SEQUENTIAL CONFIDENCE INTERVALS

WITH BETA PROTECTION IN ONE-PARAMETER FAMILIES*

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A sequential confidence interval (CI) for a real parameter γ of the form $[L,\infty)$ is proposed, based on a consistent and asymptotically normal sequence $T_1, T_2, \ldots$ of real valued statistics. This CI is required to satisfy the coverage probability $P_\gamma\{L \leq \gamma\} \geq 1 - \alpha$ for every γ, and to provide beta protection at $\phi(\gamma)$: $P_\gamma\{L \leq \phi(\gamma)\} \leq \beta$ for every γ, where α, β, and the function $\phi(\gamma) < \gamma$ are given. It is shown that this can be achieved (under certain regularity assumptions) with a stopping time of the form N = least integer $n \geq r + c^2\tau^2(T_n)$ and a terminal decision $L = \rho(T_N)$, in which the functions τ and ρ depend on ϕ and the asymptotic variance σ^2. Asymptotic values are derived for $P_\gamma\{L > \gamma\}$ and $P_\gamma\{L \leq \phi(\gamma)\}$ as γ varies over values for which $\tau(\gamma) \to \infty$.

1. Introduction.

Let T_1, $T_2, \ldots$ be a sequence of real valued random variables whose joint distribution P_γ depends on a parameter γ with values in an interval Γ. Suppose a one-sided confidence interval (CI) for γ is desired of the form $[L,\infty]$, in which $L = L(T_1,T_2,\ldots)$, that satisfies the two conditions

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$$\text{(1.1)} \qquad P_\gamma\{L \leq \gamma\} \geq 1 - \alpha, \quad \gamma \in \Gamma,$$

$$\text{(1.2)} \qquad P_\gamma\{L \leq \phi(\gamma)\} \leq \beta, \quad \gamma \in \Gamma,$$

in which α, β, and $\phi(\gamma) < \gamma$ are chosen by the experimenter. Here (1.1) is the usual coverage requirement, and (1.2) is a precision requirement that takes the place of the more customary prescribed width of a fixed width sequential CI. Motivation of this approach appears in Wijsman (1981). We shall describe (1.2) as beta protection at $\phi(\gamma)$. In Wijsman (1981, 1982) an example is worked out when γ is a normal mean; in Wijsman (1983) $\gamma = \mu/\sigma$ in a normal population; in Juhlin (1985) γ is the mean in a scale parameter exponential distribution. In the present paper a few general results will be proved for a family of models that includes the above mentioned examples as well as many others.

The sequence $\{T_n\}$ will be assumed consistent for γ and the function ϕ in (1.2) will be assumed to be such that there is no fixed sample size CI that satisfies both (1.1) and (1.2). A sequential procedure will be defined by first choosing a stopping time N on heuristic grounds, in the spirit of Chow and Robbins (1965). Whatever the choice of N, a terminal decision rule will be defined with help of a function ρ on Γ such that $\phi(\gamma) < \rho(\gamma) < \gamma$, and then putting $L(T_1, T_2, \ldots) = \rho(T_N)$. The resulting procedure will be denoted (N,ρ). The requirements (1.1) and (1.2) can then be written as

$$\text{(1.3)} \qquad P_\gamma\{\rho(T_N) \leq \gamma\} \geq 1 - \alpha, \quad \gamma \in \Gamma,$$

$$\text{(1.4)} \qquad P_\gamma\{\rho(T_N) \leq \phi(\gamma)\} \leq \beta, \quad \gamma \in \Gamma.$$

Since the search for a stopping time N is guided primarily by considering values of γ for which large average sample sizes are needed, it is to be expected that the distribution of T_n for large n is important. We shall assume $n^{1/2}(T_n - \gamma)$ to be asymptotically $N(0,\sigma^2(\gamma))$. Let $\delta(\gamma) = \gamma - \phi(\gamma)$ be the precision gap and temporarily put $\tau(\gamma) = \sigma(\gamma)/\delta(\gamma)$. Also, temporarily take

$\rho(\gamma) = \gamma - \zeta\delta(\gamma)$, with $0 < \zeta < 1$ still to be chosen. Then heuristics as in Wijsman (1983) suggest the following type of stopping time:

(1.5) $$N = \text{least integer } n \geq r + c^2\tau^2(T_n),$$

in which $c > 0$ and integer $r \geq 0$ still have to be chosen.

The temporary definitions of τ and ρ in the previous paragraph are not entirely satisfactory since the resulting procedure is not necessarily preserved under nonlinear transformations. Yet, for any $f: \Gamma \to \tilde{\Gamma}$ that is one-to-one and differentiable in both directions, with bounded derivatives, the problem in terms of $\tilde{\gamma} = f(\gamma)$ and the sequence $\tilde{T}_n = f(T_n)$ is the same as the original problem. It would be desirable if the procedure $(\tilde{N},\tilde{\rho})$ in the new problem would have the property $\tilde{N} = N$, $\tilde{\rho} = f(\rho)$ (<u>equivariant</u> interval estimator). This can indeed be achieved by the following definitions of τ and ρ (where $\tau^{-1}(\gamma)$ means $1/\tau(\gamma)$, and similarly $\sigma^{-1}(\gamma)$):

(1.6) $$\tau^{-1}(\gamma) = \int_{\phi(\gamma)}^{\gamma} \sigma^{-1}(x)dx, \ \gamma \in \Gamma,$$

(1.7) $$\int_{\rho(\gamma)}^{\gamma} \sigma^{-1}(x)dx = \zeta\tau^{-1}(\gamma), \quad \gamma \in \Gamma,$$

for any choice of $0 < \zeta < 1$. (Note that the asymptotic variance transforms as $\tilde{\sigma}^2(\tilde{\gamma}) = f'^2(\gamma)\sigma^2(\gamma)$; this causes $|\sigma^{-1}(x)dx|$ to be invariant and, as a result, $\tilde{\tau}(\tilde{\gamma}) = \tau(\gamma)$.)

2. <u>Assumptions</u>.

Throughout, Γ is an interval of the real line R. If Γ does not contain both endpoints (e.g., if $\Gamma = R$, or if $\Gamma = (0,\infty)$) let $\bar{\Gamma}$ be its compactification in the usual topology. For instance, if $\Gamma = R$, then $\bar{\Gamma}$ is R with $\pm\infty$ added. One of the important tools needed is Anscombe's (1952) theorem. This uses the notion of uniform continuity in probability (u.c.i.p.); see also Woodroofe (1982), Section 1.3. We shall require these concepts in a version that is uniform in $\gamma \in \Gamma$. The phrase "uniformly in $\gamma \in \Gamma$" will occur often and will be abbreviated "u.i.γ."

ASSUMPTION A.

(i) There is a fixed probability space $(\Omega,\mathbf{A},P)$ and a family $\{T_{n\gamma}: n = 1,2,\ldots, \gamma \in \Gamma\}$ of random variables defined on Ω, such that the distribution of $(T_1,T_2,\ldots)$ under P_γ is the same as that of $(T_{1\gamma},T_{2\gamma},\ldots)$ under P; furthermore, $P\{T_{n\gamma} \in \Gamma\} = 1$, $n = 1,2,\ldots$, $\gamma \in \Gamma$;

(ii) for every $\gamma \in \Gamma$, $T_{n\gamma} \overset{a.e.}{\to} \gamma$ as $n \to \infty$;

(iii) there is a function σ on Γ such that $\sigma^{-1}(\gamma)n^{1/2}(T_{n\gamma} - \gamma) \overset{d}{\to} N(0,1)$ as $n \to \infty$ u.i.γ.;

(iv) the sequence $\{\sigma^{-1}(\gamma)n^{1/2}(T_{n\gamma} - \gamma): n = 1,2,\ldots\}$ is u.c.i.p., u.i.γ.;

(v) the function σ is positive on the interior of Γ; both σ and ϕ (defined in Section 1) are continuously differentiable, with derivatives bounded in absolute value;

(vi) define τ on Γ by (1.6) and let γ^* stand for any endpoint of Γ at which $\tau(\gamma) \to \infty$ as $\gamma \to \gamma^*$; there is at least one such γ^*;

(vii) there is $b > 0$ such that $\tau(\gamma) \geq b$, $\gamma \in \Gamma$;

(viii) $T_{n\gamma} \overset{a.e.}{\to} \gamma^*$ as $\gamma \to \gamma^*$ uniformly in $n=1,2,\ldots$;

(ix) for every $\gamma_0 \in \bar{\Gamma}$, $\tau(T_{n\gamma})/\tau(\gamma) \overset{a.e.}{\to} 1$ as $n \to \infty$, $\gamma \to \gamma_0$.

Assumption A(i) is made for convenience and seems to be satisfied in all examples studied. Assumption A(vi) guarantees that no fixed sample size CI is able to achieve both (1.3) and (1.4). A(vii) simplifies statements and proofs of certain results and is not an essential restriction. A(ix) is automatically true for any $\gamma_0 \in \Gamma$, by A(ii) and the continuity of τ. Verification of A(ix) is therefore only needed for any γ_0 that is an endpoint of Γ but is not in Γ.

The stopping time of N of (1.5) will be considered depending on c (whereas r will stay fixed throughout). In terms of the $T_{n\gamma}$ (1.5) can be written as

(2.1) $$N = N_{c\gamma} = \text{least integer } n \geqslant r + c^2\tau^2(T_{n\gamma}).$$

The dependence of N on c and γ will usually be suppressed in the notation. It is clear that N is finite with probability one for each $c > 0$ and $\gamma \in \Gamma$. For by Assumption A(ii) and the continuity of τ (using the continuity of ϕ and (1.6)) the sequence on the right hand side of the inequality in (2.1) is bounded a.e. as $n \to \infty$.

3. The main theorems.

Let $\phi(\gamma) < \gamma$ be given; choose $0 < \zeta < 1$ and define $\tau(\gamma)$ and $\rho(\gamma)$ by (1.6) and (1.7). Furthermore, let c and r in (1.5) be chosen. Then let (N,ρ) be the procedure that takes N observations, defined by (1.5), and estimates γ by the CI $[\rho(T_N),\infty)$. For probability computations we may define N by (2.1). Then the error probabilities of $(N,\rho) = (N_{c\gamma},\rho)$ as functions of $c > 0$ and $\gamma \in \Gamma$ are

(3.1) $$\alpha(c,\gamma) \equiv P_\gamma\{\rho(T_N) > \gamma\} = P\{\rho(T_{N\gamma}) > \gamma\},$$

(3.2) $$\beta(c,\gamma) \equiv P_\gamma\{\rho(T_N) \leqslant \phi(\gamma)\} = P\{\rho(T_{N\gamma}) \leqslant \phi(\gamma)\}.$$

THEOREM 3.1. Under Assumption A, for any given $0 < \zeta < 1$, α, $\beta > 0$, and integer $r \geqslant 0$, there exists $c_1 > 0$ such that $c \geqslant c_1$ implies $\alpha(c,\gamma) \leqslant \alpha$, $\beta(c,\gamma) \leqslant \beta$ for every $\gamma \in \Gamma$.

THEOREM 3.2. Under Assumption A, for any given $0 < \zeta < 1$, $c > 0$, and integer $r \geqslant 0$, if $N = N_{c\gamma}$ then $\alpha(c,\gamma) \to 1 - \Phi(\zeta c)$ and $\beta(c,\gamma) \to 1 - \Phi((1 - \zeta)c)$ as $\gamma \to \gamma^*$, where γ^* is defined in A(vi) and Φ is the standard normal distribution function.

Theorem 3.1 shows that the proposed type of procedure (N_c,ρ) is capable of satisfying both (1.3) and (1.4) provided c is large enough. Theorem 3.2 deals with the values of the error probabilities for γ near γ where N tends

t^*_0 be large. The theorem shows that if we would want $\alpha(c,\gamma) \to \alpha$ and $\beta(c,\gamma) \to \beta$ as $\gamma \to \gamma^*$, then we should choose $\zeta c = z_\alpha$, $(1-\zeta)c = z_\beta$, where z_α is the upper α-point of $N(0,1)$. Taking $\zeta = z_\alpha/(z_\alpha + z_\beta)$ is reasonable, but the value $c = z_\alpha + z_\beta$ can be expected to be too small to guarantee $\alpha(c,\gamma) \leq \alpha$, $\beta(c,\gamma) \leq \beta$ for all γ.

The proofs of the theorems will be preceded by several lemmas.

LEMMA 3.1. $N \to \infty$ a.e. as $c \to \infty$ u.i.γ.

Proof. By (2.1) and A(vii), $N \geq c^2b^2$ if $T_{n\gamma} \in \Gamma$, $n=1,2,\ldots$.

LEMMA 3.2. If $c_1 > 0$, then $N \xrightarrow{a.e.} \infty$ as $\gamma \to \gamma^*$ uniformly in $c \geq c_1$.

Proof. Let n_0 be an arbitrary positive integer and $c \geq c_1$. Then

$$[N \leq n_0] \subset \bigcup_{n=1}^{n_0} [n \geq r + c_1^2\tau^2(T_{n\gamma})]. \tag{3.3}$$

By A(vi) and A(viii), $\tau(T_{n\gamma}) \xrightarrow{a.e.} \infty$ as $\gamma \to \gamma^*$, $n=1,2,\ldots$. Thus, except for a set of P-probability 0, the right-hand side of (3.3) converges to the empty set as $\gamma \to \gamma^*$.

LEMMA 3.3. $\tau(T_{N\gamma})/\tau(\gamma) \xrightarrow{a.e.} 1$ as $c \to \infty$ u.i.γ. or as $\gamma \to \gamma^*$ for any $c > 0$.

Proof. To prove the second assertion use Lemma 3.2 and A(ix), observing that $\gamma^* \in \bar{\Gamma}$. To prove the first assertion take any $\gamma_0 \in \bar{\Gamma}$, then by Lemma 3.1 and A(ix) we have $\tau(T_{N\gamma})/\tau(\gamma) \xrightarrow{a.e.} 1$ as $c \to \infty$, $\gamma \to \gamma_0$. The compactness of $\bar{\Gamma}$ and a standard compactness argument finishes the proof.

LEMMA 3.4. Lemma 3.3 is valid with N replaced by N - 1.

Proof. The proof of Lemma 3.3 is unchanged when replacing N by N - 1:

as $N \to \infty$ so does $N - 1$.

LEMMA 3.5. $N/c^2\tau^2(\gamma) \xrightarrow{a.e.} 1$ as $c \to \infty$ u.i.γ. or as $\gamma \to \gamma^*$ for any $c > 0$.

Proof. From (2.1) we have the double inequality

$$r + c^2\tau^2(T_{N\gamma}) \leq N < 1 + r + c^2\tau^2(T_{N-1,\gamma}). \tag{3.4}$$

Divide both sides of (3.4) by $c^2\tau^2(\gamma)$. Then use A(vii) and Lemmas 3.3 and 3.4.

LEMMA 3.6. $\sigma^{-1}(\gamma)N^{1/2}(T_{N\gamma} - \gamma) \xrightarrow{d} N(0,1)$ as $c \to \infty$ u.i.γ. or as $\gamma \to \gamma^*$ for any $c > 0$.

Proof. Use A(iii), A(iv), Lemma 3.5, and a uniform (in γ) version of Anscombe's (1952) theorem.

LEMMA 3.7. $\sigma^{-1}(\gamma)(T_{N\gamma} - \gamma) \xrightarrow{P} 0$ as $c \to \infty$ u.i.γ. or as $\gamma \to \gamma^*$ for any $c > 0$.

Proof. This follows immediately from Lemma 3.6 and Lemmas 3.1 and 3.2.

LEMMA 3.8. $\sigma(T_{N\gamma})/\sigma(\gamma) \xrightarrow{P} 1$ as $c \to \infty$ u.i.γ. or as $\gamma \to \gamma^*$ for any $c > 0$.

Proof. Write $\sigma(T_{N\gamma})/\sigma(\gamma) - 1 = \sigma'(t)\sigma^{-1}(\gamma)(T_{N\gamma}-\gamma)$ with t between $T_{N\gamma}$ and γ. Since $|\sigma'(t)|$ is bounded by A(v), the result follows from Lemma 3.7.

LEMMA 3.9. There is a constant $0 < B < \infty$ such that

$$\sigma^{-1}(\gamma)\tau(\gamma)[\gamma - \rho(\gamma)] \geq \zeta B^{-1}, \quad \gamma \in \Gamma, \tag{3.5}$$

$$\sigma^{-1}(\gamma)\tau(\gamma)[\rho(\gamma) - \phi(\gamma)] \geq (1 - \zeta)B^{-1}, \quad \gamma \in \Gamma. \tag{3.6}$$

Proof. We shall first show that there is $0 < B < \infty$ such that

$$(3.7)\quad \sigma(\gamma)(x_2 - x_1)^{-1} \int_{x_1}^{x_2} \sigma^{-1}(x)ds \leq B \quad \text{if } \phi(\gamma) \leq x_1 < x_2 \leq \gamma, \quad \gamma \in \Gamma.$$

By A(v) there is $0 < d < \infty$ such that $\sigma'(\gamma) \leq d$, $\gamma \in \Gamma$. Take $\phi(\gamma) \leq x_1 < x_2 \leq \gamma$ arbitrary, then

$$\log(\sigma(x_2)/\sigma(x_1)) = \int_{x_1}^{x_2} \sigma'(x)dx \leq d \int_{x_1}^{x_2} \sigma^{-1}(x)dx$$
$$\leq d\tau^{-1}(\gamma) \quad \text{(by (1.6))} \leq db^{-1} \text{ by A(vii)}.$$

Thus, $\sigma(x_2)/\sigma(x_1) \leq \exp(db^{-1}) = B$, say. In particular, take $x_2 = \gamma$ and x_1 any x for which $\phi(\gamma) \leq x \leq \gamma$, then $\sigma(\gamma)/\sigma(x) \leq B$ and (3.7) follows. Next (3.5) follows from (3.7) by taking $x_1 = \rho(\gamma)$, $x_2 = \gamma$, by using (1.7), and inverting the resulting inequality. Similarly, (3.6) follows from (3.7) by taking $x_1 = \phi(\gamma)$, $x_2 = \rho(\gamma)$, and using (1.6) and (1.7).

LEMMA 3.10. As $\gamma \to \gamma^*$,

$$(3.8)\qquad \sigma^{-1}(\gamma)\tau(\gamma)[\gamma - \phi(\gamma)] \to 1,$$

$$(3.9)\qquad \sigma^{-1}(\gamma)\tau(\gamma)[\gamma - \rho(\gamma)] \to \zeta.$$

Proof. We shall show first that as $\gamma \to \gamma^*$,

uniformly in x_1, x_2, if $\phi(\gamma) \leq x_1 < x_2 \leq \gamma$. By A(v) there exists $0 < d < \infty$ such that $|\sigma'(\gamma)| \leq d$, $\gamma \in \Gamma$. Take $\varepsilon > 0$ arbitrary. Since by A(vi) $\tau^{-1}(\gamma) \to 0$ as $\gamma \to \gamma^*$, there is a neighborhood V of γ^* such that for all $\gamma \in V$, $\tau^{-1}(\gamma) < \varepsilon d^{-1}$. Then for $\gamma \in V$ and $\phi(\gamma) \leq x_1 < x_2 \leq \gamma$,

$$|\log(\sigma(x_2)/\sigma(x_1))| = |\int_{x_1}^{x_2}\sigma^{-1}(x)\sigma'(x)dx| \leq d\tau^{-1}(\gamma) < \varepsilon.$$

Thus, $\sigma(\gamma)/\sigma(x)$ appearing in (3.10) is bounded between $e^{-\varepsilon}$ and e^{ε}, and the same must be true for the left-hand side of (3.10). Then (3.8) follows from (3.10) by taking in (3.10) $x_1 = \phi(\gamma)$, $x_2 = \gamma$, and inverting. Similarly, (3.9) follows by taking in (3.10) $x_1 = \rho(\gamma)$, $x_2 = \gamma$, and using (1.7).

Proof of Theorem 3.1. We shall keep $0 < \zeta < 1$ and integer $r \geq 0$ fixed and shall prove that $\alpha(c,\gamma)$ and $\beta(c,\gamma)$ converge to 0 uniformly in γ as $c \to \infty$. Write (3.1) in the form

$$\alpha(c,\gamma) = P\{T^*_{N\gamma} - A(N,\gamma) > 0\}, \tag{3.11}$$

in which

$$T^*_{N\gamma} = \sigma^{-1}(\gamma)N^{1/2}(T_{N\gamma} - \gamma) \tag{3.12}$$

and

$$A(N,\gamma) = \sigma^{-1}(\gamma)N^{1/2}[T_{N\gamma} - \rho(T_{N\gamma})]. \tag{3.13}$$

By (2.1), $A(N,\gamma) \geq c\sigma^{-1}(\gamma)\tau(T_{N\gamma})[T_{N\gamma} - \rho(T_{N\gamma})]$. Since by Lemma 3.8, $\sigma^{-1}(\gamma)\sigma(T_{N\gamma}) \xrightarrow{P} 1$ as $c \to \infty$ u.i.γ., and by (3.5)

$$\sigma^{-1}(T_{N\gamma})\tau(T_{N\gamma})[T_{N\gamma} - \rho(T_{N\gamma})] \geq \zeta B^{-1} > 0 \text{ a.e.}$$

(using $T_{N\gamma} \in \Gamma$ a.e. by A(i)) we have $A(N,\gamma) \xrightarrow{P} \infty$ as $c \to \infty$ u.i.γ. Furthermore, $T^*_{N\gamma}$ is bounded in probability as $c \to \infty$ u.i.γ by Lemma 3.6. It follows that $T^*_{N\gamma} - A(N,\gamma) \xrightarrow{P} -\infty$ as $c \to \infty$ u.i.γ so that $\alpha(c,\gamma) \longrightarrow 0$ as $c \longrightarrow \infty$ u.i.γ.

Similarly, we write (3.2) in the form

$$\beta(c,\gamma) = P\{\phi^*(T_{N\gamma}) + B(N,\gamma) < 0\}, \tag{3.14}$$

in which

$$\phi^*(T_{N\gamma}) = \sigma^{-1}(\gamma)N^{1/2}[\phi(T_{N\gamma}) - \phi(\gamma)] \tag{3.15}$$

and

$$B(N,\gamma) = \sigma^{-1}(\gamma)N^{1/2}[\rho(T_{N\gamma}) - \phi(T_{N\gamma})]. \tag{3.16}$$

From (3.15) we obtain $\phi^*(T_{N\gamma}) = \phi'(t)T^*_{N\gamma}$, using (3.12), with t between $T_{N\gamma}$ and γ. Since by A(v) $|\phi'(t)|$ is bounded above, and $T^*_{N\gamma}$ is bounded in probability, so is $\phi^*(T_{N\gamma})$, as $c \to \infty$ u.i.γ. The term $B(N,\gamma)$ of (3.16) is treated in the same way as $A(N,\gamma)$ of (3.13), using (3.6). It follows that
$\phi^*(T_{N\gamma}) + B(N,\gamma) \xrightarrow{P} \infty$ so that $\beta(c,\gamma) \to 0$ as $c \to \infty$ u.i.γ.

<u>Proof of Theorem 3.2</u>. Write $\alpha(c,\gamma)$ in the form (3.11), then $T^*_{N\gamma} \xrightarrow{d} N(0,1)$ as $\gamma \to \gamma^*$ by Lemma 3.6. Furthermore, $A(N,\gamma) \xrightarrow{P} \zeta c$ as $\gamma \to \gamma^*$, using Lemmas 3.3, 3.5, and 3.8, and using (3.9) after observing that $T_{N\gamma} \xrightarrow{a.e.} \gamma^*$ by A(viii). Therefore, the right-hand side of (3.11) converges to $P\{N(0,1) > \zeta c\}$.

Write $\beta(c,\gamma)$ in the form

$$\beta(c,\gamma) = P\{T^*_{N\gamma} < A(N,\gamma) - C(N,\gamma)\}, \tag{3.17}$$

in which $C(N,\gamma) = \sigma^{-1}(\gamma)N^{1/2}(\gamma - \phi(\gamma))$. Here $A(N,\gamma) \xrightarrow{P} \zeta c$ as before, and $C(N,\gamma) \xrightarrow{a.e.} c$, using (3.8) and Lemma 3.5. Thus, the right-hand side of (3.17) converges to $P\{N(0,1) < -(1-\zeta)c\}$.

4. Applications.

Only a few of these will be indicated.

4.1 Translation parameter.

Let $X_1,X_2,\ldots$ be i.i.d. with known distribution except for the unknown mean γ. Suppose the X_i have finite variance, which we may assume to be unity. A possible choice of T_n is $\bar{X}_n = n^{-1}\Sigma_1^n X_i$. (This is the appropriate choice in the normal translation parameter problem since then $\{T_n\}$ is a sufficient sequence.) We may set $T_{n\gamma} = \bar{Z}_n + \gamma$ in which $Z_1,Z_2,\ldots$ are i.i.d. with known distribution that has mean 0, variance 1. According to (1.6) and (1.7), and using $\sigma(\gamma) \equiv 1$, we have $\tau^{-1}(\gamma) = \gamma - \phi(\gamma) = \delta(\gamma)$, and $\rho(\gamma) = \gamma - \zeta\delta(\gamma)$. Here $\Gamma = R$, and if $\delta(\gamma) \to 0$ as $\gamma \to \pm\infty$, then $\gamma^* = \pm\infty$. Assumption A(v) requires δ to be differentiable, with bounded derivative, and A(ix) puts a further mild condition on δ; the latter is for instance satisfied if $\delta(x+y)/\delta(x) \to 1$ as $x \to \pm\infty$, $y \to 0$. The uniformity in γ of A(iii) and A(iv) is obvious since $T_{n\gamma} - \gamma = \bar{Z}_n$ does not involve γ. The estimation-oriented procedure in Wijsman (1982) is of the form (N,ρ) defined in Section 3 of the present paper.

4.2. Scale parameter.

Let $X_i = \gamma Z_i$, $\gamma \in (0,\infty)$, in which $Z_1,Z_2\ldots$ are i.i.d. with known distribution supported on $(0,\infty)$ and having mean 1 and variance σ_0^2. Suppose one chooses $T_n = \bar{X}_n$, so $T_{n\gamma} = \gamma\bar{Z}_n$. This would for instance be the appropriate choice if the Z_i have density e^{-z}, $z > 0$, as studied by Juhlin (1985). Here $\sigma(\gamma) = \gamma\sigma_0$ and the uniformity in A(iii) and (iv) is again obvious since γ drops out. The functions τ and ρ defined by (1.6) and (1.7) are $\tau(\gamma) = -\sigma_0[\log(1 - \gamma^{-1}\delta(\gamma))]^{-1}$ and $\rho(\gamma) = \gamma[1 - \gamma^{-1}\delta(\gamma)]^{\zeta}$. Here $\gamma^* = 0$ or ∞ if $\gamma^{-1}\delta(\gamma) \to 0$ as $\gamma \to 0$ or ∞. Assumption A(ix) will be satisfied for instance if $\delta(xy)/\delta(x) \to 1$ if $y \to 1$ and $x \to 0$ or ∞.

4.3 Translation-scale parameter.

Let $X_i = \mu + \sigma Z_i$, $Z_1,Z_2,\ldots$ being i.i.d. with known distribution.

If $\gamma = \mu/\sigma$ then one may base inference on the invariant sequence $T_n = \bar{X}_n/s_n$, where s_n^2 is the sample variance of $X_1,\ldots,X_n$. This is studied in Wijsman (1983) for standard normal Z's, in which case $\{T_n\}$ is invariantly sufficient. We may put $T_{n\gamma} = (\gamma + \bar{Z}_n)/s_n$ where now $s_n^2 = (n-1)^{-1} \sum_1^n (Z_i-\bar{Z}_n)^2$. Verification of Assumption A, especially A(iii) and (iv), requires more care and will be treated in a separate study. Another problem is provided by considering $\gamma = \sigma^2$, in which case we may take $T_{n\gamma} = \gamma s_n^2$.

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