CONSTRAINED STOCHASTIC APPROXIMATION

VIA THE THEORY OF LARGE DEVIATIONS

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Let G be a compact convex subset of R^r, $\{\xi_i\}$ a bounded random sequence, and $\Pi_G(x)$ the projection of x onto G. We obtain asymptotic properties of the projected stochastic approximation algorithm $X^\varepsilon_{n+1} = \Pi_G(X^\varepsilon_n + \varepsilon b(X^\varepsilon_n,\xi_n))$ (or $X_{n+1} = \Pi_G(X_n + a_n b(X_n,\xi_n)), a_n \to 0$), via the theory of large deviations. The action functionals and their properties are obtained, as is the mean exit time from a neighborhood of a stable point of the 'mean' algorithm. The usual methods for obtaining the 'asymptotic normality' of suitably centered and normed sequences in stochastic approximation do not work here - and, in fact, this (asymptotic normality) property would not usually hold. The large deviations approach provides a useful alternative. Even for the unconstrained case,for many applications the large deviations estimates seem to be more useful than those based on the 'local linearization' which leads to the asymptotic normality.

1. Introduction.

Let $G = \{x:q_i(x) \leq 0, i \leq k\}$ be a compact convex subset of R^r which is the closure of its interior, where the $q_i(\cdot)$ are continuously differentiable.

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Let $\{\xi_n\}$ be a bounded random sequence and $b(.,\xi)$ uniformly (in ξ) Lipschitz continuous. Define $\Pi_G(x)$ to be a point in G nearest to x. The projected stochastic approximation (SA) algorithm (1.1) arises frequently in control and communications theory, and elsewhere.

$$X^\varepsilon_{n+1} = \Pi_G(X^\varepsilon_n + \varepsilon b(X^\varepsilon_n,\xi_n)) \tag{1.1}$$

There is literature (Ermoliov, 1976; Kushner and Clark, 1978; Kushner and Shwartz, 1984; and Pflug, 1986) on the asymptotic locations of $\{X^\varepsilon_n\}$ (as $\varepsilon \to 0$, $\varepsilon n \to \infty$). For a vector v, define the <u>projected</u> v <u>at</u> $x \in G$ <u>by</u> $\Pi_G(x,v) = \lim_{\Delta \downarrow 0}[\Pi_G(x+\Delta v) - x]/\Delta$. Then, under appropriate conditions, the limit points of (1.1) are those of the 'projected' ODE

$$\dot{x} = \Pi_G(x,\overline{b}(x)), \tag{1.2}$$

where $\overline{b}(x) = \lim_N \frac{1}{N}\sum_0^N Eb(x,\xi_n)$. We work with the case of constant ε; similar results are obtainable for the SA process when a_n replaces ε, $0 < a_n \to 0$, and $\Sigma\, a_n = \infty$. See Section 4.

Rate of convergence results for (1.1) are unavailable. In the unconstrained case, one usually works as follows: Let $X^\varepsilon_n \to \Theta$ in distribution as $\varepsilon \to 0$, $\varepsilon n \to \infty$ and set $U^\varepsilon_n = (X^\varepsilon_n - \Theta)/\sqrt{\varepsilon}$. Let $U^\varepsilon(.)$ be the piecewise constant continuous parameter interpolation with interval ε satisfying $U^\varepsilon(n\varepsilon) = U^\varepsilon_n$. Then, using a linearization about Θ, one tries to prove that $U^\varepsilon(t_\varepsilon+.)$ converges weakly to a Gauss-Markov process as $\varepsilon \to 0$ (and $t_\varepsilon \to \infty$ fast enough). This procedure cannot be used for (1.1) when $\Theta \in \partial G$, the boundary of G. Also, being a local result, it does not fully exploit the dynamics of (1.1) and one could not obtain from it (in any case) estimates of the statistics of the escape times from a neighborhood of Θ or similar quantities of interest in the applications. Here we use the theory of large deviations to obtain such statistics.

In Section 2 assumptions are stated. Section 3 contains a statement of the main results. A brief outline of the development is given in Section 4, and one part of the proof is in Section 5. Owing to lack of space, many details are omitted, and the reader is referred to Dupuis and Kushner (1985). An outline of the result for a projected version of the classical Robbins-Monro algorithm is in Section 4.

2. Assumptions.

Suppose that the limit $\bar{b}(.)$ defined above exists uniformly in $x \in G$. For notational convenience, let the T/Δ and Δ/ε below take integer values. Suppose that there is a real valued function $H(.,.)$, continuously differentiable in its first argument (it is obviously continuous) such that for each $\Delta > 0$,

$$\sum_{0}^{T/\Delta-1} \Delta H(\alpha_i, x_i) = \lim_{N} \frac{\Delta}{N} \log E \exp \sum_{i=0}^{T/\Delta-1} \alpha_i' \sum_{j=iN}^{iN+N-1} b(x_i, \xi_j). \tag{2.1}$$

The limit exists as required if $\{\xi_j\}$ is a finite state ergodic Markov chain (Freidlin, 1978), or if $\xi_j = \sum_k g_{j-k}\psi_k$, where $g_j = 0$ for $j < 0$, $\Sigma|g_j|j < \infty$, and $\{\psi_k\}$ are i.i.d. and bounded (Dupuis and Kushner, 1985a). Define the (lower semi-continuous (l.s.c.)) dual L and action functional S: $L(\beta,x) = \sup_\alpha[\beta'\alpha - H(\alpha,x)]$; $S(T,\phi) = \int_0^T L(\dot{\phi}(s),\phi(s))ds$ for $\phi(.)$ absolutely continuous in $C_x[0,T]$, the set of G-valued continuous functions on $[0,T]$ with initial value x, and set $S(T,\phi) = \infty$ for other $\phi(.) \in C_x[0,T]$. Define the bounded convex sets $U(x) = \{\beta: L(\beta,x) < \infty\}$, and suppose that $U(.)$ is continuous in the Hausdorff topology.

For the sake of simplicity in the discussion here, we make the non-degeneracy assumption: $\underline{\lim}_N \frac{1}{N} \operatorname{cov} \sum_1^N [b(x,\xi_j) - \bar{b}(x)]$ is uniformly positive definite in G (a more general case is in Dupuis and Kushner (1985)). We state some facts. For each $\delta > 0$, $L(.,.)$ is uniformly continuous on $\{\beta,x: \beta \in U(x), d(\beta,\partial U(x)) \geq \delta, x \in G\} \equiv U^\delta(x)$, where $d(.,.)$ always denotes distance in the sup norm sense. $H(.,x)$ is strictly convex in a neighborhood of $\alpha = 0$, uniformly in x. $L(\beta,x) = 0$ iff $\beta = \bar{b}(x)$. There is a neighborhood N of $\{0\}$ such that $N + \bar{b}(x) \subseteq U(x)$ for all $x \in G$, and such that $L(\bar{b}(x) + .,x)$ is strictly convex in N, uniformly for $x \in G$. $L(\bar{b}(x) + u,x) = o(u)$ uniformly for $x \in G$.

Define $B(x,\beta) = \{v:\Pi_G(x,v) = \Pi_G(x,\beta)\}$, the set of 'velocities' with the same projection at x as β has. Since $B(.,.)$ is u.s.c. (in the Hausdorff topology) and $L(.,.)$ is l.s.c., the function defined by

$$L_G(\beta,x) = \inf_{v\in B(x,\beta)} L(v,x) \tag{2.2}$$

is l.s.c. For $\phi(.)$ absolutely continuous in $C_x[0,T]$ define

$$S_G(T,\phi) = \int_0^T L_G(\dot{\phi}(s),\phi(s))ds, \tag{2.3}$$

and set $S_G(T,\phi) = \infty$ otherwise.

3. Main results.

Define $x^\varepsilon(.)$ to be the piecewise linear interpolation of $\{X_n^\varepsilon\}$ with interpolation interval $\varepsilon > 0$. Let $E_x(P_x)$ denote expectation (probability) conditioned on $X_0^\varepsilon = x$.

THEOREM 1. Under the assumptions of Section 2, $S_G(T,.)$ is an action functional for $x^\varepsilon(.)$, i.e., it is l.s.c., the level sets $\Phi_x(s) = \{\phi \in C_x[0,T] : S_G(T,\phi) \leq s\}$ are compact, and for $A \subset C_x[0,T]$, with interior A^0 and closure $\bar{A}$,

$$\begin{aligned} -\inf_{\phi\in A^0} S_G(T,\phi) &\leq \underline{\lim}_{\varepsilon} \varepsilon \log P_x\{x^\varepsilon(.) \in A\} \\ &\leq \overline{\lim}_{\varepsilon} \varepsilon \log P_x\{x^\varepsilon(.) \in A\} \\ &\leq - \inf_{\phi\in\bar{A}} S_G(T,\phi). \end{aligned}$$

Remarks. In a rough sense, Theorem 1 says that for small δ and ε

$$P_x\{d(x^\varepsilon(.),\phi(.)) < \delta\}$$

is approximately given by

$$\exp - \varepsilon^{-1} S_G(T,\phi),$$

and that for a set of paths A, the overwhelming contribution to the probability of $x^\varepsilon(.)$ being in A comes from small neighborhoods of the paths minimizing $S_G(T,.)$ over A. It should also be noted that owing to the Lipschitz conditions it can be shown that the limit (3.1) holds uniformly for $x \in G$.

Let Θ be an asymptotically stable point of (1.2). Consider a neighborhood D of Θ with smooth boundary and whose closure is in the domain of attraction of $\overline{D}$ (all sets and neighborhoods are relative to G). Under the two following additional conditions, an estimate of the escape time from D can be obtained. We redefine $\overline{b}(.)$ and $H(.,.)$: Let M denote a stopping time for $\{X_n^\varepsilon\}$, let B_M denote the associated stopped σ-algebra, and suppose that the limits in (3.2), (3.3) below exist uniformly in M and x and that $H(.,x)$ is continuously differentiable. This will be the case for the two classes of processes listed below (2.1).

$$\overline{b}(x) = \lim_n \frac{1}{n} E_{B_M} \sum_M^{n+M-1} b(x,\xi_j) \tag{3.2}$$

$$H(\alpha,x) = \lim_n \frac{1}{n} \log E_{B_M} \exp \alpha' \sum_M^{n+M-1} b(x,\xi_j) \tag{3.3}$$

Define

$$S_D(\Theta) = \inf\{S_G(T,\phi)\colon \phi(.) \in C_\Theta[0,T],\ \phi(T) \in \partial D,\ T < \infty\}. \tag{3.4}$$

Let D_δ denote a δ-neighborhood of D with $D_0 = D$. Then clearly, $S_{D_\delta}(\Theta)$ decreases as $\delta \downarrow 0$. We assume that $S_{D_\delta}(\Theta) \downarrow S_D(\Theta)$ as $\delta \downarrow 0$. If this condition doesn't hold for D it will always hold for some small perturbation of D. If $D \subset G^0$, then the condition is implied by the non-degeneracy assumption.

THEOREM 2. Under the assumptions of Theorem 1 and the two assumptions above,

$$\lim_{\varepsilon} \varepsilon \log E_x \tau_D^{\varepsilon} = S_D(\Theta),$$

where $\tau_D^{\varepsilon} = \inf(t : x^{\varepsilon}(t) \notin D\}$.

In 'typical' cases, the exit of $x^{\varepsilon}(.)$ from D is along the boundary ∂G. Refer to Figure 1 for one such case. For small ε, the process is initially driven by the 'projected' dynamics into a small neighborhood of Θ, with overwhelming probability. From there, a rare burst of the 'correct' noises may drive the process from D, and in this case it is most likely that the process will follow close to a path for which the infimum in (3.4) is achieved, or at least nearly achieved. Owing to the (possible) discontinuity of $L_G(\beta,x)$ on ∂G, in a situation such as that depicted in Figure 1, the optimal (or most likely) exit paths will lie on ∂G. The infima in (2.2) together with that in (3.4) essentially gives us the 'cheapest' way out of D.

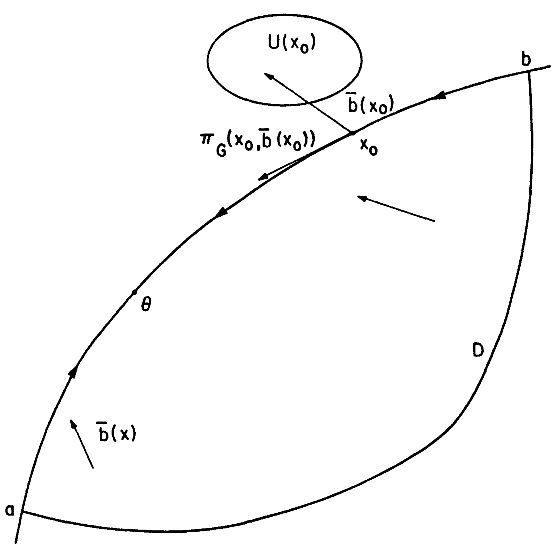

Figure 1. Example of Escape and Flow Lines.
P(Escape is along boundary) $\xrightarrow{\varepsilon}$ 1.

Extensions. As for the cases in Freidlin (1978) and Freidlin and Ventsell (1984), one can get information on the most likely locations of the exit of $x^{\varepsilon}(.)$ from D. Let there be a finite number of points $y_1,\dots,y_q \in \partial D$ such that

$$\inf_{T>0}\inf_{\phi\in A_i} S_G(T,\phi) = \inf_{T>0}\inf_{\phi\in A} S_G(T,\phi),$$

where $A_i = \{\phi(.) \in C_\Theta[0,T] : \phi(T) = y_i\}$, and $A = \{\phi(.) \in C_\Theta[0,T] : \phi(T) \in \partial D\}$. Then for $x \in D$,

$$\lim_{\varepsilon\to 0} P_x\{d(x^{\varepsilon}(\tau_D^{\varepsilon}), \cup_1^q y_i) < \delta\} = 1,$$

for any $\delta > 0$.

If (1.2) has many invariant sets in G, one can develop an analog of the results in Freidlin and Ventsell (1984) concerning the mean time to move from a neighborhood of one such set, or group of sets, to another.

The Robbins-Monro Process. There are also similar results for the stochastic approximation case where $0 < a_n \to 0$ replaces ε and $\sum_n a_n = \infty$ (Dupuis and Kushner, 1985). See Dupuis and Kushner (1985a) and Korostelev (1984) for a general discussion of such stochastic approximations via large deviations methods. The results for the unconstrained Robbins-Monro case in Dupuis and Kushner (1985a) can readily be extended to the constrained case via the methods discussed here. Define

$$X_{j+1} = \Pi_G(X_j + a_j b(X_j,\xi_j)),$$

where we assume the same conditions on $\{\xi_j\}$ and $b(.,.)$ as in Section 2. Define $t_n = \sum_0^{n-1} a_i$, $m(t) = \max\{n: t_n \leq t\}$, and the shifted process

$$X_{j+1}^n = \Pi_G(X_j^n + a_j b(X_j^n,\xi_j)),\ j \geq n,\ X_n^n = x,$$

$$x^n(t) = \frac{X_{j+1}^n(t-t_j+t_n)-X_j^n(t-t_{j+1}+t_n)}{t_{j+1}-t_j} \quad \text{on} \quad [t_j-t_n,t_{j+1}-t_n)$$

$$\tau_D^n = \inf\{t:x^n(t) \notin D\}.$$

Hence $x^n(.)$ is the linear interpolation of the sequence X_j^n with the decreasing interpolation intervals a_j. As $n \to \infty$, $x^n(.)$ represents the tail of the original sequence X_j, but started at x on the n-th step.

For $a_n = 1/n$, use the action functional

$$S_G(T,\phi) = \int_0^T e^s L_G(\phi_s,\dot{\phi}_s)ds,$$

and for $a_n = 1/n^T$, $T < 1$, use $S_G(T,\phi) = \int_0^T L_G(\phi_s,\dot{\phi}_s)ds$. Then for $A \subset C_x[0,T]$,

$$-\inf_{\phi \in A^0} S_G(T,\phi) \leq \underline{\lim}_n\, a_n \log P_x\{x^n(.)\in A\}$$

$$\leq \overline{\lim}_n\, a_n \log P_x\{x^n(.) \in A\} \leq -\inf_{\phi\in\bar{A}} S_G(T,\phi).$$

Define Θ as above, and for $x \in D$ let $A = \{\phi(.):\phi(0) = x,\phi(t) \notin D, \text{ some } t \leq T\}$. Then, under the 'continuity' condition on ∂D just above Theorem 2,

$$\lim_n a_n \log P_\Theta\{\tau_D^n \leq T\} = -\inf_{\phi\in A} S_G(T,\phi),$$

where τ_D^n is the escape time of $x^n(.)$ from D.

The values of the limits above provide useful information on the dependence of the performance of the algorithm on $b(.,.)$, on the sequence $\{a_n\}$, and on the statistics of $\{\xi_n\}$, and some of this information is obtainable without even solving the variational problem. It is also possible to use these estimates to prove w.p.1. convergence results.

The e^s appears due to the time varying scaling. We have $\lim_n a_{m(t_n+t)}/a_n = h_1(t)$, where $h_1(t) = e^{-t}$ for $a_n = 1/n$ and $h_1(t) = 1$

otherwise. In order to obtain the correct H-functional, a natural replacement for the right-hand side of (2.1) is

$$\lim_n a_n \log E \exp \sum_{i=0}^{T/\Delta-1} \alpha_i' \sum_{j=m(t_n+i\Delta)}^{m(t_n+i\Delta+\Delta)-1} a_j b(x_i,\xi_j)/a_n,$$

which equals

$$\int_0^T h_1^{-1}(t)H(h_1(t)\alpha(t),x(t))dt,$$

where $\alpha(.)$ and $x(.)$ are the piecewise constant interpolations (interval Δ) of $\{\alpha_i\}$ and $\{x_i\}$, respectively. Then our dual is $L(\beta,x,t) = h_1^{-1}(t)L(\beta,x)$.

4. Outline of the Method of Proof for Theorem 1.

It is difficult to work with $x^\varepsilon(.)$ directly, owing to the projection, so we work with a sequence of approximations. We start with a process (4.1) below, for which the large deviations result is well known. For this case, there is neither projection nor 'feedback'. We then define a sequence of more complicated processes, which get closer to (1.1), and for each the large deviations result can be obtained from the one preceeding it. Let $\psi(.)$ and $\phi(.)$ denote arbitrary functions in $C_x[0,T]$. Define $I_N = \{j: N\Delta \leq j\varepsilon < N\Delta+\Delta\}$ and $\psi_j^\Delta = \psi(n\Delta)$ for $j \in I_n$. Define $T_n^{\varepsilon,\psi,\Delta}$ by

$$T_n^{\varepsilon,\psi,\Delta} = x + \sum_0^{n-1} b(\psi_j^\Delta,\xi_j), \tag{4.1}$$

define

$$S^{\psi,\Delta}(T,\phi) = \sum_0^{N-1} \Delta L\left(\frac{\phi(i\Delta+\Delta)-\phi(i\Delta)}{\Delta},\ \psi(i\Delta)\right), \tag{4.2}$$

and let $T^{\varepsilon,\psi,\Delta}(.)$ be the piecewise constant interpolation of $\{T_n^{\varepsilon,\psi,\Delta}, n\Delta < T\}$. It follows from Freidlin's method (Freidlin, 1978) that under (2.1) that (4.2) is an action functional for the random vector $\{T^{\varepsilon,\psi,\Delta}(i\Delta),\ 0 < i \leq N\}$, for each Δ.

(We are actually comparing $\{T^{\varepsilon,\psi,\Delta}(i\Delta),\ 0 < i \leq N\}$ with the vector $\{\phi(i\Delta),\ 0 < i \leq N\}$ when we say that $S^{\psi,\Delta}(T,\phi)$ is an action functional. We keep the simpler notation since we shall eventually consider the limit as $\Delta \to 0$.) Now define

$$S_G^{\psi,\Delta}(T,\phi) = \inf_{f\in F_\Delta(\phi)} S^{\psi,\Delta}(T,f),$$

where $F_\Delta(\phi) = \{f:\Pi_G(\phi(i\Delta) + f(i\Delta+\Delta) - f(i\Delta)) = \phi(i\Delta+\Delta),\ i \leq N-1\}$. We can write

$$S_G^{\psi,\Delta}(T,\phi) = \sum_0^{N-1} \Delta \inf_{f\in F_\Delta(\phi)} L\left(\frac{f(i\Delta+\Delta)-f(i\Delta)}{\Delta},\psi(i\Delta)\right). \tag{4.3}$$

By the contraction principle (Varadhan, 1984, p.5), (4.3) is an action functional for the process $\{X_i^{\varepsilon,\psi,\Delta}, i\Delta<T\}$ defined by a projected form of (4.1), but where the projection occurs only at each $i\Delta/\varepsilon$ step.

$$X_{i+1}^{\varepsilon,\psi,\Delta} = \Pi_G(X_i^{\varepsilon,\psi,\Delta} + \varepsilon \sum_{j\in I_i} b(\psi_j^\Delta,\xi_j))$$

We next work with the 'feedback' form of $\{X_i^{\varepsilon,\psi,\Delta}\}$. Define $X_{ik}^{\varepsilon,\Delta}$ by ($\Delta/\varepsilon=k, X^{\varepsilon,\Delta}(.)$=interpolation, interval=Δ)

$$X_{i\ k+k}^{\varepsilon,\Delta} = \Pi_G(X_{ik}^{\varepsilon,\Delta} + \varepsilon \sum_{j\in I_i} b(X_{ik}^{\varepsilon,\Delta},\xi_j)),$$

($X_0^{\varepsilon,\Delta} = X_0^{\varepsilon,\psi,\Delta} = X_0^\varepsilon = x$).

Let $\psi^\Delta(.)$ (respectively, $x^{\varepsilon,\psi,\Delta}(.)$) denote the piecewise constant interpolation of $\{\psi(i\Delta), i\Delta<T\}$, (respectively, $\{X_i^{\varepsilon,\psi,\Delta}, i\Delta<T\}$). Then it can be shown that for any $\delta > 0$ there are $\delta_i > 0$ (which go to zero as $\delta \to 0$) such that

$$d(X^{\varepsilon,\psi,\Delta}(.),\psi^\Delta(.)) < \delta_2 \Rightarrow d(X^{\varepsilon,\Delta}(.),\psi^\Delta(.)) < \delta \Rightarrow d(X^{\varepsilon,\psi,\Delta}(.),\psi^\Delta(.)) < \delta_1. \tag{4.4}$$

It follows from (4.4) that $S_G^\Delta(T,\phi) \equiv S_G^{\phi,\Delta}(T,\phi)$ is an action functional for

$\{X^{\varepsilon,\Delta}(i\Delta), i\Delta \leq T\}$.

We can also show that for each $\delta > 0$ there are $\delta_1 > 0$, $\delta_2 > 0$ (which go to zero as $\delta > 0$) such that for each $\phi(.) \in C_x[0,T]$ and for small enough $\Delta > 0$,

$$d(X^{\varepsilon,\Delta}(.),\phi^{\Delta}(.)) < \delta_2 \Rightarrow d(X^{\varepsilon}(.),\varepsilon^{\phi}(.)) < \delta \tag{4.5}$$

$$\Rightarrow d(X^{\varepsilon,\Delta}(.),\phi^{\Delta}(.)) < \delta_1.$$

Theorem 1 follows from (4.5), the assertion below (4.4) and Lemma 1 below (Dupuis and Kushner, 1985). The method of proof of Lemma 1 parallels that in Freidlin (1978) for an unconstrained-continuous parameter case, with the appropriate alterations made where the boundary and projection play a role. In the next section, we prove that $S_G(T,.)$ is l.s.c.

Given $A \subset C_x[0,T]$, define

$A^{\Delta} = \{(v_1,\ldots,v_N) = (\phi(\Delta),\ldots,\phi(N\Delta)) : \phi(.) \in A\}$.

LEMMA 1. Assume the conditions of Theorem 1. $S_G(T,.)$ is l.s.c. For each $A \subset C_x[0,T]$,

$$\underline{\lim}_{\Delta} \inf_{\phi \in A^{\Delta}} S_G^{\Delta}(T,\phi) \geq \inf_{\phi \in A} S_G(T,\phi).$$

For each $\phi(.)$ for which $S_G(T,\phi) < \infty$, there are piecewise constant (on intervals of length Δ) functions $\phi_{\Delta}(.), \psi_{\Delta}(.)$ converging uniformly to $\phi(.)$ on $[0,T]$ and such that

$$\overline{\lim}_{\Delta} S_G^{\psi_{\Delta},\Delta}(T,\phi_{\Delta}) \leq S_G(T,\phi).$$

5. The Lower Semi-continuity of $S_G(T,.)$.

Let $\phi_n(.) \to \phi(.)$ in $C_x[0,T]$. The infimizing v is attained in

$$\inf_{v \in B(\phi^n(s),\dot{\phi}^n(s))} L(v,\phi^n(s)) ,$$

and we can choose a measurable minimizer which we write as $\tilde{v}^n(s) = v^n(s) + \bar{b}(\phi^n(s))$. By extracting a convergent subsequence we can assume that there is an absolutely continuous (since the U(x) are bounded) V(.) such that

$$\int_0^t v^n(s)ds \to V(t) \equiv \int_0^T v(s)ds .$$

Let $\delta > 0$. Using the uniform continuity of L(.,.) on $\{(\beta,x) : \beta \in U^\delta(x),\ x \in G\}$ we have

$$\underline{\lim}_n S_G(T,\phi_n) = \underline{\lim}_n \int_0^T L(\bar{b}(\phi^n(s)) + v^n(s),\phi^n(s))ds$$

$$\geq \underline{\lim}_\delta\ \underline{\lim}_n \int_0^T L(\bar{b}(\phi^n(s)) + (1-\delta)v^n(s),\phi^n(s))ds$$

$$(4.6) \qquad = \underline{\lim}_\delta\ \underline{\lim}_n \int_0^T L(\bar{b}(\phi(s)) + (1-\delta)v^n(s),\phi(s))ds$$

$$= \underline{\lim}_\delta\ \underline{\lim}_\Delta \{\underline{\lim}_n \int_0^T L(\bar{b}(\phi(s)) + (1-\delta)v^n(s),\phi^\Delta(s))ds - \alpha_\delta^\Delta\}$$

$$\geq \underline{\lim}_\delta\ \underline{\lim}_\Delta \{\underline{\lim}_n \sum_0^{N-1} \Delta\, L(\frac{1}{\Delta} \int_{i\Delta}^{i\Delta+\Delta} [\bar{b}(\phi(s)) + (1-\delta)v^n(s)]ds,\phi(i\Delta)) - \alpha_\delta^\Delta\},$$

where $\alpha_\delta^\Delta \to 0$ as $\Delta \to 0$. The first inequality uses the fact that $L(\bar{b}(x)+v,x)$ is a convex function of v attaining its minimum at v = 0, and the last inequality follows from Jensen's inequality and the convexity of L(.,x).

By the l.s.c. of L(.,.) and Fatou's lemma, we can continue the string of inequalities in (4.6) as

$$(4.7) \qquad \geq \underline{\lim}_\delta \int_0^T L(\bar{b}(\phi(s) + (1-\delta)v(s),\phi(s))ds$$

$$\geq \int_0^T L(\bar{b}(\phi(s)) + v(s),\phi(s))ds .$$

If $\Pi_G(\phi(s),\bar{b}(\phi(s)) + v(s)) = \dot{\phi}(s)$ for almost all $s < T$ we are done, since in that case (from almost all s) $\bar{b}(\phi(s) + v(s)) \in B(\phi(s), \dot{\phi}(s))$ and

$$L(\bar{b}(\phi(s)) + v(s),\phi(s)) \geq \inf_{v \in B(\phi(s),\dot{\phi}(s))} L(v,\phi(s)) = L_G(\dot{\phi}(s),\phi(s)).$$

See Dupuis and Kushner (1985) for a proof of this 'projection' property.

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