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THE CIRCULAR FUNCTIONAL MODEL

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Consistent estimation of the structural parameters in a circular functional relationship between two variables observed with error is given, and a practical procedure for computing estimates is proposed.

1. Introduction. The circular relationship model is defined as follows. Let $x_i = \xi + \rho\cos\tau_i + \delta_i,\ y_i = \eta + \rho\sin\tau_i + \varepsilon_i \quad (i = 1, \cdots, n)$, where the n pairs $(\delta_1, \varepsilon_1), \cdots, (\delta_n, \varepsilon_n)$ are independent and normally distributed with zero mean and covariance matrix $\sigma^2 I$ (I being the identity matrix of order 2), ξ, η, ρ and σ^2 are the structural parameters, and the τ_i are either incidental parameters in the functional model or independent and identically distributed in the structural model. Chan (1965) used the method of minimum squared distance to estimate the structural parameters in the functional model and considered consistency of these estimates. Anderson (1981) examined the maximum likelihood estimation of the parameters, assuming a uniform distribution over $[0, 2\pi)$ for the τ_i, and highlighted the difficulties involved in an adequate analysis of the functional model. Berman (1983) estimated the parameters of a circle when angular differences between successive data points are known. Both of them considered the data set of the Brogar ring given by Thom and Thom (1973). In §2, we modify the estimating equations of Chan (1965, equations (8)) to yield a set of consistent estimators of the structural parameters without invoking the additional assumptions of a uniform distribution for the τ_i or of known angular differences, and indicate the way to derive their asymptotic covariance matrix. Our method of estimation can also be applied to data that lie on a small section of an arc, as, e.g., given in the survey data of the stone 'circles' at Avebury by Thom, Thom and Foord (1976). A practical procedure for computing these estimates is provided in § 3 and illustrated by the numerical examples given in Chan (1965, §6) and a Monte Carlo simulation, and by the above two data sets in archaeology.

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2. Consistent Estimation. Denote by r_i the distance

$$\sqrt{(x_i-\xi)^2+(y_i-\eta)^2}$$

and write $\bar{r}=\Sigma r_i/n$. Under the normality assumption on (δ_i,ε_i), r_i^2/σ^2 has a noncentral χ^2 distribution with noncentrality parameter ρ^2, and we have (Chan, 1965, equation (3))

$$E(r_i)=\left(\frac{1}{2}\pi\right)^{\frac{1}{2}}\sigma_1F_1\left(-\frac{1}{2};1;-\frac{1}{2}\rho^2/\sigma^2\right),$$

where ${}_1F_1$ denotes the confluent hypergeometric function; for simplicity, the right hand expression will be written as $f(\rho^2)$. Denote by f' the derivative of f with respect to ρ^2.

By minimizing the sum of squared shortest distances from the observations to the circumference of the circle, its centre (ξ,η) can be estimated by the solution $(\hat{\xi},\hat{\eta})$ of the system of equations

$$\begin{aligned}\Sigma(r_i-\bar{r})(x_i-\xi)/r_i&=0,\\ \Sigma(r_i-\bar{r})(y_i-\eta)/r_i&=0.\end{aligned}$$

Since (Chan, 1965, equation (12))

$$\begin{aligned}E\left\{\frac{1}{n-1}\Sigma(r_i-\bar{r})(x_i-\xi)/r_i\right\}&=(n^{-1}\Sigma\cos\tau_i)\rho\{1-2f(\rho^2)f'(\rho^2)\},\\ E\left\{\frac{1}{n-1}\Sigma(r_i-\bar{r})(y_i-\eta)/r_i\right\}&=(n^{-1}\Sigma\sin\tau_i)\rho\{1-2f(\rho^2)f'(\rho^2)\},\end{aligned}$$

and $1-2f(\rho^2)f'(\rho^2)\neq 0$, the estimator $(\hat{\xi},\hat{\eta})$ is in general not consistent in view of the Theorem in Chan (1965, §4). On noting that

$$E(\bar{x})=(n^{-1}\Sigma\cos\tau_i)\rho+\xi,\quad E(\bar{y})=(n^{-1}\Sigma\sin\tau_i)\rho+\eta,$$

we would consider the following consistent estimating equations:

$$\begin{aligned}&\bar{r}n^{-1}\Sigma(x_i-\xi)/r_i+(\xi-\bar{x})n^{-1}\{2(n-1)f(\rho^2)f'(\rho^2)+1\}=0,\\ &\bar{r}n^{-1}\Sigma(y_i-n)/r_i+(\eta-\bar{y})n^{-1}\{2(n-1)f(\rho^2)f'(\rho^2)+1\}=0,\\ &\bar{r}-f(\rho^2)=0,\\ &(n-1)^{-1}\Sigma(r_i-\bar{r})^2-2\sigma^2-\rho^2+f^2(\rho^2)=0;\end{aligned}\tag{2.1}$$

the left hand sides of the first two equations in (2.1) have now zero expectations, and again by the Theorem in Chan (1965, §4), these ensure the consistency of the resulting estimators; the last two equations are introduced because the first two equations involve also the parameters ρ and σ; and the left hand sides, $\psi_i(\theta)$, $i=1,\cdots,4$, say, of equations (2.1) with $\theta=(\theta_1,\theta_2,\theta_3,\theta_4)=$

$(\xi, \eta, \rho, \sigma)$, all have zero expectations. It is interesting to notice the similarity between equations (2.1) and the maximum likelihood equations for the structural model given by Anderson (1981, equations (4.2) - (4.5)) in which the existence and uniqueness of maximum likelihood estimate are considered. In our simulation studies (given in §3), no difficulty was encountered in solving equations (2.1) using the Newton-Raphson algorithm.

Under some mild conditions on the limiting behavior of the sequence $(\tau_1, \tau_2, \cdots)$, it can be proved, using arguments similar to those of Mak (1982), that the estimators $\tilde{\theta}$ obtained as solutions of equations (2.1) are consistent and asymptotically normal with asymptotic covariance matrix $P^{-1}Q(P^{-1})'$, where $P = [E(\partial\psi_i/\partial\theta_j)]$ and $Q = [E(\psi_i\psi_j)]$. To evaluate the 4×4 matrix P, the formulae given in §5.4 of Chan (1965) may be used. The evaluation of the elements of the matrix Q involves certain complications, which we omit here. However, both P and Q can be estimated numerically by the method indicated in §3 of Chan & Mak (1984).

3. A Practical Procedure. In solving equations (2.1), we encountered the difficulties of evaluating the confluent hypergeometric function ${}_1F_1$ and its inverse. Tables of the confluent hypergeometric function are available, for example, in Slater (1960). For ρ/σ large, Slater (1960, equation (4.1.2)) gives an asymptotic series for the simplified version f:

$$f(\rho^2) = \rho + \frac{1}{2}\sigma^2\rho^{-1} + \frac{1}{8}\sigma^4\rho^{-3} + \frac{3}{16}\sigma^6\rho^{-5} + \cdots, \qquad \rho/\sigma > 5$$

and thus we obtain the useful approximations

$$f'(\rho^2) = \frac{1}{2}\rho^{-1} - \frac{1}{4}\sigma^2\rho^{-3} - \frac{3}{16}\sigma^4\rho^{-5} - \cdots,$$
$$2f(\rho^2)f'(\rho^2) = 1 - \frac{1}{2}(\sigma/\rho)^4 + \cdots.$$

Moreover, we have the approximations:

$$f(\rho^2) \doteq \sqrt{\rho^2 + \sigma^2},$$
$$2\sigma^2 + \rho^2 - f^2(\rho^2) \doteq \sigma^2.$$

For practical considerations, equations (2.1) can thus be approximated by

$$\begin{aligned}
&\bar{r}n^{-1}\Sigma(x_i - \xi)/r_i + (\xi - \bar{x})n^{-1}\{(n-1)(1 - \tfrac{1}{2}\sigma^4/\rho^4) + 1\} = 0, \\
&\bar{r}n^{-1}\Sigma(y_i - \eta)/r_i + (\eta - \bar{y})n^{-1}\{(n-1)(1 - \tfrac{1}{2}\sigma^4/\rho^4) + 1\} = 0, \\
&\bar{r} - \sqrt{\rho^2 + \sigma^2} = 0, \\
&(n-3)^{-1}\Sigma(r_i - \bar{r})^2 - \sigma^2 = 0.
\end{aligned} \tag{3.1}$$

Using the non-linear Newton-Raphson iterative method, we solve for $\theta = (\xi, \eta, \rho, \sigma)$ equations (3.1) as follows: Write

$$\begin{aligned} p_i &= x_i - \xi, & g_i &= y_i - \eta, & r_i &= (p_i^2 + g_i^2)^{\frac{1}{2}}, \\ z_i &= r_i^{-1}, & s_i &= p_i z_i, & t_i &= g_i z_i, \\ u_i &= p_i^2 z_i^3, & v_i &= p_i g_i z_i^3, & w_i &= g_i^2 z_i^3, \end{aligned}$$

$p, g, \cdots, w$ for their arithmetic means, respectively (that is, $p = \Sigma p_i/n$, etc. and note that $r = \bar{r}$), and

$$\begin{aligned} c &= n^{-1}\{(n-1)(1 - \frac{1}{2}\sigma^4/\rho^4) + 1\}, \\ p_0 &= c(\bar{x} - \xi), \\ g_0 &= c(\bar{y} - \eta). \end{aligned}$$

If $\theta^{(k)} = (\xi^{(k)}, \eta^{(k)}, \rho^{(k)}, \sigma^{(k)})$ represents the k-th iteration for θ, the $(k+1)$-th iteration is

$$\begin{aligned} \sigma^{(k+1)} =& \sqrt{(n-3)^{-1}\Sigma(r_i - \bar{r})^2}\big|_{\theta=\theta^{(k)}}, \\ \rho^{(k+1)} =& \sqrt{\bar{r}^2 - \sigma^2}\big|_{\theta=\theta^{(k)}}, \\ \xi^{(k+1)} =& \xi^{(k)} \\ &+ \frac{c(p_0 - rs) - t(p_0 t - g_0 s) - r(p_0 u + g_0 v) + r^2(su + tv)}{c(c - rz - s^2 - t^2) + r(s^2 u + 2stv + t^2 w) + r^2(wu - v^2)}\Big|_{\theta=\theta^{(k)}}, \\ \eta^{(k+1)} =& \eta^{(k)} \\ &+ \frac{c(g_0 - rt) - s(g_0 s - p_0 t) - r(g_0 w + p_0 v) + r^2(tw + sv)}{c^2 - c(rz + s^2 + t^2) + r(s^2 u + 2stv + t^2 w) + r^2(wu - v^2)}\Big|_{\theta=\theta^{(k)}}. \end{aligned} \tag{3.2}$$

A convenient initial value $\theta^{(0)} = (\xi^{(0)}, \eta^{(0)}, \rho^{(0)}, \sigma^{(0)})$ in the iterative procedure is (Anderson, 1981, §5)

$$\xi^{(0)} = \bar{x}, \qquad \eta^{(0)} = \bar{y}, \qquad \rho^{(0)} = n^{-1}\Sigma\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\}^{\frac{1}{2}}$$

and

$$\sigma^{(0)} = \sqrt{n^{-1}\Sigma\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2 - (\rho^{(0)})^2\}}.$$

Consider the sets of simulated data given as examples in Chan (1965, §6):

EXAMPLE (a). n =12, θ = (0, 0, 10, 1) and the τ_i are simulated from a uniform distribution over the interval [0, 2π) but regarded as incidental parameters. The old estimate using the method of minimum squared distance (Chan, 1965) is $\hat{\theta}$ = (-0.7986, 0.1045, 9.6435, 0.7057). Using (3.2), the new estimate becomes $\tilde{\theta}$ = (-0.7986, 0.1044, 9.6435, 0.7057) which differs very little from the old estimate.

EXAMPLE (b). $n = 12, \theta = (0,\ 0,\ 20,\ 1)$ and the τ_i are confined to be uniformly distributed over the interval $[0,\ \frac{1}{2}\pi)$. The old estimate is $\hat{\theta} =$ (0.1601, -1.3399, 21.2263, 0.7357). By (3.2), the new estimate yields $\tilde{\theta} =$ (0.1588, -1.3413, 21.2263, 0.7357). There is a small difference between the two estimates of the centre $(\xi,\ \eta)$.

EXAMPLE (c). A Monte Carlo study based on 500 samples of size n = 240 was carried out on an IBM 4381 computer. $\theta = (0,\ 0,\ 7,\ 1)$ and the τ_i are confined to an arc of 135°. The results by the old and new methods are displayed below.

θ	Mean		Mean squared error	
	Old (c=1)	New	Old(c=1)	New
ξ	0.014753	0.002264	0.031931	0.031955
η	0.022858	-0.007524	0.121362	0.122234
ρ	6.989020	7.014830	0.092544	0.093797
σ	0.988243	0.988262	0.002291	0.002291

The new estimate of the centre seemed to yield a smaller bias than the old estimate.

Consider the two megalithic stone rings for which accurate survey data are available:

EXAMPLE (d). The ring of Brogar in the Orkney Islands surveyed by Thom and Thom (1973). If the two outliers, stones 7 and 8, are removed, then n =33, and our estimate is $(\tilde{\rho}, \tilde{\sigma}^2) = (170.404,\ 1.4822)$ which is quite close to Anderson's estimate of (170.41, 1.3574). By the method indicated at the end of §2, the approximate 95 per cent confidence interval for the diameter is 340.81 ± 0.71 ft. Anderson and Berman gave as the confidence intervals 340.82 ± 0.81 ft and 340.76 ± 1.88 ft, respectively. With stones 7 and 8 included (n = 35), our analysis yields an approximate 95 per cent confidence interval of 339.935 ± 1.268 ft as against their intervals of 339.94 ± 1.21 ft and 339.96 ± 1.90 ft, respectively.

EXAMPLE (e). The stone 'circles' at Avebury surveyed by Thom, Thom and Foord (1976) who split the data up into four circular arcs each of which is only 20°. By our method, the estimates $\tilde{\theta}$ with n = 7, 16, 10 for the first three arcs are respectively (530.84, 650.96, 638.78, 2.367), (1472.00, 1553.37, 1840.44, 1.945) and (794.98, 516.48, 782.83, 3.001) which are all very close to the old estimates due to the small values in the ratio $\tilde{\sigma}/\tilde{\rho}$. For the fourth arc, the iterative process diverged and the difficulties were also discussed in Anderson (1981).

Finally, in the circular functional model of § 1, the incidental parameters τ_i may simply be estimated by

$$\tilde{\tau}_i = \arctan\{(y_i - \tilde{\eta})/(x_i - \tilde{\xi})\}.$$

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