac{\nu_n}{n}\} + P\{|\frac{\nu_n}{n} - p| > \varepsilon\} = a_n + b_n$$

say, where $\varepsilon > 0$ is chosen such that $2\varepsilon < F(\xi_p + \alpha) - p$. Then, by Markov's inequality, we get

$$a_n \leq P\{F_n(\xi_p + \alpha) - F(\xi_p + \alpha) < -\varepsilon\} \leq \frac{1}{\varepsilon^4} E\{F_n(\xi_p + \alpha) - F(\xi_p + \alpha)\}^4 \tag{2.7}$$

and, since $E\{F_n(\xi_p + \alpha) - F(\xi_p + \alpha)\}^4 \leq 3/n^2$, from (2.4), (2.6) and (2.7), we obtain

$$\sum_{n=1}^{\infty} P\{X_{n,\nu_n} > \xi_p + \alpha\} \leq \frac{3}{\varepsilon^4} \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} P\{|\frac{\nu_n}{n} - p| > \varepsilon\} < \infty . \tag{2.8}$$

Similarly, we have

$$\sum_{n=1}^{\infty} P\{X_{n,\nu_n} < \xi_p - \alpha\} < \infty . \tag{2.9}$$

Theorem 2.2 follows by combining (2.8) and (2.9).

Remark 2.1. Note that if $\{k_n\}$ is the numerical sequence defined by $k_n = np$ if np is an integer and $k_n = [np]+1$ if np is not an integer, then, if $\nu_n \equiv k_n$, $n \geq 1$, X_{n,ν_n} reduces to the usual sample p-th quantile $\hat{\xi}_p$. In this case, condition (2.4) is easily checked and Theorem 2.2 yields the well-known strong consistency of the sample quantile for estimation of ξ_p (see Serfling (1980), Theorem 2.3.1).

3. The Asymptotic Distribution of X_{n,ν_n} in Regular Cases.

A well-known result on the sample quantile $\hat{\xi}_{np}$ asserts that if $F(\xi_p) = p$, F is differentiable at ξ_p and $F'(\xi_p) > 0$ (i.e. the regular cases), then

$$\frac{n^{1/2}(\hat{\xi}_{np} - \xi_p)F'(\xi_p)}{[p(1-p)]^{1/2}} \xrightarrow{\mathcal{D}} N(0,1) .$$

In the present context, it would be of interest to know whether or not in regular cases the limiting distribution of X_{n,ν_n} is the same as that of $\hat{\xi}_{np}$. In seeking a solution to this problem we would like to impose minimal assumptions on the parent distribution and allow dependence between $\{\nu_n\}$ and the original variates $\{X_n\}$ (an independence assumption will rarely be fulfilled in interesting cases). Here the main obstacle is to overcome the random factor introduced by ν_n and we have to develop new methods of proof.

The main result of this section is contained in the following theorem.

THEOREM 3.1. (Central Limit Theorem) If

(i) $F(\xi_p) = p$, F is differentiable at ξ_p and $F'(\xi_p) > 0$,

and

(ii) $n^{1/2}(\frac{\nu_n}{n} - p) \xrightarrow{P} c$, for some constant c,

then

$$n^{1/2}(X_{n,\nu_n} - \xi_p)F'(\xi_p)\sigma_p^{-1} \xrightarrow{\mathcal{D}} N\,(c\sigma_p^{-1},\ 1) \tag{3.1}$$

where $\sigma_p^2 = p(1-p)$, $p \in (0,1)$.

Remark 3.1. (3.1) shows that the constant c in (ii) above has a direct influence on the asymptotic mean of the (normalized) X_{n,ν_n}. For nonrandom central order statistics, this fact was noticed by Serfling (1980, p. 94) who emphasized its importance in the treatment of confidence intervals for quantiles.

Proof of Theorem 3.1. Our approach to proving that (3.1) holds is to show that the asymptotic distribution of $n^{1/2}(X_{n,\nu_n} - \xi_p)$ coincides with the asymptotic distribution of $[F'(\xi_p)]^{-1}\{n^{-1/2}\sum_{i=1}^{n} W_i + n^{1/2}(\frac{\nu_n}{n} - p)\}$, where $\{W_i\}_{1\leq i\leq n}$ are i.i.d.r.v.'s with mean 0 and variance σ_p^2.

To this end, set

$$G_n(x) = P\{n^{1/2}(X_{n,\nu_n} - \xi_p) \leq x\}, \quad x \in \mathbb{R}$$

and, notice that

$$\begin{aligned} G_n(x) &= P\{F_n(\xi_p + xn^{-1/2}) \geq \frac{\nu_n}{n}\} \\ &= P\{n^{1/2}[p - F_n(\xi_p)] - \rho_n(x) \leq F'(\xi_p)x - n^{1/2}(\frac{\nu_n}{n} - p)\} \end{aligned} \tag{3.2}$$

where

$$\rho_n(x) = n^{1/2}\{F_n(\xi_p + xn^{-1/2}) - F_n(\xi_p)\} - F'(\xi_p)x .$$

We now show that for each $x \in \mathbb{R}$,

$$\rho_n(x) \xrightarrow{P} 0 . \tag{3.3}$$

Let $u(t) = 1$ if $t \geq 0$ and $= 0$ if $t < 0$. In proving (3.3), we may assume $x \neq 0$. Write

$$\rho_n(x) = n^{-1/2}\sum_{i=1}^{n} Z_{ni} - F'(\xi_p)x \tag{3.4}$$

where $Z_{ni} = u(\xi_p + xn^{-1/2} - X_i) - u(\xi_p - X_i)$, $1 \leq i \leq n$, $n \geq 1$ are row-wise independent random variables with

$$p_n = E(Z_{ni}) = F(\xi_p + xn^{-1/2}) - p \text{ and } Var(Z_{ni}) = |p_n|(1 - |p_n|)$$

Then, since $p_n \to 0$ as $n \to \infty$, we obtain by Chebyshev's inequality, that for any $\varepsilon > 0$,

$$P\{|\sum_{i=1}^{n} [Z_{ni} - E(Z_{ni})]| > \varepsilon n^{1/2}\} \leq \frac{|p_n|(1-|p_n|)}{\varepsilon^2} \to 0 \text{ , as } n \to \infty$$

entailing

$$n^{-1/2} \sum_{i=1}^{n} Z_{ni} - n^{1/2}[F(\xi_p + xn^{-1/2}) - p] \xrightarrow{P} 0 . \tag{3.5}$$

On the other hand,

$$n^{1/2}[F(\xi_p + xn^{-1/2}) - p] \to F'(\xi_p)x \text{ as } n \to \infty$$

which together with (3.5) implies (3.3).

Now, write $p - F_n(\xi_p) = \frac{1}{n}\sum_{i=1}^{n} W_i$, where $W_i = p - u(\xi_p - X_i)$, $1 \leq i \leq n$, $n \geq 1$. This is an average of i.i.d.r.v.'s to which according to the classical central limit theorem,

$$n^{-1/2} \sum_{i=1}^{n} W_i \xrightarrow{\mathcal{D}} N(0,\sigma_p^2) . \tag{3.6}$$

Using (3.2), (3.3) and (3.6) we deduce

$$G_n(x) \to P\{Z \leq F'(\xi_p)x\sigma_p^{-1}\} = \Phi((F'(\xi_p)x - c)\sigma_p^{-1}) \tag{3.7}$$

where Z is $N(c\sigma_p^{-1}, 1)$ r.v., and Φ is the standard normal cdf. (3.1) follows from (3.7).

Another relevant question for the asymptotic theory of random central order statistics is a Bahadur-type representation. It will be interesting (but seems difficult) to investigate whether a Bahadur representation with a strong remainder term holds for the case of X_{n,ν_n} treated here. We will nevertheless prove

THEOREM 3.2. (Weak Representation Theorem) Under the assumptions (i) and (ii) of Theorem 3.1 we have

$$X_{n,\nu_n} = \xi_p + [F'(\xi_p)]^{-1}[\frac{\nu_n}{n} - F_n(\xi_p)] + R_n \tag{3.8}$$

where

$$R_n = o_p(n^{-1/2}) \text{ as } n \to \infty . \tag{3.9}$$

Proof: Defining

$$Q_n = n^{1/2}[\frac{\nu_n}{n} - F_n(\xi_p) - (X_{n,\nu_n} - \xi_p)F'(\xi_p)], \text{ we need}$$

to show that

$$Q_n \xrightarrow{P} 0 . \tag{3.10}$$

To achieve this, take an arbitrary $\varepsilon > 0$. Then, using Theorem 3.1, choose $K > 0$ sufficiently large such that

$$P\{n^{1/2}|X_{n,\nu_n} - \xi_p| > K\} < \varepsilon/2, \text{ for } n \geq n_o . \tag{3.11}$$

Now, partition the interval $[-K,K]$ into $m - 1$ intervals $-K = \Delta_1 < \Delta_2 < \dots < \Delta_m = K$ such that

$$\Delta_i - \Delta_{i-1} < \varepsilon_o[2F'(\xi_p)]^{-1}, \text{ for } i = 2,3,\dots, m \text{ and } \varepsilon_o > 0 . \tag{3.12}$$

Now, set

$$\Pi_{n,i} = \{\Delta_{i-1} \leq n^{1/2}(X_{n,\nu} - \xi_p) \leq \Delta_i\}, \ i = 2,\dots,m; \ n \geq 1$$

and use (3.11) to get

$$(3.13) \qquad P\{Q_n > \varepsilon_o\} \leq \sum_{i=2}^{m} P\{[Q_n > \varepsilon_o] \cap \Pi_{n,i}\} + \varepsilon/2 .$$

Since $X_{n,\nu_n} \leq \xi_p + n^{-1/2}\Delta_i$ entails $\nu_n/n \leq F_n(\xi_p + n^{-1/2}\Delta_i)$, by using the monotonicity of F_n and (3.12), we have for $i = 2,\ldots,m$

$$\{[Q_n > \varepsilon_o] \cap \Pi_{n,i}\} \subset \{n^{1/2}[F_n(\xi_p + n^{-1/2}\Delta_i) - F_n(\xi_p)] - F'(\xi_p)\Delta_i > \varepsilon_o/2\}$$

which, together with (3.13), entails

(3.14)

$$P\{Q_n > \varepsilon_o\} \leq \sum_{i=2}^{m} P\{n^{1/2}[F_n(\xi_p + n^{-1/2}\Delta_i) - F_n(\xi_p)] - F'(\xi_p)\Delta_i > \varepsilon_o/2\} + \varepsilon/2.$$

Now, by (3.3) with $x = \Delta_i$ for $i = 2,\ldots,m$, the sum in the right-hand side of (3.14) is $< \varepsilon/2$ for sufficiently large n, proving that $P\{Q_n > \varepsilon_o\} < \varepsilon \ \forall\, n \geq n_1$. Similarly $P\{Q_n < -\varepsilon_o\} < \varepsilon \ \forall\, n \geq n_2$. The proof follows.

Remark 3.2. Note that if $c = 0$ in the condition (ii) of Theorem 3.1 (in which case the limiting law of the normalized X_{n,ν_n} is standard normal), then (3.8) and (3.9) reduce to

$$(3.15) \qquad X_{n,\nu_n} = \xi_p + [F'(\xi_p)]^{-1}[p - F_n(\xi_p)] + o_p(n^{-1/2}) ,$$

showing that asymptotically, X_{n,ν_n} may be represented as an average of i.i.d.r.v.'s. In this form (3.15) represents a generalization of a result due to Ghosh (1971) (see also Serfling (1980), p. 92) and David (1981), pp. 254-256)).

4. The Berry-Esséen Bound for X_{n,ν_n} in Regular Cases. Recently interest has

been focused on the convergence to normality for sample quantiles $\hat{\xi}_{np}$ in regular cases. For such situations, under the assumption that F has a bounded second derivative on $\mathbb{R}$, Reiss (1974) and Serfling (1980, Theorem C, p. 81) derived independently the Berry-Esséen bound $O(n^{-1/2})$. In this section we study this problem in the case of normalized random central order statistic X_{n,ν_n} under assumptions somewhat weaker than those of Reiss and Serfling (cit. op.). Our results not only include the results of these authors as a special case but also extend their results to cover general random rank sequences $\{\nu_n\}$. To be precise, we seek bounds on the quantity

$$\Delta_n = \sup_x |P\{n^{1/2}(X_{n,\nu_n} - \xi_p)\, F'(\xi_p) \leq x\sigma_p\} - \Phi(x)|$$

for the case when the limit law in (3.1) is standard normal (i.e., when $c = 0$). To solve this problem we would need to impose a relatively stronger version of the condition (ii) of Theorem 3.1 (with $c = 0$), namely $P\{n^{1/2}|\frac{\nu_n}{n} - p| > \varepsilon_n\} \leq \delta_n$, for some numerical sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ converging to zero. The "exact" order of approximation for Δ_n will then be obtained when $\varepsilon_n = O(n^{-1/2}) = \delta_n$.

In what follows $C_1, C_2, C_3, \ldots$ will denote positive constants.

THEOREM 4.1. (Rates of Convergence in the CLT) Let ε_n and δ_n be positive constants such that $\varepsilon_n \leq 1$ and

$$P\{n^{1/2}\,|\,\frac{\nu_n}{n} - p\,| > \varepsilon_n\} \leq \delta_n\,,\ n \geq 1 \tag{4.1}$$

where $\{\nu_n\}$ is a random central rank sequence and $p \in (0,1)$. Assume that F"exists and is bounded in the interval $J = [\xi_p - K, \xi_p + K]$ (for some $K > 0$). Let $M = \sup_{x\in J} |F''(x)|$ and $K_o = \min\{F'(\xi_p)M^{-1}, K\}$. Then

$$\Delta_n \leq C_1 n^{-1/2} + C_2 n^{-1} + C_3\varepsilon_n + \delta_n \tag{4.2}$$

where

$$C_1 = 4\sigma_p^{-2} + [1.19625 + 8(2\pi)^{-1/2} e^{-1}]\sigma_p^{-1} + 4\sigma_p M(2\pi)^{-1/2} e^{-1} [F'(\xi_p)]^{-2},$$

$$C_2 = 24\sigma_p^{-2} + (2\pi e)^{-1/2} \sigma_p^2 K_o^{-2} [F'(\xi_p)]^{-2}$$

and

$$C_3 = 1.5(2\pi)^{-1/2} \sigma_p^{-1} .$$

Remark 4.1. (Example) Let $\{X_n\}$ be a given sequence of i.i.d.r.v.'s with cdf F. Let $\{\delta_n\}$ be a sequence of positive constants such that $\delta_n \to 0$ as $n \to \infty$. Consider the "quantile" sequence $\{k_n\}$ defined in Remark 2.1. Define $\nu_n : \Omega \to \{1,\ldots,n\}$ by $\nu_n(\omega) = k_n$ if $\omega \in \Omega_n$, where $\{\Omega_n\}$ is a sequence of events chosen such that $P\{\Omega_n\} = 1 - \delta_n$ (the definition of ν_n on Ω_n^c may be made arbitrarily provided $\nu_n \in \{1,\ldots,n\}$ and is measurable).

Then, since $|\frac{k_n}{n} - p| < \frac{1}{n}$, it follows that

$$\{|\frac{\nu_n}{n} - p| \geq \frac{1}{n}\} \subset \Omega_n^c$$

which guarantees that

$$P\{n^{1/2} |\frac{\nu_n}{n} - p| \geq n^{-1/2}\} \leq P(\Omega_n^c) = \delta_n .$$

Thus, condition (4.1) is fulfilled with $\varepsilon_n = n^{-1/2}$. We remark that according to our Theorem 4.1, if $\delta_n = O(n^{-1/2})$ as $n \to \infty$, the (exact) Berry-Esséen approximation order for Δ_n is $O(n^{-1/2})$. In particular, if $\delta_n \equiv 0$, $n \geq 1$, the theorem of Reiss and Serfling will follow as a special case.

To prove Theorem 4.1 we need a sequence of lemmas.

LEMMA 4.1. If

$$(4.3) \qquad |F(x) - p| < \sigma_p^2/2$$

then

$$(4.4) \qquad |\sigma_p \sigma_{F(x)}^{-1} - 1| < \sigma_p^{-2}|F(x) - p|, \text{ where } \sigma_{F(x)}^2 = F(x)(1 - F(x)) .$$

Proof: Let $d(x) = |\sigma_p \sigma_{F(x)}^{-1} - 1|$. Note that under (4.3), $0 < F(x) < 1$, so that d(x) is well defined. By setting $\lambda = \sigma_p^{-2}(F(x) - p)$, d(x) simplifies, after some calculation, to

$$(4.5) \qquad |\lambda| \cdot \frac{|\lambda\sigma_p^2 - (1-2p)|}{[1+(1-2p)\lambda-\sigma_p^2 \lambda^2] + [1+(1-2p)\lambda-\sigma_p^2\lambda^2]^{1/2}} .$$

Now, since $|\lambda| < 1/2$ and $p \in (0,1)$, it is easily seen that $|\lambda\sigma_p^2 - (1 - 2p)| < 1$, while

$$1 + (1-2p)\lambda - \sigma_p^2\lambda^2 > 1 - |\lambda| - \lambda^2/4 > 7/16.$$

Thus, according to (4.5), we have

$$d(x) < |\lambda| \frac{1}{\frac{7}{16} + (\frac{7}{16})^{1/2}} < |\lambda|$$

and (4.4) obtains.

The following well-known lemma (cf. Petrov (1975), p. 16) is also needed.

LEMMA 4.2. Let $\{V_n\}$ and $\{W_n\}$ be two sequences of random variables. If $\{a_n\}$ is a sequence of positive constants, then

$$\sup_x |P\{V_n + W_n \le x\} - \Phi(x)| \le \sup_x |P\{V_n \le x\} - \Phi(x)| \tag{4.6}$$
$$+ P\{|W_n| \ge a_n\} + (2\pi)^{-1/2} a_n .$$

The next result gives an approximation of the true distribution of random central order statistics.

LEMMA 4.3. Assume that (4.1) holds. Then $\forall\ n > 1$,

(4.7)
$$D_n = \sup_x |P\{X_{n,\nu_n} < x\} - \Phi(n^{1/2}(F(x) - p)\sigma_p^{-1})| < C_4 n^{-1/2} + C_5 n^{-1} + C_3 \varepsilon_n + \delta_n$$

where C_3 is defined in Theorem 4.1, $C_4 = 4\sigma_p^{-2} + [1.19625 + 8(2\pi)^{-1/2} e^{-1}]\sigma_p^{-1}$ and $C_5 = 24\sigma_p^{-2}$.

Remark 4.2. As a consequence of the fact that in Lemma 4.3 absolutely no conditions are imposed on the distribution function F, the estimate (4.7) will also play a key role in the study of the asymptotic law of X_{n,ν_n} in nonregular cases; (see Section 5).

Remark 4.3. The following example shows that for a given sequence $\{\varepsilon_n\}$ such that $\varepsilon_n = O(n^{-1/2})$, the rate of convergence in (4.7) cannot be sharpened even if the assumption (4.1) is maximally sharpened to

$$P\{n^{1/2}|\frac{\nu_n}{n} - p| > \varepsilon_n\} = 0 . \tag{4.1*}$$

Let $\{X_n\}$ be a sequence of i.i.d. symmetric Bernoulli r.v.'s with $P\{X_1 = 1\} = P\{X_1 = -1\} = 1/2$. By setting $\nu_n = [\frac{n}{2}]$, it is easily seen that $(4.1)^*$ is satisfied with $p = 1/2$ and $\varepsilon_n = n^{-1/2}$. Now, if n is even, we have

$$P\{X_{n,\nu_n} < 0\} = P\{F_n(0) > \frac{\nu_n}{n}\} = P\{\sum_{i=1}^{n} u(-X_i) > \frac{n}{2}\}$$

and since $\sum_{i=1}^{n} u(-X_i) \overset{\mathcal{D}}{\sim} B(n, \frac{1}{2})$ (binomial), with D_n as defined in (4.7), we have

$$D_n \geq \tfrac{1}{2} P\{B(n, \tfrac{1}{2}) = \tfrac{n}{2}\} = \frac{1}{2^{n+1}} \frac{n!}{[(\frac{n}{2})!]^2} = d_n, \text{ say.}$$

Then, using Stirling's formula, it is easily seen that $d_n \sim 1/(2\pi n)^{1/2}$ as $n \to \infty$. This shows that the assumption (4.1) in Lemma 4.3 with $\varepsilon_n = O(n^{-1/2}) = \delta_n$ is the most reasonable assumption to obtain the "correct" Berry-Esséen type bound $O(n^{-1/2})$.

Proof of Lemma 4.3. We estimate D_n by splitting it into two parts, namely for $x \in I$ and for $x \in I^c$, where $I = \{x: |F(x) - p| \leq \sigma_p^2/2\}$.

(i) Let $x \in I$. Set

$$S_n(x) = n^{-1/2} \sigma_{F(x)}^{-1} \sum_{i=1}^{n} [u(x - X_i] - F(x)].$$

Then,

$$\begin{aligned} P\{X_{n,\nu_n} \leq x\} &= P\{F_n(x) \geq \tfrac{\nu_n}{n}\} \\ &= P\{S_n(x) - n^{1/2}(\tfrac{\nu_n}{n} - p)\sigma_{F(x)}^{-1} \geq n^{1/2}(p - F(x))\sigma_{F(x)}^{-1}\}. \end{aligned} \tag{4.8}$$

In view of Lemma 4.2 and (4.8), we have

(4.9)

$$\begin{aligned} D_{n,1}(x) &= |P\{X_{n,\nu_n} \leq x\} - \Phi(n^{1/2}(F(x) - p)\sigma_{F(x)}^{-1})| \\ &\leq \sup_{t \in \mathbb{R}} |P\{S_n(x) \leq t\} - \Phi(t)| + P\{|n^{1/2}(\tfrac{\nu_n}{n} - p)| \geq a_n \sigma_{F(x)}\} + a_n/(2\pi)^{1/2}. \end{aligned}$$

for any sequence of positive constants $\{a_n\}$. Now, since $x \in I$, using Lemma 4.1, we get

$$(4.10) \qquad \sigma_p \sigma_{F(x)}^{-1} \leq 1.5$$

and by setting $a_n = 1.5\varepsilon_n \sigma_p^{-1}$, we deduce from (4.1) and (4.9), that

$$(4.11) \qquad D_{n,1}(x) \leq \sup_{t \in \mathbb{R}} |P\{S_n(x) \leq t\} - \Phi(t)| + 1.5\varepsilon_n (2\pi)^{-1/2} \sigma_p^{-1} + \delta_n .$$

Applying the Berry-Esséen theorem (with the sharpest constant 0.7975 given by van Beeck (1972)) the first term in the right-hand side of (4.11) is bounded by $(0.7975)\sigma_{F(x)}^{-3} E|u(x - X_1) - F(x)|^3 n^{-1/2}$ and since $E|u(x - X_1) - F(x)|^3 \leq \sigma_{F(x)}^2$, (4.11) implies

$$(4.12) \qquad D_{n,1}(x) \leq (0.7975)\sigma_{F(x)}^{-1} n^{-1/2} + 1.5\varepsilon_n (2\pi)^{-1/2} \sigma_p^{-1} + \delta_n .$$

Consequently, using (4.10) once again, we get from (4.12) that

$$(4.13) \qquad D_{n,1}(x) \leq (1.19625)\sigma_p^{-1} n^{-1/2} + 1.5\varepsilon_n (2\pi)^{-1/2} \sigma_p^{-1} + \delta_n .$$

Consider now

$$D_{n,2}(x) = |\Phi(n^{1/2}(F(x) - p)\sigma_{F(x)}^{-1}) - \Phi(n^{1/2}(F(x) - p)\sigma_p^{-1})| .$$

The mean value theorem yields

$$(4.14) \qquad D_{n,2}(x) = n^{1/2} |F(x) - p| \sigma_p^{-1} |\sigma_p \sigma_{F(x)}^{-1} - 1| (2\pi)^{-1/2} \exp\left\{ \frac{-n(F(x) - p)^2}{2\sigma_p^2} [1 + \theta(\sigma_p \sigma_{F}^{-1}(x) - 1)]^2 \right.$$

for some $0 < \theta < 1$.

Now Lemma 4.1 implies that

$$|1 + \theta(\sigma_p \sigma_{F(x)}^{-1} - 1)| \geq 1 - |\sigma_p \sigma_{F(x)}^{-1} - 1| \geq 1/2$$

which together with (4.14) and another application of Lemma 4.1 gives

$$(4.15) \qquad D_{n,2}(x) \leq n^{1/2}(F(x) - p)^2 (2\pi)^{-1/2} \sigma_p^{-3} \exp\{-\tfrac{1}{8} n(F(x) - p)^2 \sigma_p^{-2}\}.$$

Therefore, since

$$(4.16) \qquad \sup_{t>0} t \exp(-ta) = (ae)^{-1}, \ a > 0$$

we obtain, from (4.15) that

$$(4.17) \qquad D_{n,2}(x) \leq 8(2\pi)^{-1/2} (\sigma_p e)^{-1} n^{-1/2}.$$

Combining (4.17) and (4.13), we conclude that

$$(4.18) \qquad \sup_{x \in I} |P\{X_{n,\nu_n} \leq x\} - \Phi(n^{1/2}(F(x) - p)\sigma_p^{-1})|$$

$$\leq \{1.19625 + 8(2\pi)^{-1/2} e^{-1}\}\sigma_p^{-1} n^{-1/2} + 1/5(2\pi)^{-1/2} \sigma_p^{-1} \varepsilon_n + \delta_n .$$

(ii) <u>Let $x \in I^c$.</u> In order to estimate $\Phi(n^{1/2}(F(x) - p)\sigma_p^{-1})$ for x in I^c, we shall use the well-known inequality:

$$(4.19) \qquad 1 - \Phi(t) \leq (2\pi)^{-1/2} t^{-1} \exp(-t^2/2) \leq (2\pi e)^{-1/2} t^{-2}, \ t > 0.$$

Set $\lambda = \sigma_p^{-2}(F(x) - p)$ (as in the proof of Lemma 4.1) and assume that $\lambda \leq -1/2$. Then, according to (4.19), we have

$$(4.20) \qquad \Phi(n^{1/2}(F(x)-p)\sigma_p^{-1}) \leq 4(2\pi e)^{-1/2} \sigma_p^{-2} n^{-1}.$$

On the other hand, note that

$$P\{X_{n,\nu_n} \leqslant x\} = P\{F_n(x) \geqslant \frac{\nu_n}{n},\ n^{1/2}\,|\frac{\nu_n}{n} - p| < \varepsilon_n\}$$

$$(4.21)\qquad + P\{F_n(x) \geqslant \frac{\nu_n}{n},\ n^{1/2}\,|\frac{\nu_n}{n} - p\,| \geqslant \varepsilon_n\}$$

$$\leqslant P\{F_n(x) > p - \varepsilon_n n^{-1/2}\} + \delta_n$$

where the last inequality follows from (4.1).

Observe that, in proving (4.7), we may without loss of generality assume that

$$(4.22)\qquad n^{1/2}\sigma_p^2 \geqslant 4\ ,$$

since if (4.22) does not hold, then the bound in (4.7) applies trivially. Now, by Chebyshev's inequality, we have

$$(4.23)\qquad P\{F_n(x) > p - \varepsilon_n n^{-1/2}\} \leqslant \sigma^2_{F(x)}(p - F(x) - \varepsilon_n n^{-1/2})^{-2} n^{-1}\ .$$

To estimate the right-hand side of (4.23), we see that, since $\lambda \leqslant -1/2$,

$$(4.24)\qquad \sigma^2_{F(x)}\sigma_p^{-2} = 1 + (1-2p)\lambda - \sigma_p^2\lambda^2 \leqslant 1 + |\lambda| \leqslant 6\lambda^2$$

and

$$(4.25)\qquad \sigma_p^2 + \varepsilon_n\lambda^{-1}n^{-1/2} \geqslant \sigma_p^2 - 2\varepsilon_n n^{-1/2} \geqslant \sigma_p^2/2$$

where the last inequality of (4.25) follows from (4.22).

Thus, using (4.23)-(4.25), we derive

$$P\{F_n(x) > p - \varepsilon_n n^{-1/2}\} \leq 24\sigma_p^{-2} n^{-1} . \tag{4.26}$$

Since, for $\alpha, \beta > 0$, $|\alpha - \beta| \leq \max(\alpha, \beta)$, by combining (4.20), (4.21) and (4.26), we find that for $\lambda \leq -1/2$,

$$|P\{X_{n,\nu_n} \leq x\} - \Phi(n^{1/2}(F(x) - p)\sigma_p^{-1})| \leq 24\sigma_p^{-2} n^{-1} + \delta_n . \tag{4.27}$$

Repeating the argument in (4.20) and (4.26) for $\lambda \geq 1/2$, we see that (4.27) continues to hold in this case also. (4.7) now follows by combining (4.27) and (4.18). The proof follows.

Proof of Theorem 4.1. Our main result (4.2) follows readily from the estimate (4.7) and the following two lemmas.

LEMMA 4.4. If $|\alpha x| \leq 1/2$, then

$$|\Phi(x + \alpha x^2) - \Phi(x)| \leq 8|\alpha|(2\pi)^{-1/2} e^{-1} . \tag{4.28}$$

LEMMA 4.5. Under the assumptions of Theorem 4.1, we have

(4.29)
$$E_n = \sup_{x \in R} | \Phi(n^{1/2}(F(x) - p)\sigma_p^{-1}) - \Phi(n^{1/2}(x - \xi_p)F'(\xi_p)\sigma_p^{-1})| \leq C_6 n^{-1/2} + C_7 n^{-1}$$

where

$$C_6 = 4M\sigma_p(2\pi)^{-1/2} e^{-1}[F'(\xi_p)]^{-2} \text{ and } C_7 = \sigma_p^2(2\pi e)^{-1/2} K_o^{-2}[F'(\xi_p)]^{-2} .$$

Proof of Lemma 4.4. By the mean value theorem, we have with $y = x + \alpha x^2$,

$$\Phi(y) - \Phi(x) = (2\pi)^{-1/2} \alpha x^2 \exp[-1/2 x^2[1 + \theta\alpha x]^2\} , \quad 0 < \theta < 1 ,$$

and, since $|1+\theta\alpha x| \geq 1 - |\alpha x| \geq 1/2$, we obtain

$$|\Phi(y) - \Phi(x)| \leq (2\pi)^{-1/2} |\alpha| x^2 \exp(-\frac{1}{8} x^2)$$

which, together with (4.16) yields (4.28).

Proof of Lemma 4.5. If $|x - \xi_p| \leq K_o$, then by using a second order Taylor expansion, we may express

$$E_n(x) = |\Phi(n^{1/2}(F(x) - p)\sigma_p^{-1}) - \Phi(n^{1/2}(x - \xi_p)F'(\xi_p)\sigma_p^{-1})|$$

as

$$(4.30) \qquad E_n(x) = |\Phi(n^{1/2}(x-\xi_p)F'(\xi_p)\sigma_p^{-1} + 1/2\, n^{1/2}(x-\xi_p)^2 F''(\xi_p+\theta(x-\xi_p))\sigma_p^{-1})$$

$$- \Phi(n^{1/2}(x-\xi_p)F'(\xi_p)\sigma_p^{-1})| \ , \ 0 < \theta < 1 \ ,$$

and, by using Lemma 4.4, we obtain

$$(4.31) \qquad \sup_{|x-\xi_p|\leq K_o} E_n(x) \leq 4M\sigma_p(2\pi)^{-1/2} e^{-1}[F'(\xi_p)]^{-2} n^{-1/2} .$$

Assume on the other hand that $|x - \xi_p| \geq K_o$. If $x \leq \xi_p - K_o$, then, according to (4.19) we have

$$(4.32) \qquad \Phi(n^{1/2}(x-\xi_p)F'(\xi_p)\sigma_p^{-1}) \leq \sigma_p^2(2\pi e)^{-1/2} K_o^{-2}[F'(\xi_p)]^{-2} n^{-1} .$$

Now, since $x \leq \xi_p - K_o$, by using (4.31) (with $x = \xi_p - K_o$) together with (4.32) we deduce that

(4.33)

$$\Phi(n^{1/2}(F(x)-p)\sigma_p^{-1}) \leq 4M\sigma_p(2\pi)^{-1/2} e^{-1}[F'(\xi_p)]^{-2} n^{-1/2} + \sigma_p^2(2\pi e)^{-1/2} K_o^{-2}[F'(\xi_p)]^{-2} n^{-1}$$

Thus, from (4.32) and (4.33), we obtain for $x \leq \xi_p - K_o$, that

(4.34) $$E_n(x) \leq C_6 n^{-1/2} + C_7 n^{-1}$$

where C_6 and C_7 are defined in Lemma 4.5.

Finally, since a similar statement holds for $x \geq \xi_p + K_o$ we deduce that

(4.35) $$\sup_{|x-\xi_p| \geq K_o} E_n(x) \leq C_6 n^{-1/2} + C_7 n^{-1} .$$

Lemma 4.5 now follows immediately from (4.31) and (4.35).

Remark 4.4. Apropos the regularity conditions on F in Theorem 4.1, the requirements concerning F" may be dropped. In many situations, F is not sufficiently smooth at ξ_p and expansion (4.30) is inappropriate. Yet an estimate like (4.29) may still be valid under modified assumptions. Specifically, assume that F is differentiable at ξ_p, $(F(\xi_p) = p)$ with $F'(\xi_p) > 0$, and

(4.36) $$|F(\xi_p + h) - p - hF'(\xi_p)| = O(h^2) \quad \text{as} \quad h \to 0 .$$

Then, with E_n defined by (4.29), we may show that

(4.37) $$E_n \leq C_8 n^{-1/2} + C_9 n^{-1}$$

for some positive constants C_8 and C_9 to be specified below. To establish (4.37), we write

$$\alpha(x) = F(x) - p - (x - \xi_p)F'(\xi_p) ,$$

and use (4.36) to infer that $\lim_{x \to \xi_p} \alpha(x) (x-\xi_p)^{-1} = 0$.

Now pick $\delta > 0$ such that if $|x - \xi_p| \leqslant \delta$,

$$|\alpha(x)(x - \xi_p)^{-1}[F'(\xi_p)]^{-1}| \leqslant 1/2$$

and let $K_1 > 0$ be such that if $|x - \xi_p| \leqslant \delta$, then

$$|\alpha(x)| \leqslant K_1(x - \xi_p)^2 .$$

Then, by putting together the estimates found by the method of Lemma 4.5 we note that (4.37) holds with

$$C_8 = 8K_1\sigma_p(2\pi)^{-1/2}e^{-1}[F'(\xi_p)]^{-2}$$

and

$$C_9 = \sigma_p^2 (2\pi e)^{-1}\delta^{-2}[F'(\xi_p)]^{-2} .$$

The details are omitted. We may also note that if $F''(\xi_p)$ exists, then by using Young's form of Taylor's theorem (cf. Hardy (1952)) we have

$$F(\xi_p + h) = p + hF'(\xi_p) + \frac{h^2}{2} F''(\xi_p) + o(h^2) \quad \text{as} \quad h \to 0 ,$$

and consequently (4.36) obtains. In any case, (4.36) implies (4.37) which together with Lemma 4.3 guarantees that

$$\Delta_n \leqslant O(n^{-1/2}) + O(\varepsilon_n) + \delta_n , \text{ as } n \to \infty .$$

5. Limit Law and Berry-Esséen Rates for X_{n,ν_n} in Nonregular Cases.

In studying the asymptotic law of X_{n,ν_n} , of considerable interest are those distributions for which $F'(\xi_p) = 0$ and are not covered by Theorem 3.1.

Our next results are tailored to just these cases. In theorem 5.1 we shall assume that $F(\xi_p) = p$ and that

(5.1) $$\lim_{h\to 0} [F(\xi_p+h) - F(\xi_p)]h^{-\rho} = M > 0 \text{ , for some odd integer } \rho > 1 \text{ .}$$

Condition (5.1) is considerably less restrictive than its formulation makes it appear. The generality it confers is discussed by Chanda (1975) who considered nonrandom central order statistics X_{n,k_n} and showed that under (5.1), if $k_n/n = p + o(n^{-1/2})$ as $n \to \infty$, then

(5.2) $$Mn^{1/2}(X_{n,k_n} - \xi_p)^{\rho} \xrightarrow{\mathcal{D}} N(0,\sigma_p^2) \text{ .}$$

In this section we generalize (5.2) for X_{n,ν_n} and then study the accuracy of the corresponding normal approximation. To the best of our knowledge the Berry-Esséen type problem has been investigated <u>even</u> for the non-random case.

THEOREM 5.1. (Central Limit Theorem) If

(5.3) $$n^{1/2}(\frac{\nu_n}{n} - p) \xrightarrow{P} 0 \text{ ,}$$

and (5.1) is satisfied, then

(5.4) $$Mn^{1/2}(X_{n,\nu_n} - \xi_p)^{\rho} \xrightarrow{\mathcal{D}} N(0,\sigma_p^2) \text{ .}$$

<u>Proof</u>: Define

(5.5) $$\Gamma_n(x) = \Phi(n^{1/2}(F(x) - p)\sigma_p^{-1}) - \Phi(Mn^{1/2}(x - \xi_p)^{\rho}\sigma_p^{-1})$$

and

(5.6) $$\alpha(x) = (F(x) - p)(x - \xi_p)^{-\rho} \text{ .}$$

Then, since $\alpha(x) \to M > 0$ as $x \to \xi_p$, given $0 < \varepsilon < 1/2$, there exists $\eta > 0$, such that if $|x - \xi_p| \leqslant \eta$,

$$|\alpha(x)M^{-1} - 1| < \varepsilon . \tag{5.7}$$

First we estimate $\sup_{x \in \mathbb{R}} |\Gamma_n(x)|$, and to this end, we distinguish the following cases.

<u>Case (i)</u>: $|x - \xi_p| \leqslant \eta$. Using the mean value theorem, we obtain

$$\Gamma_n(x) = (2\pi)^{-1/2} \sigma_p^{-1} n^{1/2} (x - \xi_p)^{\rho}(\alpha(x)-M)\exp\{-1/2\, M^2\sigma_p^{-2} n(x-\xi_p)^{2\rho}[1+\theta(\alpha(x)M^{-1}-1]^2\}$$

for some $0 < \theta < 1$, and proceeding as in the derivation of (4.31), we obtain with the help of the inequality $t \exp(-t^2) \leqslant (2e)^{-1/2}$, $t > 0$ that, for $|x - \xi_p| \leqslant \eta$,

$$|\Gamma_n(x)| \leqslant 2^{1/2} (\pi e)^{-1/2} \varepsilon \tag{5.8}$$

<u>Case (ii)</u>: $|x - \xi_p| \geqslant \eta$. Assume that $x \leqslant \xi_p - \eta$. Then, by using (4.19), we deduce

(5.9)
$$\Phi(Mn^{1/2} (x-\xi_p)^{\rho}\sigma_p^{-1}) \leqslant \sigma_p^2(2\pi e)^{-1/2} M^{-2}(x-\xi_p)^{-2\rho} n^{-1} \leqslant \sigma_p^2(2\pi e)^{-1/2} M^{-2}\eta^{-2\rho} n^{-1} .$$

On the other hand, defining $x_1 = \xi_p - \eta$, we have

$$\Phi(n^{1/2} (F(x)-p)\sigma_p^{-1}) \leqslant \Phi(n^{1/2} (F(x_1) - p)\sigma_p^{-1})$$

$$\leqslant |\Gamma_n(x_1)| + \Phi(Mn^{1/2} (x_1 - \xi_p)^{\rho}\, \sigma_p^{-1})$$

which together with (5.8) (for $x = x_1$) and (5.9) implies

(5.10) $$|\Gamma_n(x)| \leq 2^{1/2}(\pi e)^{-1/2}\varepsilon + \sigma_p^2(2\pi e)^{-1/2} M^{-2}\eta^{-2\rho} n^{-1} .$$

Thus, putting together the estimates obtained so far, we have

(5.11) $$\sup_{x\in \mathbb{R}} |\Gamma_n(x)| \leq 2^{1/2}(\pi e)^{-1/2}\varepsilon + \sigma_p^2 (2\pi e)^{-1/2} M^{-2}\eta^{-2\rho} n^{-1}$$

Now, condition (5.3) implies that

$$P\{n^{1/2} | \frac{\nu_n}{n} - p | \geq \varepsilon\} \leq \varepsilon , \forall n \geq n_o(\varepsilon).$$

Therefore, with D_n and C_i ($i = 1,2,3$) as defined in Lemma 4.3, the technique leading to (4.7) may be used to prove that for $n \geq n_o(\varepsilon)$,

(5.12) $$D_n \leq C_4 n^{-1/2} + C_5 n^{-1} + (C_3 + 1)\varepsilon$$

Now, combining (5.11) and (5.12) we obtain (via the inversion $y = Mn^{1/2}(x - \xi_p)^\rho \sigma_p^{-1}$ which is in order since ρ is an odd integer ≥ 1) that

(5.13) $$\sup_{y\in \mathbb{R}} |P\{Mn^{1/2}(X_{n,\nu_n} - \xi_p)^\rho \leq y\sigma_p\} - \Phi(y)| \leq C_4 n^{-1/2} + C_{10} n^{-1} + C_{11}\varepsilon$$

where

$$C_{10} = C_5 + \sigma_p^2 (2\pi e)^{-1/2} M^{-2}\eta^{-2\rho} \text{ and } C_{11} = C_3 + 1 + 2^{1/2}(\pi e)^{-1/2} .$$

(5.13) guarantees (5.4) and the theorem is proved.

Remark 5.1. Note that if (5.1) holds with $\rho = 1$, then $M = F'(\xi_p)$ and in this case, by the preceding proof, one gets an alternative proof of (3.1) when $c = 0$.

To study the speed of convergence apropos to the preceding theorem we shall need somewhat stronger assumptions than those imposed in Theorem 5.1. Precisely, we assume (4.1) and

(5.14) $$F(\xi_\rho + h) - F(\xi_p) = Mh^\rho + O(h^{2\rho}) \quad \text{as} \quad h \to 0$$

for some $M > 0$ and odd integer $\rho > 1$.

Then, we have

THEOREM 5.2. (Rates of Convergence in the CLT) Under (4.1) and (5.14)

(5.15)
$$\sup_x |P\{Mn^{1/2}(X_{n,\nu_n} - \xi_p)^\rho \leq x\sigma_p\} - \Phi(x)| \leq C_{12}n^{-1/2} + C_{13}n^{-1} + C_3\varepsilon_n + \delta_n$$

where

$$C_{12} = C_4 + 8\sigma_p K_o(2\pi)^{-1/2}e^{-1}M^{-2} \text{ and } C_{13} = C_5 + \sigma_p^2(2\pi e)^{-1/2}\eta^{-2\rho}M^{-2}$$

Proof: (outline) Let

$$\beta(x) = F(x) - F(\xi_p) - M(x - \xi_p)^\rho.$$

In view of (5.14), select $\eta_o > 0$ and $K_2 > 0$ such that if $|x - \xi_p| \leq \eta_o$,

(5.16) $$|\beta(x)| \leq K_2(x - \xi_p)^{2\rho}.$$

Then, choose $0 < \eta < \eta_o$ sufficiently small, so that

(5.17) $$\eta^\rho \leq M(2K_2)^{-1}.$$

To estimate $\sup_x |\Gamma_n(x)|$ (with $\Gamma_n(x)$ defined by (5.5)), we proceed as in the proof of the preceding theorem and break $\mathbb{R}$ into two parts for which we have different arguments.

For $|x - \xi_p| \leq \eta$, by the mean value theorem, we get

$$
\begin{aligned}
\Gamma_n(x) &= \Phi(Mn^{1/2}(x-\xi_p)^\rho \sigma_p^{-1} + n^{1/2}\beta(x)\sigma_p^{-1}) - \Phi(Mn^{1/2}(x - \xi_p)^\rho \sigma_p^{-1}) \\
&= n^{1/2}\beta(x)\,(2\pi)^{-1/2}\sigma_p^{-1} \exp\{-\tfrac{1}{2}M^2 n(x-\xi_p)^{2\rho}\sigma_p^{-2}[1+\theta\beta(x)M^{-1}(x-\xi_p)^{-\rho}]^2\}
\end{aligned} \tag{5.18}
$$

for some $0 < \theta < 1$.

Now, from (5.16) and (5.17), we obtain

$$
\begin{aligned}
|1 + \theta\beta(x)M^{-1}(x-\xi_p)^{-\rho}| &\geq 1 - M^{-1}|\beta(x)(x-\xi_p)^{-\rho}| \geq 1 - K_2 M^{-1}|x-\xi_p|^\rho \\
&\geq 1 - K_2 M^{-1}\eta^\rho \geq 1/2 ,
\end{aligned}
$$

which together with (5.18) entails (via (4.16))

$$
|\Gamma_n(x)| \leq 8K_2\sigma_p(2\pi)^{-1/2}e^{-1}M^{-2}n^{-1/2} . \tag{5.19}
$$

On the other hand, if $|x - \xi_p| \geq \eta$, proceeding as in the derivation of (5.10), we obtain

$$
|\Gamma_n(x)| \leq 8K_2\sigma_p(2\pi)^{-1/2}e^{-1}M^{-2}n^{-1/2} + \sigma_p^2\,(2\pi e)^{-1/2}\eta^{-2\rho}M^{-1}n^{-1}. \tag{5.20}
$$

The proof of the theorem follows by combining the estimates (5.19) and (5.20) with Lemma 4.3 (again via the inversion $y = Mn^{1/2}(x-\xi_p)^\rho\sigma_p^{-1})$.

Remark 5.2. Regarding the variation of F at ξ_p in nonregular cases, we may consider a condition more general than (5.1), namely

$$
\lim_{h\to 0} |F(\xi_p + h) - F(\xi_p)|\,|h|^{-\rho} = M > 0 \tag{5.1)*}
$$

for some $\rho > 0$ (not necessarily an integer).

Then, using arguments similar to those in the proof of Theorem 5.1, it can be shown (details omitted) that under (5.1*) and (5.3),

(5.21) $$Mn^{1/2}|X_{n,\nu_n} - \xi_p|^\rho \sigma_p^{-1} \xrightarrow{\mathcal{D}} |N(0,1)| \ .$$

This result again is a generalization of a result of Chanda (1975).

Finally, we note that under stronger assumptions, an error bound of the approximation (5.21) can also be obtained. To avoid repetition, we only state an analog of Theorem 5.2. Assume that (4.1) holds and

(5.14)* $$|F(\xi_p + h) - F(\xi_p)||h|^{-\rho} = M + O(|h|^\rho) \quad \text{as} \quad h \to 0$$

for some $M > 0$ and $\rho > 0$. Then

(5.22) $$\sup_x |P\{Mn^{1/2}|X_{n,\nu_n} - \xi_p|^\rho \leq x\sigma_p\} - \Phi^*(x)| = O(n^{-1/2}) + O(\varepsilon_n) + \delta_n$$

where Φ^* denotes the distribution function of a $|N(0,1)|$ random variable.

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