ESSAY II. CONTINUATION OF AN EXAMPLE OF C. DELLACHERIE

1. THE PROCESS R_t .

We consider a single occurrence in continuous time $t \geq 0$ which happens at an instant $T_* \geq 0$ which may be random. For example, T_* may be the failure time of some mechanical apparatus. Analytically, the entire situation is described simply by the distribution function $F(x) = P\{T_* \leq x\}$. We restrict F only by $F(0-) = 0$ and $F(\infty) \leq 1$, and we define $T_* = \infty$ where T_* is not finite, so that $P\{T_* = \infty\} = 1 - F(\infty)$. Without risk of confusion, we speak of the "occurrence of T_*," thus identifying the event with its instant.

From the viewpoint of an observer waiting for T_* to occur, the situation presents itself not as a distribution function but as a stochastic process, and as such it provides a basic example of general methods. Thus we associate with T_* the process

$$R_t = I_{[T_*,\infty]}(t) \ , \quad -\infty \leq t \leq \infty, \tag{1.0}$$

where I denotes the usual indicator function.

This process was studied by C. Dellacherie (1972), and by C. S. Chou and P. A. Meyer (1975). The closely related process $T_* \wedge t$ was also studied briefly by C. Dellacherie and P. A. Meyer (1975), who corrected some errors in [4]. Since we require some preliminary results from [4], we use that formulation in large part. However, our purpose is to study R_t in terms of its prediction process, as defined in F. B. Knight (1975) and P. A. Meyer (1976). This dictates that $\{T_* = 0\}$ and $\{T_* = \infty\}$ be permitted to have positive probability, which in turn makes it useful to set $R_t = 0$ for $-\infty \leq t < 0$. Thus we introduce the probability space $(\Omega, \mathcal{F}^o, P)$ where $\Omega = [0,\infty]$, $\mathcal{F}^o$ is the Borel σ-field, and $P(dx) = F(dx)$, and we define $T_*(x) = x$ on Ω, so that $R_r(x) = I_{[x,\infty]}(t)$, $-\infty \leq t \leq \infty$. Then the σ-field $\mathcal{F}_t^o$ generated by R_s, $s \leq t$, is $\{\Phi,\Omega\}$ for $t < 0$, and is that generated by the atom $(t,\infty]$ and the Borel sets of $[0,t]$ for $t \geq 0$.

As an example of the "general theory of processes," R_t was replaced in [4] by the supermartingale $X_t = E(R_\infty - R_t | \mathcal{F}_t^o) = I_{[0,T_*)}(t)$, which was even a potential since $P\{T_* < \infty\} = 1$ was assumed. In the present case, the argument of [4, Chap. 5, T56] transfers with no substantial change to provide the Doob-Meyer decomposition of R_t. We need the usual augmented σ-fields $F_t (= F_t^P)$ generated by F_t^o and all P-null sets in the completion of P on F^o. Observe that $F_t = F_{t+}$, where for any adapted family of σ-fields G_t we set $G_{t+} = \bigcap_{s>t} G_s$ and $G_{t-} = \bigvee_{s<t} G_s$.

THEOREM 1.1. The unique F_t-previsible increasing process R_t^* such that $R_0^* = P\{R_0 = 1\}$ and $R_t - R_t^*$ is a martingale is given by

$$R_t^* = 0, \quad -\infty \leq t < 0,$$

$$R_t^* = \int_{0-}^{T_* \wedge t} (1 - F(u-))^{-1} \, dF(u), \quad 0 \leq t < \infty,$$

$$R_\infty^* = \begin{cases} R_{\infty-}^* & \text{on } \{T_* < \infty\} \\ R_{\infty-}^* + 1 & \text{on } \{T_* = \infty\}. \end{cases}$$

REMARK. Uniqueness means unique up to a fixed P-null set.

In the present note we will go one step farther, and study R_t as an example in the theory of Markov processes (as well as of martingales). Indeed, a general feature of the prediction process construction is that it permits any process to be viewed as a homogeneous Markov process--more specifically, as a right process in the sense of P. A. Meyer and having still additional structure. It may be said here that R_t provides a more or less prototypical example of the prediction process of a positive pure-jump submartingale. The behavior of this prediction process depends, in turn, on classification of the stopping times of F_t, which accordingly is our next concern. However, the reader may prefer to skip this rather technical discussion, and go directly to Section 2 where the results are applied. The connections with Essay I are postponed until the end of the present essay, for reasons stated there.

We recall that a stopping time T is "totally inaccessible" if for every increasing sequence of stopping times T_n one has $P\{\lim_{n\to\infty} T_n = T < \infty\} = 0$, and "previsible" if $P\{T = 0\} = 0$ or 1, and if when $P\{T = 0\} = 0$ there exist T_n with $1 = P\{T_n < T\} = P\{\lim_{n\to\infty} T_n = T\}$. For the remaining concepts in our classification, as well as its existence and uniqueness, we refer to [5, Chap. IV, Theorem 81]. According to the basic representation theorem of our particular situation ([4, III, T53]) a random time T is an F_t-stopping time if and only if for some $s \leq \infty$,

$$\text{(1.1)} \qquad P\{\{T_* \le s \wedge T\} \cup \{T_* > s = T\}\} = 1 \ .$$

We note that s is unique unless $P\{T_* > T\} = 0$, and then we may choose $s = \infty$. The classification of stopping times depends on:

THEOREM 1.2. The accessible part of a stopping time T is given by

$$T_A = \begin{cases} T & \text{on } A \\ \infty & \text{on } A^c \end{cases}, \quad \text{where}$$

$$\text{(1.2)} \qquad A = \{T > T_*\} \cup \{T = s < T_*\} \cup \bigcup_{s_k \le s} \{T = T_* = s_k\}$$

where s_k enumerate the values with $P\{T_* = s_k\} > 0$.

REMARK. It is easy to see that this set is unique up to a P-null set even if s is not unique.

PROOF. We have $\{T = 0\} = \{T = 0 = T_*\} \cup \{T = 0 < T_*\}$, hence if $P\{T = 0\} > 0$ then either 0 is an s_k or $s = 0$. In either case $\{T = 0\}$ is in (1.2), as it should be. Now let T_n be any nondecreasing sequence of stopping times, and let $T_\infty = \lim_{n\to\infty} T_n$. If we assume that $P\{T_n < T_\infty\} = 1$ for all n (thus T_∞ is previsible) and let s_n correspond to T_n as in (1.1) with $s_n = \infty$ whenever possible, then we see that $\lim_{n\to\infty} s_n = s_\infty$ exists, and satisfies (1.1) for T_∞. Then we have $\{T_* < s_\infty\} \subset \{T_* < T_\infty\}$ up to a P-null set, and therefore

$$\text{(1.3)} \qquad P\{\{T_* < s_\infty \wedge T_\infty\} \cup \{s_\infty = T_* \wedge T_\infty\}\} = 1 \ .$$

Conversely, if a stopping time T satisfies (1.3) for some s_∞ and $P\{T > 0\} = 1$, then we can construct a sequence $T_n \longrightarrow T$, $P\{T_n < T\} = 1$, as follows. If $s_\infty = \infty$ then $1 = P\{\{T_* < T\} \cup \{T_* = T = \infty\}\}$ and writing $T = f(T_*)$ on Ω we can define $T_n = f_n(T_*)$ where f_n are any measurable functions with $f_n(\infty) = n$, and for $x < \infty$, $x < f_n(x) < f(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$. If $0 < s_\infty < \infty$, then we define for $n^{-1} < s_\infty$

$$T_n = \begin{cases} f_n(T_*) & \text{on } \{T_* \le s_\infty - n^{-1}\} \cup \{T_* = s_\infty < T\} \\ s_\infty - n^{-1} & \text{elsewhere} \end{cases}$$

and observe that T_n satisfies (1.1) with $s = s_\infty - n^{-1}$. Finally, if $s_\infty = 0$ then $P\{T_* = 0\} = 1$ and T is equivalent to a positive constant. It follows that (1.3) characterizes the previsible stopping times T with $P\{T > 0\} = 1$.

Next we observe that for constant c, any T is accessible on a set of the form $\{T = c\}$, hence on $\{T = s\} \cup \bigcup_{s_k \le s} \{T = T_* = s_k\}$. It remains to show that the rest of the accessible part is given by $\{T > T_*\}$. That this is contained in the accessible part follows by writing $T_n = f_n(T_*)$ as in the preceeding paragraph. On the other hand, by (1.1) we have $\{T \leq T_*\} = \{T = T_*\} \cup \{T_* > s = T\}$ up to a P-null set, hence the only part of $\{T \leq T_*\}$ not already found excessible is $\{T = T_* \neq s_k$ for all $k\}$. To see that this last is not accessible, note that for any previsible stopping time $T_\infty > 0$, (1.3) implies that the set $\{T = T_* = T_\infty\}$ is contained in $\{T = T_* = T_\infty = s_\infty\}$ up to a P-null set, where s_∞ corresponds to T_∞ as in (1.3). Therefore, only sets $\{T = T_* = s_k\}$ of positive probability can be in the accessible part, and the proof is complete.

COROLLARY 1.3. A stopping time T is: a) totally inaccessible if and only if $P\{T > T_*\} = 0$ and $P\{s = T \leq T_*\} = 0$ for $0 \leq s \leq \infty$, b) previsible if and only if $P\{T = 0\} = 0$ or 1 and, for some s, $P\{\{T_* < s \wedge T\} \cup \{s = T_* \wedge T\}\} = 1$.

PROOF. Part b) is just (1.3), so we need only prove a). The condition is obviously sufficient by Theorem 1.2. On the other hand, if $P\{s = T \leq T_*\} > 0$ for some s, then either s corresponds to T as in (1.1) and $P\{s = T < T_*\} > 0$, or else s is one of the s_k's in Theorem 1.2 and $P\{T = T_* = s_k\} > 0$. In either case, T is partially accessible.

COROLLARY 1.4. If $P\{T_* = s\} = 0$ for all $s \leq \infty$, then T_* is totally inaccessible and a stopping time T is previsible if and only if $P\{T = T_*\} = 0$. Furthermore, the necessary and sufficient condition that $\mathcal{F}_t$ be free of times of discontinuity is that, for all $s \geq 0$, $P\{T_* > s\} > 0$ implies that $P\{T_* = s\} = 0$.

REMARK. It is known from [4, Chap. III, T51] that absence of times of discontinuity is equivalent to the previsibility of all T whose accessible part is Ω (up to a P-null set).

PROOF. The first assertion is immediate from Theorem 1.2. For the second, assume $P\{T = T_*\} = 0$, and let s correspond to T as in (1.1). Since $P\{T_* = s\} = 0$, we have $P\{T_* = s \wedge T\} = 0$, hence T satisfies Corollary 1.3 b). Conversely, if $P\{T = T_*\} > 0$ then T is inaccessible on this set, hence not previsible.

It remains to prove the last assertion. Assume that the condition holds; i.e., that the distribution of T_* has not atoms except perhaps its maximal value, and suppose that the accessible part of T is Ω . Let s correspond to T as in (1.1). If $P\{T_* = s\} > 0$, then by Theorem 1.2 we have $1 = P\{\{T > T_*\} \cup \{T = T_* = s\}\}$, and since $T_* \leq s$ holds on $\{T > T_*\}$, T is previsible by Corollary 1.3 b). If, on the other hand,

$P\{T_* = s\} = 0$, then if $P\{T = T_* = s_k\} > 0$ for any s_k, we see from $P\{T_* > s_k\} = 0$ and (1.1) that $s_k < s$, and hence s_k may replace s in (1.1). Thus either we have the former case, or $P\{T = T_* = s\} = 0$ for all s. Then since $T > T_*$ implies $T_* < s$ except on a P-null set, and $1 = P\{\{T > T_*\} \cup \{T = s < T_*\}\}$, by Corollary 1.3 b) T is again previsible. Thus (see the Remark) $\mathcal{F}_t$ is free of discontinuities. The converse is obvious, since $P\{T_* > s\} > 0$ and $P\{T_* = s\} > 0$ imply $\mathcal{F}_{s-} \neq \mathcal{F}_s$.

2. THE PREDICTION PROCESS OF R_t.

We turn now to the construction of the prediction process of R_t, which we will denote by Z_t. According to its definition, the values of Z_t are the conditional probability distributions of $T_{t+(\cdot)}$ given $\mathcal{F}_t$ (we recall that $\mathcal{F}_t = \mathcal{F}_{t+}$). Clearly such distributions can be specified by the conditional distribution of $T_* - t$ given $\mathcal{F}_t$, whence they have the same form as F. Thus, writing $Z_t(x) = Z_t(x,w)$ for the corresponding distribution function, we have $Z_t(0) = 1$ if $t \geq T_*$ or $F(t) = 1$, while $Z_t(x) = (F(t + x) - F(t))/(1 - F(t))$ otherwise. The left-limit process of Z_t, in a suitable topology to be specified, is $Z_{t-}(0) = 1$ if $t > T_*$ or $F(t-) = 1$, and $Z_{t-}(x) = (F(t + x) - F(t-))/(1-F(t-))$ otherwise.

The prediction process may be used to best advantage only by introducing it as a Markov process in its own right, instead of confining it to the probability space of R_t (this represents a partial shift of the author's views from those expressed in [9]). This is because there are technical difficulties in carrying out the theory of additive functionals of the prediction process if it is defined on the original probability space Ω (as noted by R. K. Getoor (1978)). On the other hand, once we free ourselves from this restriction, the theory becomes comparatively straightforward. Furthermore, in a sense to be made precise, nothing concerning the process R_t is lost in the transition. Therefore, we introduce formally both a new state space and a new probability space.

DEFINITION 2.1. The prediction state space of R_t is the space $(E_Z, \mathcal{E}_Z)$ where

$$E_Z = \{(F(t+\cdot) - F(t))/1 - F(t)\ ,\ -\infty < t < \infty:\ F(t) \neq 1\ ;$$

$$(F(t+\cdot) - F(t-))/(1 - F(t-))\ ,\ -\infty < t < \infty:\ F(t-) \neq 1\ ;$$

$$F_{-\infty},\ \text{and}\ F_{+\infty}\}\ ,\ \text{with}\ F_{-\infty}(x) \equiv 0\ ,\ F_{+\infty}(x) \equiv 1\ ,$$

and $\mathcal{E}_Z$ is the σ-field generated by the functions $G(x)$, $0 \leq x \leq \infty$, as G varies on E_Z. We denote elements of E_Z of the first two types by F_t and F_{t-} respectively (although, with this notation, they are not necessarily distinct). We let E_Z^+ denote $\{F_{-\infty}, F_{+\infty}, F_t, -\infty < t < \infty\}$.

In the present very specialized situation, it is natural to introduce in E_Z the topology of weak convergence of measures on Ω, when Ω is considered as a subset of the space D with the Skorokhod J_1-topology (Billingsley, [2], Chapter 3). Specifically, to each $x \in \Omega$ we associate the element of D given by $f_x(s) = R_t(x)$ with $s = \frac{1}{2}(1 + \frac{2}{\pi} \arctan t)$, $-\infty \leq t \leq \infty$. We note that $f_x(s) = 0$ for $0 \leq s < \frac{1}{2}$, and that convergence in D of f_x is the same as convergence of x in the extended topology of $[0,\infty]$. It therefore follows that the continuous functions on Ω in the D-topology are just $C[0,\infty]$, and weak convergence of probabilities on Ω becomes simply weak convergence of the corresponding distribution functions F on $[0, \infty]$. In particular, we note that E_Z is a Borel set and that $\mathcal{E}_Z$ is a Borel σ-field generated by this (metrizable) topology on E_Z. Furthermore, since F_t is right-continuous for $t < \min\{s: F(s) = 1\}$, with left limits F_{t-} for $t > 0$, it is clear that Z_t is right-continuous with left limits in this topology. In fact, the space E_Z is "almost" compact, the only limit points not necessarily included being those of F_t obtained as $t \longrightarrow +\infty$. This set is trivial if either $F(\infty) < 1$ or $F(t) = 1$ for some $t < \infty$, but in general it cannot be avoided.

We turn next to the <u>prediction probability space</u> for the process Z_t, using the same notation Z_t for the process on the new space.

DEFINITION 2.2. Let $(\Omega_Z, \mathcal{F}_Z, Z_t)$ consist of

a) The space of all paths $z(t)$, $0 \leq t < \infty$, with values in E_Z^+ and which are right-continuous, with left limits for $t > 0$, in the topology of weak convergence,

b) The coordinate σ-field generated on Ω_Z by $\{z(t) \in A\}$, $t \geq 0$, $A \in \mathcal{E}_Z$.

c) The coordinate functions $Z_t = Z_t(z) = z(t)$.

We observe that the original $F(=F_{0-})$ is in Ω_Z, and that the process on Ω given by F_t for $0 \leq t < T_*$ and by F_∞ for $t \geq t_*$ has its paths as points in Ω_Z. Hence we can define a probability P on $(\Omega_Z, \mathcal{F}_Z)$ such that the joint distributions of $Z(t)$ are the same as those of the above process on Ω. Furthermore, to every $z \in E_Z$ we can associate in the same way probability P^z on $(\Omega_z, \mathcal{F}_z)$, by using z in the role of F as the distribution of T_*. Thus the points $z \in E_Z$ correspond to probabilities for Z_t. If $z = F_t$ for some t,

$-\infty \le t \le \infty$, then $P^z\{Z_0 = z\} = 1$. However, if $z = F_{t-} \neq F_t$, so that $F(t) - F(t-) > 0$, then $P^z\{Z_0 = F_t\} = 1$.

We are now in a position to view the family $\{P^x, z \in E_Z\}$ as a Markov process on $(\Omega_Z, \mathcal{F}_Z)$. The points z such that $z = F_{t-} \neq F_t$ are the "branching points" of this process, in the terminology of Walsh and Meyer [13]. The transition function $q(t,z,A)$ of the process is such that for each (t,z) the probability is concentrated on at most two points. Precisely, we have

DEFINITION 2.3. The transition function of Z_t is given by $q(t,z,A)$, $t \ge 0$, $z \in E_Z$, $A \in \mathcal{E}_Z$, where

i) $q(t,F_\infty,\{F_\infty\}) = 1$, $t \ge 0$

ii) $q(t,z,\{F_\infty\}) = 1 - q(t,z,\{F_{s+t}\}) = F_s(t)$ if $z = F_s$ and $1 > F_s(t)(=F(s+t))$, $t \ge 0$,

iii) $q(t,z,\{F_\infty\}) = 1 - q(t,z,\{F_{s+t}\}) = F_{s-}(t)$ if $z = F_{s-} \neq F_s$ and $1 > F_s(t)$, $t > 0$,

iv) $q(t,z,\{F_\infty\}) = 1$ in cases ii) and iii) if $1 = F_s(t)$,

v) $q(0,z,\{F_\infty\}) = 1 - q(0,z,\{F_s\}) = F_{s-}(0)$ in case iii).

It follows from the general theory of [9] and [11] (or can easily be seen directly) that $(\Omega_Z, \mathcal{F}_Z, Z_t, P^z)$ becomes a right process on E_Z^+ in the sense of P. A. Meyer, with transition function q, when we include the canonical translation operators θ_t^z and σ-fields $\mathcal{F}_t^z$. Of course, both E_Z^+ and q are Borel, so the general U-space set-up of Getoor [6] is unnecessary (this is quite generally true for the prediction process). Furthermore, the process has unique left limits Z_{t-} in E_Z, $t > 0$.

It is important to observe that probabilistically nothing is lost by considering (Z_t, P^F) in place of (R_t, P). Thus we introduce on E_Z^+ the Borel function

$$\varphi(G) = \begin{cases} 0 & \text{if } G \neq F_\infty \\ 1 & \text{if } G = F_\infty \end{cases} . \tag{2.1}$$

Then $\varphi(Z_t)$ is P^F-equivalent to R_t in joint distribution, and is right-continuous with left limits. Hence it is a valid replacement for R_t. The σ-fields $\mathcal{F}_t^{o,Z}$ generated by Z_s, $s \le t$, are of course larger than those generated by $\varphi(Z_s)$, $s \le t$. But the entire difference can be traced to the fact that $\varphi(Z_0)$ does not determine Z_0. Thus for each initial point z the above two fields have the same P^z-completion, and hence Z_t and $\varphi(Z_t)$ generate the same completed σ-fields $\mathcal{F}_t^Z$.

One basic feature of the prediction process which gives insight into the given process is its times of discontinuity. The analogue of the jump time T_* on Ω_Z is of course the stopping time

$$T_{Z,*} = \inf \{t: Z_t = f_\infty\} . \tag{2.2}$$

However, this is not necessarily a time of discontinuity for Z_t under P^F, and by no means the only one. By Theorem 1.2 the accessible part of $T_{Z,*}$ under P^F consists of $\bigcup_k \{T_{Z,*} = s_k\}$, where the s_k enumerate the jump points of F. But while R_t is discontinuous at $t = s_k$ with probability $F(s_k) - F(s_{k-})$, Z_t is discontinuous at $t = s_k$ with probability $1 - F(s_k-) (= P^F\{T_{Z,*} \geq s_k\})$ unless $F(s_k) = 1$, when it is continuous (since Z_{s_k} is then $\mathcal{F}_{s_k-}$-measurable). On the other hand, at the totally inaccessible part of $T_{Z,*}$ (i.e. the part where F is continuous), Z_t like R_t has an inaccessible jump. It is clear that Z_t is continuous except at $\bigcup_k \{s_k\} \cup \{T_{Z,*}\}$ hence we have classified its discontinuities under P^F, and for other $z \in E_Z$ the situation is analogous. Thus, the conclusion which roughly emerges is that Z_t has the same totally inaccessible jumps as R_t but it has additional accessible jumps at times when R_t has a positive (but unrealized) potentiality for a jump.

This distinction in the behavior of R_t and Z_t at the previsible times s_k disappears when we replace R_t by the martingale $R_t - R_t^*$ of Theorem 1.1. More generally, we introduce on Ω_Z the <u>previsible additive functional</u>

$$A_t = \int_0^{T_{Z,*} \wedge t} (1 - G(u-))^{-1}\, d\, G(u) \quad \text{on} \quad \{Z_0 = G\}\ , \quad G \in E_Z^+ . \tag{2.3}$$

(previsibility is clear since A_t is a Borel function of the previsible process $T_{Z,*} \wedge t)$. The process $\varphi(Z_t) - \varphi(Z_0) - Z_t$ is now seen to be a <u>martingale additive functional</u> of Z_t . More importantly, one easily checks that $\varphi(Z_t) - \varphi(Z_0) = A_t$ and Z_t have the <u>same times of discontinuity</u> for each P^G . This is an expression of the general fact that a right-continuous martingale has its times of discontinuity contained in those of its prediction process, as proved in F. Knight [10, Lemma 1.5]. However, the application is not direct because the prediction process of $\varphi(Z_t) - A_t$ for fixed $G = Z_0$ has a different (and less convenient) state space than E_Z, and it cannot be identified with Z_t . For example, if $F(s) - F(s-) = 1$ for some s then $\varphi(Z_t) - A_t \equiv 0$ for P^F while Z_t, although continuous, is not constant.

We consider finally the Lévy system of Z_t, and its relevance to R_t and $T_t - R_t^*$. By definition [1, Corollary 5.2] this is a pair (N,H) where $N(x,dy)$ is a kernel on $(E_Z, \mathcal{E}_Z)$, $N(x,\{x\}) = 0$, and H is a previsible additive functional such that for $0 \le f(x,y) \in \mathcal{E}_Z \times \mathcal{E}_Z$, with $f(z,z) = 0$,

$$(2.4) \qquad E^\cdot\Big(\sum_{0<s\le t} f(Z_{s-},Z_s)\Big) = E^\cdot \int_0^t dH_s\Big(\int_E N(Z_{s-},dy)\, f(Z_{s-},dy)\Big) .$$

In the present case, although Z_t does not satisfy all the hypotheses of [1, Cor. 5.2] it is easy to specify such a system explicitly. One has only to take $H_t = A_t$ from (2.3) and then define

$$(2.5) \qquad N(x,dy) = \begin{cases} q(0,x,dy) & \text{for } x = F_{t-} \ne F_t, \quad -\infty < t < \infty \\ \delta(F_\infty) & \text{otherwise}, \quad x \ne F_\infty , \end{cases}$$

where $\delta(F_\infty)$ is the unit mass at F_∞ (we define $N(F_\infty,\cdot)$ in any convenient way). As a compensator for the discontinuities of Z_t, the Lévy system is here more relevant to $R_t - R_t^*$ than to R_t, for the reasons of the preceeding paragraph. Thus we have an analogous "Lévy system" for $R_t - R_t^*$ in the form (N^*, R_t^*) where

$$(2.6) \qquad \begin{aligned} N^*(-R^*_{s_j-},\{-R^*_{s_j} + 1\}) &= F_{s_j-}(0) \\ &= 1 - N^*(-R^*_{s_j-}, \{-R^*_{s_j}\}) \end{aligned}$$

for $F(s_j) - F(s_j-) > 0$, and $N^*(x,B\} = I_B(x+1)$ for all $x \notin \{-R^*_{s_j-}\}$. It is clear that (2.6) is obtained from (2.5) by just substituting the jumps of $R_t - R_t^*$ for those of (Z_t,P^F) except at $t = 0$ and $t = \infty$ which are disallowed as jump times of Z_t. Since (2.6) has a role analogous to (2.4) but for the martingale $R_t - R_t^*$ instead of Z_t, it is natural to take it as the definition of a Lévy system for the martingale. Again, this is a very special case of a general existence theorem ([10, Theorem 1.3]).

3. CONNECTIONS WITH THE GENERAL PREDICTION PROCESS.

For the reader who is already familiar with Essay I, the present Section 2 is easily incorporated into that more general setting. However, it is somewhat more natural to treat all single-jump processes simultaneously, as realized by a single prediction process. This formalizes, so to speak, the essence of the underlying idea. It has been carried out by Professor John B. Walsh, who has consented to let us use the material that follows.

We take $w(t) = w_2(t)$, with all other components discarded from the notation. Let Ω_J (J for jump) be the set of functions of the form $w(x) = I_{[T,\infty)}(x)$, $0 \le T \le \infty$. Then Ω_J inherits from Ω the topology of pointwise convergence of the corresponding T. Hence it is compact. Let H_J be the set of all probability measures on Ω_J, with the weak-* topology. If we identify $h \in H_J$ with the probability distribution it assigns to T, then convergence in H_J becomes weak convergence of distribution functions on $[0,\infty]$, and H_J is compact.

For $h \in H_J$ (regarded as a measure on Ω vanishing outside Ω_J), the prediction process Z_t remains in H_J, and so does Z_{t-} for $t > 0$. Thus H_J is a complete Borel packet, in the sense of Essay I, Definition 2.1, 3). The transition function of Z_t on H_J is given above by Essay 2, Definition 2.3. The elements of $H_J \cap H_0$, regarded as distributions of T, are just F_∞ and all F with $F(0) = 0$. Thus Z_t is a right process on $H_J \cap H_0$. In fact, we have more in the present case.

PROPOSITION 3.1. Z_t is a Ray process on H_J.

PROOF. It is to be shown that $\int_0^\infty e^{-\lambda t} q_t f dt \in C(H_J)$ if $f \in C(H_J)$, where $q_t f(h) = \int f(z) q(t,h,dz)$. As before, we let $F(t) = h\{T \le t\}$. Then we have

$$\int_0^\infty e^{-\lambda t} q_t f(h) dt = f(F_\infty) \int_0^\infty e^{-\lambda t} F(t) dt + \int_0^\infty e^{-\lambda t} f\left[\frac{F(t+\cdot) - F(t)}{1 - F(t)}\right] (1-F(t)dt ,$$

where the last integrand is 0 if $F(t) = 1$. Now if $h_n \to h$, with corresponding $F_n \to F$, the first term on the right obviously converges to its limit with F in place of F_n. Also, if $F(t) < 1$ then

$$\frac{F_n(t+\cdot) - F_n(t)}{1 - F(t)}$$ has at most two weak limit points as $n \to \infty$:

$$\frac{F(t+\cdot) - F(t)}{1 - F(t)} \text{ and } \frac{F(t+\cdot) - F(t-)}{1 - F(t-)}.$$ Thus at continuity points t of F with $F(t) < 1$ it converges to the same limit. Since f is bounded it is easy to see that the contribution to the last integral for $t > \inf\{t : F(t) = 1\}$ tends to 0 as $n \to \infty$. Hence by dominated convergence, the last integrals also converge to their value at F, completing the proof.

REMARK. It follows immediately that Conjecture 2.10 of Essay I holds for H_J.

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