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REGIONS WHOSE PROBABILITIES INCREASE WITH THE CORRELATION COEFFICIENT AND SLEPIAN'S THEOREM

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Let $\mathbf{X}$ have a multivariate normal distribution. Slepian (1962) proved that the upper and lower orthants $(\mathbf{x} \le \mathbf{c})$ and $(\mathbf{x} \ge \mathbf{c})$ have the property that their probabilities are nondecreasing in each ρ_{ij}. This easily implies, in the bivariate case, that if $A = Q_1 \cup Q_3 \cup B$, where Q_1 is an upper quadrant, Q_3 is a lower quadrant, B is a disjoint union of horizontal or vertical infinite strips and the interiors of Q_1, Q_3 and B are disjoint, then $P(A)$ is nondecreasing in ρ. This paper shows that, within a broad class of bivariate regions, sets A of the type described above are the only sets whose probabilities increase with the correlation coefficient when the means and the variances of X_1, X_2 take arbitrary values. Some results are also given for the cases where the means and the variances are restricted in some way.

1. Introduction. Let $\mathbf{X} = (X_1, \ldots, X_n)$ have the multivariate normal distribution with mean vector μ, variance vector σ^2 and correlation matrix (ρ_{ij}). Slepian (1962) proved that certain orthant probabilities are nondecreasing in each ρ_{ij} separately. This result and its generalizations have several applications; see, for example, Slepian (1962), Šidák (1968) and Joag-dev, Perlman and Pitt (1983). It is natural to ask whether there are sets other than orthants whose probabilities are nondecreasing in each ρ_{ij}. In this paper, we deal mainly wth the bivariate case and obtain a result which can be considered as a partial converse to Slepian's result.

Following the number of the quadrants in the plane, we denote by Q_1 an upper quadrant of the type $x_1 \geqslant a_1$, $x_2 \geqslant a_2$. A lower quadrant will be denoted by Q_3. The term infinite horizontal strip will mean a set defined by $-\infty < x_1 < \infty$, $a_2 \leqslant x_2 \leqslant b_2$. An infinite vertical strip is defined similarly. We note that the probability of an infinite horizontal or vertical strip is constant in ρ. Therefore, the following corollary of Slepian's result is immediate. For ease of reference, we state it as a theorem.

SLEPIAN'S THEOREM. *Let $A \subset \mathcal{R}^2$ have the form $A = Q_1 \cup Q_3 \cup B$, where B is a finite disjoint union or horizontal (or vertical) infinite strips and the interiors of Q_1, Q_3 and B are disjoint. Then $P(A)$ is nondecreasing in ρ.*

In section 2, we show that, within a broad class of bivariate regions, the sets A described in Slepian's theorem are the only sets whose probabilities are nondecreasing in ρ, when the means μ_1, μ_2 and the variances σ_1^2, σ_2^2 are allowed to take arbitrary values. Such a result can be considered to be a partial converse to Slepian's theorem. When the means and variances are restricted in some way, it is possible to obtain some additional regions whose probabilities increase with ρ. Some results in this direction are given in Section 3. In Section 4, we discuss an equivalent form of Slepian's result in terms of covariances and show that its generalization based on the concept of association fails.

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2. A Partial Converse to Slepian's Theorem. In this section we first describe a broad class $\mathscr{D}$ of sets in $\mathscr{R}^2$ and then show that the only sets in $\mathscr{D}$ whose probabilities are nondecreasing in ρ for all values of μ_1, μ_2, σ_1 and σ_2 are the sets A described in Slepian's theorem.

Let $\mathscr{D}$ denote the class of all sets D with the following properties.

(1) D is a subset of $\mathscr{R}^2$ and coincides with the closure of its interior.

(2) The boundary of D consists of a finite number of line segments.

These two properties easily imply the following useful property.

(3) If **a** is a boundary point of D which is not a vertex, then there is an $\varepsilon > 0$ such that the intersection of the disc $|\mathbf{x} - \mathbf{A}| < \varepsilon$ with D is a convex set with a nonempty interior.

It is clear that $\mathscr{D}$ is a fairly broad class which includes all closed quadrants and their finite disjoint unions. It is also easy to see that every set considered in Slepian's theorem is in $\mathscr{D}$. We also note that the boundary of a set in $\mathscr{D}$ may contain line segments which are neither horizontal nor vertical.

Theorem 1 below concerns the class $\mathscr{D}$. We suspect that the theorem holds for a wider class of sets such as the class of sets with Jordan boundaries. We also believe that the heart of our proof will carry over to the more general case.

Suppose again that (X_1, X_2) has a bivariate normal distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2 and ρ.

Definition 1. A set $A \subset \mathscr{R}^2$ is called an *S-region* if $P(A)$ is nondecreasing in ρ for all values of μ_1, μ_2, σ_1^2, σ_2^2.

We need a Lemma (see Figure 1).

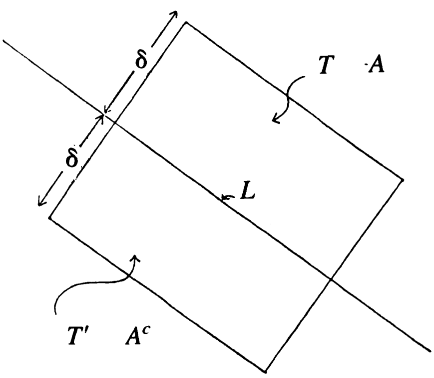


FIGURE 1. Illustration for the proof of Lemma 1.

LEMMA 1. *Let $A \subset \mathscr{R}^2$ be such that there is a line segment L in the boundary of A and a $\delta > 0$ such that* (a) *the open rectangle T with L as one side and height δ is contained in A;* (b) *the mirror image T'* of T about L is disjoint from A; and (c) *L is neither horizontal nor vertical. Then A is not an S-region.*

Proof. First assume that the slope of L is negative. By changing the origin and scales, if necessary, we may assume that

(i) L is contained in the line $x_1 + x_2 = 0$;

(ii) T is described by the conditions

$$0 < x_1 + x_2 < \delta \text{ and } |x_1 - x_2| < \eta.$$

where 2η is the length of L. The mirror image T' of T about L is then described by

$-\delta < x_1 + x_2 < 0$ and $|x_1 - x_2| < \eta$.

Write $U = X_1 + X_2$ and $V = X_1 - X_2$. Suppose that $\mu_1 = \mu_2 = \sigma_1^2 = \sigma_2^2 = \epsilon$, where $0 < 2\epsilon < \delta$. We use Z to denote a standard normal random variable.

Since $T \subset A$, we have $P(A) \geq P(T) = P[0 < U < \delta] \cdot P[|V| < \eta]$. But

$$P[0 < U < \delta] = P[-(\sqrt{2}\epsilon/\sqrt{1+\rho}) < Z < (\delta-2\epsilon)/(2\epsilon(1+\rho))^{1/2}]$$
$$\to 1 \quad \text{as } \pi \to -1.$$

and

$$P[|V| < \eta] = P[|Z| < \eta/(2\epsilon(1-\rho))^{1/2}] \to P[|Z| < \eta/2\sqrt{\epsilon}] \quad \text{as } \rho \to 1.$$

Therefore, $\lim_{\epsilon\to 0} \liminf_{\rho\to(-1)} P(A) = 1$. On the other hand T' is disjoint from A. Therefore

$$1-P(A) \geq P(T') = P[-\delta < U < 0] \cdot P[|V| < \eta].$$

Again

$$P[|V| < \eta] = P[|Z| < \eta/(2\epsilon(1-\rho))^{1/2}] \to 1 \quad \text{as } \rho \to 1.$$

and

$$P[-\delta < U < 0] = P[-(\delta+2\epsilon)/(2\epsilon(1+\rho))^{1/2} < Z < -(2\epsilon/(1+\rho))^{1/2}]$$
$$\to P[-(\delta+2\epsilon/2\sqrt{\epsilon} < Z < -\sqrt{\epsilon}] \quad \text{as } \rho \to 1.$$

Therefore,

$$\lim_{\varepsilon\to 0} \limsup_{\rho\to 1} P(A) \leqslant 1/2.$$

We thus see that, if ε is sufficiently close to zero, then

$$\liminf_{\rho\to(-1)} P(A) > \limsup_{\rho\to 1} P(A).$$

This shows that A is not an S-region. The case where the line segment L has positive slope can be handled similarly, the only change being that the mean vector is taken outside the rectangle T. The lemma is thus proved. □

We are now ready to prove a partial converse to Slepian's theorem. While the proof is somewhat long, it is elementary and is broken down into several simple steps.

THEOREM 1. *Let $D \in \mathcal{D}$ be an S-region. Then D is of the form $Q_1 \cup Q_3 \cup B$, where B is a finite disjoint union of horizontal (or vertical) infinite strips, the interiors of Q_1, Q_3 and B are disjoint and one or more of Q_1, Q_3, B may be empty.*

Proof. Recall that D satisfies the conditions (1), (2) and (3) stated at the beginning of this section.

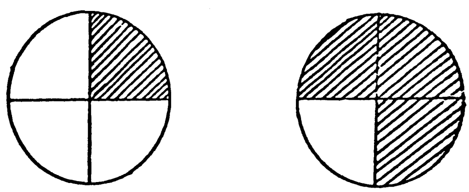
Step 1. If L is a line segment in the boundary of D which is neither vertical nor horizontal, condition (3) shows that D would satisfy the conditions of Lemma 1 and could not be an S-region. Therefore, every line segment in the boundary of D is either horizontal or vertical.

Step 2. Suppose $\mathbf{a}$ is a vertex of D. Let $\varepsilon > 0$ and let N denote the disc $|\mathbf{x} - \mathbf{a}| < \varepsilon$. The horizontal and vertical lines through $\mathbf{a}$ divide N into four parts which we denote by N_1, N_2, N_3, N_4. We claim that, if ε is sufficiently small, then $D \cap N$ is the union of one or more of the N_i's.

In view of Step 1, we can use condition (2) to choose ε so that no boundary point of D is in the interior of any one of the N_i's. Now suppose, if possible, that there are points

b and **c** in the interior of N_1 such that $\mathbf{b} \in D^c$. Then the line segment [**b**,**c**] must contain a boundary point **d** of D and this point **d** must be in the interior of N_1. But this contradicts the choice of ϵ. Thus, the interior of N_1 must be either completely contained in D or completely outside D. The same argument holds for the other N_i's. Thus, the claim at the beginning of this step is verified.

Step 3. What we have proved so far tells us that, around a vertex, the set D is either a quadrant or the union of two or three quadrants. We can thus classify the vertices conveniently into ten types to be designated as NE, NW, SW, SE, (NE)^c, (NW)^c, (SW)^c, (SE)^c, NE $\cup$ SW, NW $\cup$ SE. We illustrate two of these types in Figure 2.



(a) NE vertex (b) (SW)^c vertex

FIGURE 2. Illustration for the proof of Theorem 1 (Step 3).

Calculations similar to those in the proof of Lemma 1 show that the existence of a vertex of the type NW, SE, (NE)^c, (SW)^c or NW $\cup$ SE would contradict the fact that D is an S-region. Therefore, a vertex of D must be one of the five types NE, SW, (NW)^c, (SE)^c or NE $\cup$ SW. In what follows, we treat a NE $\cup$ SW vertex as both a NE vertex and a SW vertex.

Step 4. Any vertex of D, of one of the acceptable types in Step 3, is defined by two half lines starting at the vertex. We now show that no other vertex of D can be on any one of these defining half lines. Suppose, for instance, that $\mathbf{a} = (a_1, a_2)$ is a NE-type vertex. If there is a vertex on the half line $x_1 > a_1$, $x_2 = a_2$, then the closest such vertex (to **a**) must be either a NW vertex or a (SW)^c vertex, which is impossible; see Figure 3. Thus there cannot be any vertex on the half line $x_1 > a_1$, $x_2 = a_2$. The same argument applies to vertices of the type SW, (NW)^c and (SE)^c.

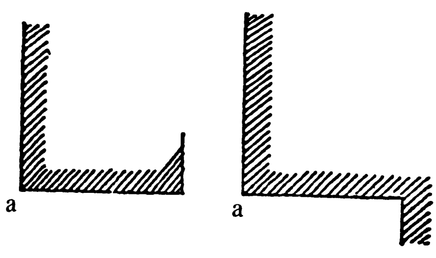


FIGURE 3. Illustration for the proof of Theorem 1 (Step 4).

Step 5. Let $\mathbf{a} = (a_1, a_2)$ be a vertex of the NE type. We show that the entire quadrant $x_1 \geq a_1$, $x_2 \geq a_2$ is contained in D. To see this, let (b_1, b_2) be a point of D^c in the open quadrant $x_1 > a_1, x_2 > a_2$. Since D^c is open, there is a neighborhood of **b** which is contained in D^c. Therefore (see Figure 4), we can start from a point **c** in such a neighborhood and proceed vertically downward to hit the set D at a point **d** which is not a vertex of D. Now, if we proceed horizontally to thge left from **d** we must hit a vertex **e**, which is a SE, (NE)^c or SE $\cup$ NW vertex. Since all these types are impossible, we have reached a contradiction. Thus the entire quadrant determined by a NE type vertex is contained in D. The same result holds for a SW type vertex.

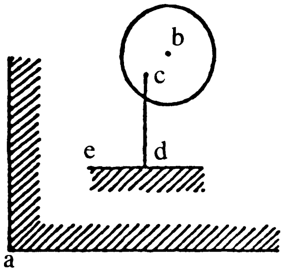


FIGURE 4. Illustration for the proof of Theorem 1 (Step 5).

Step 6. *If* $\mathbf{a} = (a_1, a_2)$ is a $(\mathrm{NW})^c$ type vertex, then one can show by following elementary arguments as above that the entire open quadrant $x_1 < a_1$, $x_2 > a_2$ is outside D. A similar result holds for a $(\mathrm{SE})^c$ type vertex.

Step 7. It follows easily from Steps 4, 5 and 6 that D can have at most one vertex of any given type. For instance, if there are two NE type vertices, then, by Step 5, it cannot happen that one of the quadrants is contained in the other. But then, we are bound to get a vertex on one of the defining half lines which is impossible by Step. 4.

Step 8. As our final step, we show that, if D has a $(\mathrm{NW})^c$ vertex or a $(\mathrm{SE})^{Cc}$ vertex, then D contains an infinite horizontal or vertical strip. We give the proof for a $(\mathrm{NW})^c$ vertex. Let $\mathbf{a} = (a_1, a_2)$ be a $(\mathrm{NW})^c$ vertex. The horizontal and vertical lines through $\mathbf{a}$ divide the plane into four quadrants, which we denote by A_1, A_2, A_3, A_4 in the usual order. By Step 6, we know that the interior of A_2 is completely outside D. If D coincides with $A_1 \cup A_3 \cup A_4$, then D clearly contains an infinite strip. So suppose that there is a point of D^c in the interior of A_1 for some $i = 1,3,4$. Three cases arise.

Case (i). Suppose we can find a point $\mathbf{b}$ of D^c in the interior of A_1; (see Figure 5). Since D^c is open, we may assume that, if we proceed leftward from $\mathbf{b}$, we would hit D at a boundary point $\mathbf{c}$ which is not a vertex. The boundary of D at $\mathbf{c}$ must be vertical. If we proceed downward from $\mathbf{c}$ and reach a vertex $\mathbf{d}$, then $\mathbf{d}$ must be either a NW vertex or a $(\mathrm{NE})^c$ vertex. Since both these types are impossible, there is no vertex on the half line $x_1 = c_1$, $x_2 \leqslant c_2$. Now proceed upward from $\mathbf{c}$. If we do not reach a vertex at all, then D clearly contains a vertical strip. So suppose that we do reach a vertex $\mathbf{e}$. In view of Step 4, $\mathbf{e}$ must be of the $(\mathrm{SE})^c$ type. We now claim that the half line $x_1 = c_1$, $x_2 > e_2$ is in the interior of $\mathbf{d}$. To see this, suppose that we proceed upward from $\mathbf{c}$ to reach a boundary point $\mathbf{f}$ of D. If $\mathbf{f}$ is a vertex, then $\mathbf{f}$ can be of either the $(\mathrm{NW})^c$ type, which is impossible by Step 7, or the $(\mathrm{NE})^c$ type, which is impossible by Step 3. Thus $\mathbf{f}$ is not a vertex. Further, the boundary of D at $\mathbf{f}$ is horizontal. Now, if we go leftward from $\mathbf{f}$, we must reach a vertex,

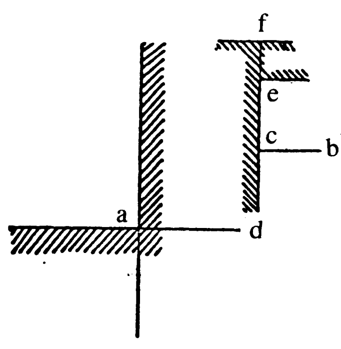


FIGURE 5. Illustration for the proof of Theorem 1 (Step 8).

which is either a SE vertex or a $(NE)^c$ vertex. This contradiction proves the above claim. We now see that, for some $\delta > 0$, the vertical strip $c_1 - \delta < x_1 \leq c_1$, $-\infty < x_2 < \infty$ must be contained in D.

Case (ii). If we can find a point $\mathbf{b}$ of D^c in the interior of A_3, then the discussion in Case (i) above easily shows that D contains an infinite horizontal strip.

Case (iii). If neither Case (i) nor Case (ii) arises, then $A_1 \cup A_3 \subset D$ and we can find a point $\mathbf{b}$ of D^c in the interior of A_4. Again, we may assume that, if we proceed leftward from $\mathbf{b}$, we would hit D at a boundary point $\mathbf{c}$ which is not a vertex. From this point on, the proof given in Case (i) applies word for word.

We have thus shown that the existence of a $(NW)^c$ vertex implies that D contains an infinite strip. The same conclusion clearly holds if D has a $(SE)^c$ vertex.

We are now ready to put everything together. If D contains an infinite horizontal (or vertical) strip, then we can remove the finite disjoint union B of all such strips from D to get a set E. Of course, if D does not contain an infinite strip, then $B = \phi$ and $E = D$. *In any case,* E is an S-region, $E \in \mathcal{D}$ and E does not contain an infinite strip. Now observe that:

(a) By Step 8, any vertex of E is a NE or a SW or a NE SW vertex, (b) by Step 7, E has at most two vertices, (c) if E has exactly one vertex, then E has the form Q_1, Q_3 or $Q_1 \cup Q_3$, (d) if E has two vertices, then one must be a NE vertex and the other a SW vertex, in this case, E has the form $Q_1 \cup Q_3$. The theorem is now completely proved. □

5. The Effect of Restrictions on Means or Variances. The results of Section 2 show that within the reasonably broad class of sets $\mathcal{D}$, the subclass of sets whose probabilities increase with ρ is rather narrow. However, it should be noted that the means and variances were completely unrestricted. One may therefore ask whether, under some restrictions on means and variances, one can identify additional sets whose probabilities increase with ρ. In this section we show that, at least in some cases, the answer is in the affirmative. We again assume that (X_1, X_2) has the bivariate normal distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2 and ρ.

Example 1. Suppose that μ_1, μ_2 are fixed. Consider the half space H defined by the inequality $a_1x_1 + a_2x_2 \geq k$, where a_1, a_2 have the same sign. Suppose that (μ_1, μ_2) is outside H. That is, $a_1\mu_1 + a_2\mu_2 < k$. Then

$$P(H) = P[Z \geq (k - a_1\mu_1 - a_2\mu_2)/((a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_1\sigma_2\rho))^{1/2}$$

which is nondecreasing in ρ because $a_1a_2 \geq 0$. The same conclusion holds if $a_1a_2 < 0$ and (μ_1, μ_2) belongs to H. We note that H does belong to the class $\mathcal{D}$.

In the case where σ_1, σ_2 are fixed and μ_1, μ_2 are allowed to vary, we have been unable to find any regions (additional to those already found in Section 2) whose probabilities increase with ρ.

The rest of this section considers the case where $\mu_1, \mu_2, \sigma_1, \sigma_2$ are all fixed. Since a change of origin does not change the variances we assume that $\mu_1 = \mu_2 = 0$. Write $\alpha = (\sigma_1/\sigma_2)$. For the given value of α, we now construct a family of sets whose probabilities increase with ρ. Consider the following conditions on a set S in $\mathcal{R}^2$.

(i) If $(x_1,x_2) \in S$, then the entire line segment joining (x_1,x_2) to $((x_1 + \alpha x_2)/2, (x_1 + \alpha x_2)/2\alpha)$ as in S.

(ii) If $(x_1,x_2) \in S$, then $(x_1 + \alpha c, x_2 + c) \in S$ for every $c > 0$.

(iii) $(x_1,x_2) \in S \Rightarrow x_1 + \alpha x_2 \geq 0$.

Condition (i) is a convexity condition. Condition (ii) says that the variables "hang together" in S. The third condition is not natural but is related to the fact that $\mu_1 = \mu_2 = 0$. We saw in Example 1 that the position of the mean vector is important in determining whether a given set has its probability increasing in ρ.

THEOREM 2. *Suppose $S \subset \mathcal{R}^2$ is a set which satisfies the conditions* (i), (ii), (iii) *stated above. Let $\mu_1 = \mu_2 = 0$ and $(\sigma_1/\sigma_2) = \alpha$. Then $P(S)$ is nondecreasing in* ρ.

Proof. Let $U_1 = (X_1 + \alpha X_2)/(\sqrt{2}\ \sigma_1)$ and $U_2 = (X_1 - \alpha X_2)/(\sqrt{2}\ \sigma_1)$. Then U_1, U_2 are independent with zero means and variances $(1 + \rho)$ and $(1 - \rho)$ respectively. Under this transformation, the set S is converted into a set T such that

(A) If $(u_1,u_2) \in T$, then the entire line segment joining (u_1,u_2) to $(u_1,0)$ is in T.

(B) If $(u_1,u_2) \in T$, then $(u_1 + t, u_2) \in T$, for all $t > 0$.

(C) $(u_1,u_2) \in T \Rightarrow u_1 \geq 0$.

Let Q_ρ denote the distribution of $\mathbf{U} = (U_1,U_2)$. Then $P(S) = Q_\rho(T)$. Now, if D_ρ denotes the 2×2 diagonal matrix whose (1,1) entry is $(1 + \rho)^{-1/2}$ and (2,2) entry is $(1 - \rho)^{-1/2}$, then $Q_\rho(T) = Q_0(D_\rho T)$. But clearly $\rho_1 < \rho_2 \Rightarrow D_{\rho_1}(T) \subset D_{\rho_2}(T)$. Therefore $Q_{\rho_1}(T) \leqslant Q_{\rho_2}(T)$, whenever $\rho_1 \leqslant \rho_2$. The theorem is thus proved. □

The generalization of Theorem 2 to the k-variate equi-correlated case is straightforward and we state it in the theorem below without proof. Again we assume that the means are zero, $\sigma_1^2 = \mathrm{Var}(X_i)$ and $\rho = \mathrm{corr}(X_i,X_j)$, for all $i \neq j$. If $\mathbf{x} = (x_1, \ldots, x_n)$, we write $x^* = (1/n)\Sigma(x_i/\sigma_i)$. We also write $\sigma = (\sigma_1, \ldots, \sigma_n)$.

THEOREM 3. *Let X have the equi-correlated multivariate normal distribution with zero means. Suppose $S \subset \mathcal{R}^n$ be such that*

(1) $\mathbf{x} \in S \Rightarrow$ *the entire line segment joining* $\mathbf{x}$ *and* $(\sigma_1 x^*, \sigma_2 x^*, \ldots, \sigma_n x^*)$ *is in* S.

(2) $\mathbf{x} \in S \Rightarrow (\mathbf{x} + c\sigma) \in S$, *for all* $c \geq 0$.

(3) $\mathbf{x} \in S \Rightarrow \Sigma(x_i/\sigma_i) \geqslant 0$.

Then $P(S)$ is a nondecreasing function of ρ.

Example 2. One may ask whether the class of regions whose probabilities are nondecreasing in ρ is closed under intersections. The answer is in the negative even if attention is restricted to "increasing" sets. To see this, suppose that $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Let $S_1 = \{(x_1,x_2): x_1 \geqslant 0\}$ and $S_2 = \{(x_1,x_2): x_2 \geqslant -(1 + \epsilon)x_1\}$, where $\epsilon > 0$. Then $P(S_1)$ and $P(S_2)$ are both nondecreasing in ρ because of Example 1. Now, if ϵ is close to zero, then $P(S_1 \cap S_2)$ is close to ½ when $\rho = -1$ and close to ⅜ when $\rho = 0$. Thus $P(S_1 \cap S_2)$ is not nondecreasing in ρ.

4. An Equivalent Form of Slepian's Result. According to Yanagimoto and Okamoto (1969), a random vector (X_1,X_2), whose distribution P_ρ depends on a parameter ρ, is said to have larger positive quadrant dependence under ρ_1 than under ρ_2 if

$$P_{\rho_1}(X_1 \leqslant x_1, X_2 \leqslant x_2) \geqslant P_{\rho_2}(X_1 \leqslant x_1, X_2 \leqslant x_2) \text{ for all } (x_1,x_2). \tag{4.1}$$

They showed that (4.1) is equivalent to

$$\mathrm{Cov}[f_1(X_1),f_2(X_2);\rho_1] \geqslant \mathrm{Cov}[f_1(X_1),f_2(X_2);\rho_2] \tag{4.2}$$

for all nondecreasing functions f_1,f_2. Now Slepian's result shows that a bivariate normal vector (X_1,X_2) satisfies (4.1) and consequently it satisfies (4.2), whenever $\rho_1 > \rho_2$. Here the concept of "positive quadrant dependence" has been given an ordering relation which agrees, at least for the bivariate normal case, with the ordering based on the weaker concept of dependence, namely, the correlation coefficient.

Observe that the functions f_1, f_2 have separate arguments. We may ask whether $\mathrm{Cov}[h_1(X_1,X_2), h_2(X_1,X_2)]$ is nondecreasing in ρ if h_1, h_2 are nondecreasing in each argument and (X_1,X_2) is bivariate normal. This question is clearly related to the concept of association introduced by Esary, Proschan and Walkup (1967). That the answer is in the negative is indicated by Example 2. This example gives a set $S_1 \cap S_2 = B$, say, such that $P(B)$ is near 3/8 when $\rho = 0$ and near 1/2 when $\rho = -1$. If h denotes the indicator function of B, then h is increasing and $\mathrm{Var}[h(X_1,X_2)]$ is larger at $\rho = -1$ than at $\rho = 0$. Another example is as follows.

Example 3. Consider two quadrants

$$Q_0 = \{(x_1,x_2): x_1 \geq 0, x_2 \geqslant 0\}$$

and

$$Q_t = \{(x_1,x_2): x_1 \geqslant 0, x_2 \geqslant t\}, \text{ where } t > 0.$$

Again let (X_1,X_2) have a bivariate normal distribution with zero means, unit variances and correlation coefficient ρ. Denote the density function of (X_1,X_2) by g. Then

$$P(Q_t) = \int_0^\infty \int_t^\infty g(x_1,x_2;\rho)\,dx_2\,dx_1 \tag{4.3}$$

Using (4.3) and the fact that

$$(\partial/\partial\rho)g = (\partial^2/\partial x_1,\partial x_2)g,$$

we get

$$\begin{aligned} &d/d\rho[P(Q_0 \cap Q_1) - P(Q_0)P(Q_t)] \\ &= g(0,t;\rho) - P(Q_0)g(0,t;\rho) - P(Q_0)g(0,0;\rho). \end{aligned} \tag{4.4}$$

When ρ is near 1, $g(0,0;\rho)$ is large, $g(0,t;\rho)$ is small and $P(Q_1)$ is bounded away from zero. Therefore the above derivative (4.4) is negative for ρ near 1. Equivalently, the indicators of Q_0 and Q_1 are nondecreasing functions whose covariance is decreasing in ρ near $\rho = 1$.

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