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ALLOCATION THROUGH STOCHASTIC SCHUR CONVEXITY AND STOCHASTIC TRANSPOSITION INCREASINGNESS[1]

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Consider a stochastic allocation problem where a total resource of R units are to be allocated among m competing facilities in a system. An allocation of r_i units to facility i results in a random response $X_i(r_i), i = 1, \ldots, m$. The system response is then defined by the random variable $Y(\mathbf{r}) = h(X_1(r_1), \ldots, X_m(r_m))$ where $h : \mathbb{R}^m \to \mathbb{R}$ is the system performance function. Let $\mathcal{S} \subset \mathbb{R}_+^m$ be the set of all feasible allocations. We are then interested in the stochastic allocation problem $\min\{Eg(Y(\mathbf{r})) : \sum_{i=1}^m r_i = R, \mathbf{r} \in \mathcal{S}\}$ for some utility function g. The aim of the paper is to obtain a partial or a full characterization of the optimal solution to this problem with minimal restriction on g. For this we introduce notions of *stochastic Schur convexity* and *stochastic transposition increasingness* and identify sufficient conditions on $X_i(r_i), i = 1, \ldots, m$ and h under which $Y(\mathbf{r})$ will be either stochastically Schur convex or transposition increasing with respect to $\mathbf{r}$. Then under appropriate condition on g it can be shown that the stochastic Schur convexity of $Y(\mathbf{r})$ will imply the optimality of balanced resource allocation and the transposition increasingness will imply a partial characterization of the optimal solution thus reducing the computational effort needed to find the optimal solution. Several examples in the telecommunication, manufacturing and reliability/performability systems are presented to illustrate the main results of this paper.

1. Introduction

Consider a system consisting of m facilities that compete for a limited resource with a capacity of R units. An allocation of r_i units to facility i results in a random response $X_i(r_i), i = 1, \ldots, m$. The overall system response

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is then defined by the random variable $Y(\mathbf{r}) = h(X_1(r_1), \ldots, X_m(r_m))$ where $h : \mathbb{R}^m \to \mathbb{R}$ is the system performance function. Let $\mathcal{S} \subset \mathbb{R}_+^m$ be the set of all feasible allocations. In this paper we are interested in the stochastic allocation problem

$$\min\{Eg(Y(\mathbf{r})) : \sum_{i=1}^{m} r_i = R, \mathbf{r} \in \mathcal{S}\}, \tag{1.1}$$

where g is an appropriate utility function chosen by the decision maker. For example consider a flexible manufacturing system consisting of m machine cells. A total of R flexible machines are available that needs to be allocated among the m machine cells. Let $X_i(r_i)$ be the stationary number of parts in cell i if r_i flexible machines are allocated to cell $i, i = 1, \ldots, m$. Suppose the performance of the system is measured by the total number of parts in it: i.e. $h(\mathbf{x}) = \sum_{i=1}^m x_i, \mathbf{x} \in \mathcal{Z}_+^m$. A stochastic allocation problem for this scenario is then to obtain, if possible, an optimal allocation that minimizes $\sum_{i=1}^m X_i(r_i)$ in the usual stochastic sense: i.e. we restrict g to be an increasing but an arbitrary function.

The purpose of this paper is to obtain a partial or a full characterization of the optimal solution to the stochastic allocation problem with minimal restriction on g. For this we define notions of *stochastic Schur convexity* (Section 2) and *stochastic transposition increasingness* (Section 3) and find sufficient conditions on $X_i(r_i), i = 1, \ldots, m$ and h under which $Y(\mathbf{r})$ will be either stochastically Schur convex or transposition increasing with respect to $\mathbf{r}$. Then under appropriate condition on g it will be shown that the Schur convexity of $Y(\mathbf{r})$ will imply the optimality of balanced resource allocation (Section 2) and the transposition increasingness will imply a partial characterization of the optimal solution thus reducing the computational effort needed to obtain the optimal solution (Section 3). Several examples in the telecommunication, manufacturing and reliability/performability systems are presented in Section 4 to illustrate the main results of this paper.

2. Stochastic Schur Convexity

In this section we will define stochastic Schur convexity and give sufficient conditions on $X_i(r_i), i = 1, \ldots, m$ and h under which $Y(\mathbf{r})$ is stochastically schur convex with respect to $\mathbf{r}$. For this we will need the following definitions of majorization and Schur functions (e.g. see Marshall and Olkin (1979) for more details). For any $\mathbf{x} \in \mathbb{R}^m, x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[m]}$ denotes the decreasing rearrangement of the coordinates of $\mathbf{x}$ and for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m, \mathbf{x} \geq \mathbf{y}$ denotes the usual coordinatewise ordering. Throughout this paper the terms 'increasing' and 'decreasing' are not used in the strict sense.

DEFINITION 2.1 Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Then $\mathbf{x}$ *majorizes* $\mathbf{y}$ if

$$(2.1) \qquad \sum_{i=1}^{k} x_{[i]} \geq \sum_{i=1}^{k} y_{[i]}, k = 1, \ldots, m, \text{ and } \sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]},$$

or equivalently

$$(2.2) \qquad \sum_{i=k}^{m} x_{[i]} \leq \sum_{i=k}^{m} y_{[i]}, k = 1, \ldots, m, \text{ and } \sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]}.$$

We denote this $\mathbf{x} \geq_m \mathbf{y}$. When the requirement of the equality $\sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]}$ is dropped from (2.1) [(2.2)] we say that $\mathbf{x}$ *weakly sub-majorizes* [*sup-majorizes*] $\mathbf{y}$ and is denoted $\mathbf{x} \geq_{wm} [\geq^{wm}]\mathbf{y}$.

The following lemma (e.g. see Marshall and Olkin (1979)) allows one to simplify the analysis of majorization, often making it sufficient to prove the desired result just for the two dimensional case.

LEMMA 2.2

(i) $\mathbf{x} \geq_m \mathbf{y} \Leftrightarrow$ *there exist a finite number (say k) of vectors* $\mathbf{x}^{(i)}, i = 1, \ldots, k$ *such that* $\mathbf{x} = \mathbf{x}^{(1)} \geq_m \cdots \geq_m \mathbf{x}^{(k)} = \mathbf{y}$ *and such that* $\mathbf{x}^{(i)}$ *and* $\mathbf{x}^{(i+1)}$ *differ in two coordinates only,* $i = 1, \ldots, k-1$.

(ii) $\mathbf{x} \geq_{wm} \mathbf{y} \Leftrightarrow$ *there exists a vector* $\mathbf{z}$ *such that* $\mathbf{x} \geq \mathbf{z}$ *and* $\mathbf{z} \geq_m \mathbf{y}$.

(iii) $\mathbf{x} \geq^{wm} \mathbf{y} \Leftrightarrow$ *there exists a vector* $\mathbf{z}$ *such that* $\mathbf{x} \leq \mathbf{z}$ *and* $\mathbf{z} \geq_m \mathbf{y}$.

DEFINITION 2.3 A function $\phi : \mathbb{R}^m \to \mathbb{R}$ is *Schur convex* [*concave*] if $\mathbf{x} \geq_m \mathbf{y}$ implies $\phi(\mathbf{x}) \geq [\leq]\phi(\mathbf{y})$. It is *increasing* Schur convex [concave] if it is increasing and Schur convex [concave]; i.e. (see Lemma 2.2 (ii) [(iii)]), if $\mathbf{x} \geq_{wm} [\geq^{wm}]\mathbf{y}$ implies $\phi(\mathbf{x}) \geq [\leq]\phi(\mathbf{y})$.

Note that all Schur convex and Schur concave functions are symmetric: i.e. for any permutation π of $\{1, \ldots, m\}, \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{x}_\pi = (x_{\pi(i)}, i = 1, \ldots, m)$ one has $\phi(\mathbf{x}_\pi) = \phi(\mathbf{x})$. Recall that a random variable V is said to be larger than a random variable W in the sense of the usual stochastic [increasing convex, increasing concave] ordering if $E\psi(V) \geq E\psi(W)$ for all increasing [increasing and convex, increasing and concave] functions ψ. We denote this $V \geq_{st} [\geq_{icx}, \geq_{icv}]W$ (e.g. see Ross (1983)).

DEFINITION 2.4 *(Stochastic Schur Convexity)*: A real valued random variable $Z(\mathbf{x})$ parametrized by $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^m$ is *stochastically Schur convex* in the sense of the usual stochastic [increasing convex, increasing concave] ordering if for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}, \mathbf{x} \geq_m \mathbf{y}$ implies $Z(\mathbf{x}) \geq_{st} [\geq_{icx}, \geq_{icv}] Z(\mathbf{y})$. We denote this $\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{X}\} \in S - SchurCX(st)[S - SchurCX(icx), S - SchurCX(icv)]$.

If in addition $Z(\mathbf{x})$ is stochastically increasing [decreasing] then $Z(\mathbf{x})$ is stochastically increasing [decreasing] and Schur convex. We denote this $\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{X}\} \in SI - SchurCX(st)$ $[\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{X}\} \in SD - SchurCX(st)]$ etc. That is (see Lemma 2.2 (ii) [(iii)]), for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}, \mathbf{x} \geq_{wm} [\geq^{wm}]\mathbf{y}$ implies $Z(\mathbf{x}) \geq_{st} [\geq_{st}] Z(\mathbf{y})$ etc.

If $-Z(\mathbf{x})$ is stochastically Schur convex then we say that $Z(\mathbf{x})$ is stochastically Schur concave.

As an immediate consequence of the definition of stochastic Schur convexity one has the following characterization of the optimal solution to the allocation problem (1.1).

THEOREM 2.5 *Let* $Y(\mathbf{r}) = h(X_1(r_1), \ldots, X_m(r_m))$. *Suppose* $\{Y(\mathbf{r}), \mathbf{r} \in \mathcal{S}\} \in S - SchurCX(st)[S - SchurCX(icx), S - SchurCX(icv)]$. *Then for any increasing [increasing and convex, increasing and concave] function g and* $\mathbf{r}, \mathbf{s} \in \mathcal{S}$ *one has,*

$$\mathbf{r} \geq_m \mathbf{s} \Rightarrow Eg \circ h(X_1(r_1), \ldots, X_m(r_m)) \geq Eg \circ h(X_1(s_1), \ldots, X_m(s_m)).$$

Suppose $\{Y(\mathbf{r}), \mathbf{r} \in \mathcal{S}\} \in SI - SchurCX(st)[SI - SchurCX(icx), SI - SchurCX(icv)]$. *Then for any increasing [increasing and convex, increasing and concave] function g and* $\mathbf{r}, \mathbf{s} \in \mathcal{S}$ *one has,*

$$\mathbf{r} \geq_{wm} \mathbf{s} \Rightarrow Eg \circ h(X_1(r_1), \ldots, X_m(r_m)) \geq Eg \circ h(X_1(s_1), \ldots, X_m(s_m)).$$

In either case if $\mathbf{r}^* := (\frac{R}{m}, \ldots, \frac{R}{m})$ *is in* $\mathcal{S}$ *then*

$$Eg \circ h(X_1(r_1), \ldots, X_m(r_m)) \geq Eg \circ h(X_1(\frac{R}{m}), \ldots, X_m(\frac{R}{m})),$$

i.e. $\mathbf{r}^*$ *is an optimal solution to problem* (1.1).

From the above theorem it is clear that it is worthwhile to search for sufficient conditions on $X_i(r_i), i = 1, \ldots, m$ and h for $Y(\mathbf{r})$ to be stochastically Schur convex. We shall do this next in this section (and examples where such conditions are naturally satisfied are given in Section 4.) Before that we note the following easily verified lemma.

LEMMA 2.6 *Let* $F_Z(t; \mathbf{x}) = P\{Z(\mathbf{x}) \leq t\}$ *and* $\bar{F}_Z(t; \mathbf{x}) = P\{Z(\mathbf{x}) > t\}$ *be respectively, the cumulative and survival functions of* $Z(\mathbf{x})$. *Then* $\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{X}\} \in S - SchurCX(st)[S - SchurCX(icx), S - SchurCX(icv)] \Leftrightarrow \bar{F}_Z(t; \mathbf{x})[\int_t^\infty \bar{F}_Z(s; \mathbf{x})ds, -\int_{-\infty}^t F_Z(s; \mathbf{x})ds]$ *is Schur convex in* $\mathbf{x}$.

In the remainder of this section we will assume that $\{X_i(\theta), \theta \in \Theta\}, i = 1, \ldots, m$ are m probabilistically identical and mutually independent collections of random variables. We will now present a result on stochastic majorization that extends a result of Proschan and Sethuraman (1976) for random variables with proportional hazard rates. We will need the following definition for this result (see Example 4.3 of Shaked and Shanthikumar (1990a)).

DEFINITION 2.7 (*Stochastic Convexity in the Hazard Rate*): Let $\{Z(\theta), \theta \in \Theta\}$ be a collection of absolutely continuous positive random variables with hazard rate functions $\{\gamma(\cdot;\theta), \theta \in \Theta\}$. Then $\{Z(\theta), \theta \in \Theta\}$ is said to be stochastically increasing and convex in the sense of hazard rate ordering if $\gamma(t;\theta)$ is pointwise decreasing and concave in θ for each fixed $t \in \mathbb{R}_+$. We denote this $\{Z(\theta), \theta \in \Theta\} \in SICX(hr)$.

REMARK 2.8 Observe that since $\bar{F}(t;\theta) = P\{Z(\theta) > t\} = exp\{-\int_{u=0}^{t} \gamma(u;\theta)du\}, t \in \mathbb{R}_+$ from Theorem 3.16 of Shaked and Shanthikumar (1990a) it follows that $\{Z(\theta), \theta \in \Theta\} \in SICX(hr) \Rightarrow \{Z(\theta), \theta \in \Theta\} \in SICX(st) \Rightarrow \{Z(\theta), \theta \in \Theta\} \in SICX(sp) \Rightarrow \{Z(\theta), \theta \in \Theta\} \in SICX$.

THEOREM 2.9 *Suppose $\{X_i(\theta), \theta \in \Theta\} \in SICX(hr)$. Then for any increasing and symmetric function h and $Y(\mathbf{x}) = h(X_1(x_1), \ldots, X_m(x_m))$ one has $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in SI - SchurCX(st)$. That is for any $\mathbf{x}, \mathbf{y} \in \Theta^m$, increasing symmetric function h and increasing function g,*

$$\begin{aligned} (2.3) \qquad \mathbf{x} \geq_{wm} \mathbf{y} \quad \rightarrow \quad & Eg \circ h(X_1(x_1), \ldots, X_m(x_m)) \\ \geq \quad & Eg \circ h(X_1(y_1), \ldots, X_m(y_m)). \end{aligned}$$

Since a Schur convex function is a symmetric function, one also has

$$\begin{aligned} (2.4) \qquad \mathbf{x} \geq_{wm} \mathbf{y} \quad \rightarrow \quad & (X_1(x_1), \ldots, X_m(x_m)) \\ \geq_{wm:st} \quad & (X_1(y_1), \ldots, X_m(y_m)). \end{aligned}$$

Here $\mathbf{V} \geq_{wm:st} \mathbf{W}$ stands for the weak stochastic majorization order of two random vectors $\mathbf{V}$ and $\mathbf{W}$. That is $Ef(\mathbf{V}) \geq Ef(\mathbf{W})$ for all increasing Schur convex function f (see Marshall and Olkin (1979, Chapter 11G)*).*

PROOF We will first establish the theorem for the case $m = 2$ and $\mathbf{x} \geq_m \mathbf{y}$. Then the result for the case $\mathbf{x} \geq_{wm} \mathbf{y}$ will follow from Lemma 2.2 (ii) and the stochastic monotonicity of $X_i(\theta)$ in θ. Suppose, without a loss of generality, $x_1 \leq y_1 \leq y_2 \leq x_2$ and $x_1 + x_2 = y_1 + y_2$. Define

$$(2.5) \qquad \hat{\gamma}(t;y_1) = \gamma(t;x_1) + \frac{y_1 - x_1}{x_2 - x_1}\{\gamma(t;x_2) - \gamma(t;x_1)\}, t \in \mathbb{R}_+,$$

and

$$\hat{\gamma}(t;y_2) = \gamma(t;x_1) + \frac{y_2 - x_1}{x_2 - x_1}\{\gamma(t;x_2) - \gamma(t;x_1)\}, t \in \mathbb{R}_+. \tag{2.6}$$

From the concavity of the hazard rate function $\gamma(\cdot;\theta)$ in θ it is immediate that

$$\hat{\gamma}(t;y_i) \leq \gamma(t;y_i), t \in \mathbb{R}_+, i = 1,2. \tag{2.7}$$

Let $Z_i, \hat{Z}_i, i = 1,2$ be four mutually independent random variables such that $Z_i =^{st} X_i(x_i), i = 1,2$ and $\hat{Z}_i$ has a hazard rate function $\hat{\gamma}(\cdot;y_i), i = 1,2$. Then from (2.7) one sees that $\hat{Z}_i$ is larger than $X_i(y_i)$ in the hazard rate ordering, $i = 1,2$ (e.g. see Ross (1983)). That is

$$\hat{Z}_i \geq_{hr} X_i(y_i), i = 1,2. \tag{2.8}$$

Observe that the hazard rate function of $\min\{\hat{Z}_1, \hat{Z}_2\}$ is $\sum_{i=1}^2 \hat{\gamma}(\cdot;y_i) = \sum_{i=1}^2 \gamma(\cdot;x_i)$ which is the same as that for $\min\{Z_1, Z_2\}$. Therefore

$$\min\{\hat{Z}_1, \hat{Z}_2\} =^{st} \min\{Z_1, Z_2\}. \tag{2.9}$$

For any $s,t \in \mathbb{R}_+$ such that $t \geq s$, consider

$$\begin{aligned} &P\{\max\{Z_1,Z_2\} > t \mid \min\{Z_1,Z_2\} = s\} \\ &= \{\frac{f(s;x_1)\bar{F}(s;x_2)}{f(s;x_1)\bar{F}(s;x_2) + f(s;x_2)\bar{F}(s;x_1)}\}\frac{\bar{F}(t;x_2)}{\bar{F}(s;x_2)} \\ &\quad +\{\frac{f(s;x_2)\bar{F}(s;x_1)}{f(s;x_1)\bar{F}(s;x_2) + f(s;x_2)\bar{F}(s;x_1)}\}\frac{\bar{F}(t;x_1)}{\bar{F}(s;x_1)} \\ &= \frac{\gamma(s;x_1)exp\{-\int_s^t \gamma(u;x_2)du\} + \gamma(s;x_2)exp\{-\int_s^t \gamma(u;x_1)du\}}{\gamma(s;x_1) + \gamma(s;x_2)}. \end{aligned} \tag{2.10}$$

Similarly one sees that

$$\begin{aligned} &(P\{\max\{\hat{Z}_1,\hat{Z}_2\} > t \mid \min\{\hat{Z}_1,\hat{Z}_2\} = s\} \\ &= \frac{\hat{\gamma}(s;y_1)exp\{-\int_s^t \hat{\gamma}(u;y_2)du\} + \hat{\gamma}(s;y_2)exp\{-\int_s^t \hat{\gamma}(u;y_1)du\}}{\hat{\gamma}(s;y_1) + \hat{\gamma}(s;y_2)}. \end{aligned} \tag{2.11}$$

Since $\gamma(u;x_1) \geq \hat{\gamma}(u;y_1) \geq \hat{\gamma}(u;y_2) \geq \gamma(u;x_2)$ and $\gamma(u;x_1) + \gamma(u;x_2) = \hat{\gamma}(u;y_1) + \hat{\gamma}(u;y_2)$ one has

$$\begin{gathered} \gamma(s;x_1)exp\{-\int_s^t \gamma(u;x_2)du\} + \gamma(s;x_2)exp\{-\int_s^t \gamma(u;x_1)du\} \geq \\ \hat{\gamma}(s;y_1)exp\{-\int_s^t \hat{\gamma}(u;y_2)du\} + \hat{\gamma}(s;y_2)exp\{-\int_s^t \hat{\gamma}(u;y_1)du\} \\ \text{for all } u \in \mathbb{R}_+. \end{gathered}$$

Therefore, from (2.10) and (2.11) one sees that

$$(2.12) \qquad P\{\max\{Z_1, Z_2\} > t | \min\{Z_1, Z_2\} = s\} \geq P\{\max\{\hat{Z}_1, \hat{Z}_2\} > t | \min\{\hat{Z}_1, \hat{Z}_2\} = s\}, t \in \mathbb{R}_+.$$

Now combining (2.9) and (2.12) one concludes that

$$(2.13) \quad (\min\{Z_1, Z_2\}, \max\{Z_1, Z_2\}) \geq_{st} (\min\{\hat{Z}_1, \hat{Z}_2\}, \max\{\hat{Z}_1, \hat{Z}_2\}).$$

Then from (2.8) one has

$$(\min\{\hat{Z}_1, \hat{Z}_2\}, \max\{\hat{Z}_1, \hat{Z}_2\}) \geq_{st} (\min\{X_1(y_1), X_2(y_2)\}, \max\{X_1(y_1), X_2(y_2)\})$$

and hence for any increasing symmetric function h

$$h(X_1(x_1), X_2(x_2)) \geq_{st} h(X_1(y_1), X_2(y_2)).$$

That is we have established the theorem for $m = 2$. Extension to the general case can be routinely carried out using Lemma 2.2 (i) (e.g. see Marshall and Olkin (1979), Shanthikumar (1987)).

Proschan and Sethuraman (1976) showed that if the random variables $\{X_i(\theta), \theta \in \Theta\}$ have proportional hazard rates, i.e. if $\gamma(u;\theta) = \theta\alpha(u), \theta \in \Theta$ for each fixed u, then (2.4) holds. Here we have established a stronger conclusion (2.3) with a condition weaker than the proportionality of the hazard rates.

For the above result we need the strong condition of stochastic convexity in the hazard rate on $\{X_i(\theta), \theta \in \Theta\}$. We will next present a result that is weaker than Theorem 2.9, but requires only a weaker stochastic convexity condition on $\{X_i(\theta), \theta \in \Theta\}$. For this we will need the following definitions.

DEFINITION 2.10 (*Stochastic Convexity in the Usual Stochastic Ordering*): Let $\{Z(\theta), \theta \in \Theta\}$ be a collection of random variables with survival function $\bar{F}(\cdot;\theta)$. Then $\{Z(\theta), \theta \in \Theta\}$ is said to be stochastically increasing and linear [convex, concave] in the sense of usual stochastic ordering (see Shaked and Shanthikumar 1990a) if $\bar{F}(t;\theta)$ is pointwise increasing and linear [convex, concave] in θ for each fixed t. We denote this $\{Z(\theta), \theta \in \Theta\} \in SIL(st)[SICX(st), SICV(st)]$.

DEFINITION 2.11 A function $\phi : \mathbb{R}^m \to \mathbb{R}$ is submodular [supermodular] if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ we have

$$\phi(\mathbf{x}) + \phi(\mathbf{y}) \geq [\leq] \phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}).$$

Here $\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \dots, \max\{x_m, y_m\})$ and $\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \dots, \min\{x_m, y_m\})$. Now we can present the next theorem.

THEOREM 2.12 *Suppose* $\{X_i(\theta), \theta \in \Theta\} \in SIL(st)$. *Then for any increasing symmetric submodular [supermodular] function h and* $Y(\mathbf{x}) = h(X_1(x_1), \ldots, X_m(x_m))$ *one has* $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in SI\text{-}SchurCX(icv)[SI\text{-}SchurCV(icx)]$. *That is for any* $\mathbf{x}, \mathbf{y} \in \Theta^m$, *increasing and symmetric submodular [supermodular] function h and increasing and concave [convex] function g,*

$$\text{(2.14)} \qquad \begin{aligned} \mathbf{x} \geq_{wm} [\geq^{wm}] \ \mathbf{y} \rightarrow & \ Eg \circ h(X_1(x_1), \ldots, X_m(x_m)) \\ \geq [\leq] & \ Eg \circ h(X_1(y_1), \ldots, X_m(y_m)). \end{aligned}$$

PROOF We will first establish the theorem for the case $m = 2$ and $\mathbf{x} \geq_m \mathbf{y}$. Then the result for the case $\mathbf{x} \geq_{wm} [\geq^{wm}]\mathbf{y}$ will follow from Lemma 2.2 (ii) [(iii)] and the stochastic monotonicity of $X_i(\theta)$ in θ. Suppose, without a loss of generality, $x_1 \leq y_1 \leq y_2 \leq x_2$ and $x_1 + x_2 = y_1 + y_2$. Then from Theorem 3.9 of Shaked and Shanthikumar (1990a) for a collection of $SIL(st)$ random variables, it is known that there exist four random variables $\{\hat{X}_i; \hat{Y}_i, i = 1, 2\}$ defined on a common probability space such that

$$\text{(2.15)} \qquad \begin{aligned} \hat{X}_i &=^{st} X(x_i); \hat{Y}_i =^{st} X(y_i), i = 1, 2 \\ \hat{X}_1 &= \min\{\hat{Y}_1, \hat{Y}_2\} \\ \hat{X}_2 &= \max\{\hat{Y}_1, \hat{Y}_2\}. \end{aligned}$$

Observe that $\hat{X}_1 \leq \hat{X}_2$. Therefore if $\{\hat{X}_i^{(j)}; \hat{Y}_i^{(j)}, i = 1, 2\}, j = 1, 2$ are two independent samples of $\{\hat{X}_i; \hat{Y}_i, i = 1, 2\}$ one has,

$$\text{(2.16)} \qquad \hat{X}_1^{(1)} \leq \hat{X}_2^{(1)} \text{ and } \hat{X}_1^{(2)} \leq \hat{X}_2^{(2)}.$$

Consider specific realizations $x_i^{(j)}$ of $\hat{X}_i^{(j)}$ and $y_i^{(j)}$ of $\hat{Y}_i^{(j)}, i = 1, 2; j = 1, 2$. Then by (2.16) one has $x_1^{(1)} \leq x_2^{(1)}$ and $x_1^{(2)} \leq x_2^{(2)}$. There are only the following four cases one may encounter:

$$\begin{array}{llcll} (a) & x_1^{(1)} & = & y_1^{(1)}(\Leftrightarrow x_2^{(1)} = y_2^{(1)}), & x_1^{(2)} = y_1^{(2)}(\Leftrightarrow x_2^{(2)} = y_2^{(2)}) \\ (b) & x_1^{(1)} & = & y_1^{(1)}(\Leftrightarrow x_2^{(1)} = y_2^{(1)}), & x_1^{(2)} = y_2^{(2)}(\Leftrightarrow x_2^{(2)} = y_1^{(2)}) \\ (c) & x_1^{(1)} & = & y_2^{(1)}(\Leftrightarrow x_2^{(1)} = y_1^{(1)}), & x_1^{(2)} = y_1^{(2)}(\Leftrightarrow x_2^{(2)} = y_2^{(2)}) \\ (d) & x_1^{(1)} & = & y_2^{(1)}(\Leftrightarrow x_2^{(1)} = y_1^{(1)}), & x_1^{(2)} = y_2^{(2)}(\Leftrightarrow x_2^{(2)} = y_1^{(2)}). \end{array}$$

It is easily verified that for cases (a) and (d) and any symmetric function ϕ,

$$\text{(2.17)} \qquad \phi(x_1^{(1)}, x_2^{(2)}) + \phi(x_1^{(2)}, x_2^{(1)}) = \phi(y_1^{(1)}, y_2^{(2)}) + \phi(y_1^{(2)}, y_2^{(1)}).$$

So consider case (b). If ϕ is symmetric and submodular [supermodular] one sees that

$$\text{(2.18)} \qquad \begin{aligned} & \phi(x_1^{(1)}, x_2^{(2)}) + \phi(x_1^{(2)}, x_2^{(1)}) = \phi(x_1^{(1)}, x_2^{(2)}) + \phi(x_2^{(1)}, x_1^{(2)}) \\ & \geq [\leq] \phi(x_1^{(1)}, x_1^{(2)}) + \phi(x_2^{(1)}, x_2^{(2)}) = \phi(y_1^{(1)}, y_2^{(2)}) + \phi(y_2^{(1)}, y_1^{(2)}) \\ & = \phi(y_1^{(1)}, y_2^{(2)}) + \phi(y_1^{(2)}, y_2^{(1)}). \end{aligned}$$

The first equality follows by the symmetry of ϕ, the second inequality follows from the submodularity [supermodularity] of ϕ and $x_1^{(j)} \le x_2^{(j)}, j = 1,2$, the third equality follows because of the conditions of case (b) and the final equality follows by the symmetry of ϕ. Similarly it can be shown that under case (c),

$$(2.19) \qquad \phi(x_1^{(1)}, x_2^{(2)}) + \phi(x_1^{(2)}, x_2^{(1)}) \ge [\le] \phi(y_1^{(1)}, y_2^{(2)}) + \phi(y_1^{(2)}, y_2^{(1)}).$$

Therefore

$$\phi(\hat{X}_1^{(1)}, \hat{X}_2^{(2)}) + \phi(\hat{X}_1^{(2)}, \hat{X}_2^{(1)}) \ge [\le] \phi(\hat{Y}_1^{(1)}, \hat{Y}_2^{(2)}) + \phi(\hat{Y}_1^{(2)}, \hat{Y}_2^{(1)}).$$

Hence

$$(2.20) \qquad E\phi(X_1(x_1), X_2(x_2)) \ge [\le] E\phi(X_1(y_1), X_2(y_2)).$$

Observing that if g is an increasing concave [convex] function and h is an increasing submodular [supermodular] function then $\phi = g \circ h$ is a submodular [supermodular] function the proof of the theorem for the case $m = 2$ is complete. Extension to the general case can be routinely carried out using Lemma 2.2 (i).

The following result then follows easily from Theorem 2.12.

THEOREM 2.13 *Suppose $\{X_i(\theta), \theta \in \Theta\} \in SICX(st)[SICV(st)]$. Then for any increasing symmetric submodular [supermodular] function h and $Y(\mathbf{x}) = h(X_1(x_1), \ldots, X_m(x_m))$ one has $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in SI{-}SchurCX(icv)[SI{-}SchurCV(icx)]$. That is for any $\mathbf{x}, \mathbf{y} \in \Theta^m$, increasing and symmetric submodular [supermodular] function h and increasing and concave [convex] function g,*

$$(2.21) \qquad \begin{aligned} \mathbf{x} \ge_{wm} \; [\ge^{wm}] \; \mathbf{y} \to & \; Eg \circ h(X_1(x_1), \ldots, X_m(x_m)) \\ \ge [\le] & \; Eg \circ h(X_1(y_1), \ldots, X_m(y_m)). \end{aligned}$$

Consider the case where the system performance is measured by the maximum [minimum] of the individual responses (e.g. parallel [series] reliability system.) That is $h(\mathbf{x}) = \max\{x_i, i = 1, \ldots, m\}$ [$= \min\{x_i, i = 1, \ldots, m\}$]. It is easily verified that h in this case is a submodular [supermodular] function. If we then apply the above result for this case we will have to restrict g to be increasing and concave [convex]. But the following easily verified lemma will allow us to strengthen this result.

LEMMA 2.14 *For any increasing function $\psi : \mathbb{R} \to \mathbb{R}$, the function $\phi : \mathbb{R}^m \to \mathbb{R}$ defined by $\phi(\mathbf{x}) = \psi(\max\{x_1, \ldots, x_m\})$ [$= \psi(\min\{x_1, \ldots, x_m\})$] for $\mathbf{x} \in \mathbb{R}^m$ is symmetric, increasing and submodular [supermodular].*

Combining Lemma 2.14 with Theorem 2.13 (Equation 2.20) one obtains

THEOREM 2.15 *Suppose* $\{X_i(\theta), \theta \in \Theta\} \in SICX(st)[SICV(st)]$. *Then if we define* $Y(\mathbf{x}) = \max\{X_1(x_1), \dots, X_m(x_m)\}$ $[= \min\{X_1(x_1), \dots, X_m(x_m)\}]$ *one has* $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in SI - SchurCX(st)[SI - SchurCV(st)]$. *That is for any* $\mathbf{x}, \mathbf{y} \in \Theta^m$ *and increasing function* g,

$$\begin{aligned} \mathbf{x} \geq_{wm} \mathbf{y} \quad \rightarrow \quad & Eg(\max\{X_1(x_1), \dots, X_m(x_m)\}) \\ \geq \quad & Eg(\max\{X_1(y_1), \dots, X_m(y_m)\}) \end{aligned}$$

$$\begin{aligned} [\mathbf{x} \geq^{wm} \mathbf{y} \quad \rightarrow \quad & Eg(\min\{X_1(x_1), \dots, X_m(x_m)\}) \\ \leq \quad & Eg(\min\{X_1(y_1), \dots, X_m(y_m)\})]. \end{aligned}$$

REMARK 2.16 Since $P\{\min\{X_1(x_1), \dots, X_m(x_m)\} > t\} = \Pi_{i=1}^m \bar{F}(t; x_i)$ it is immediate that $\bar{F}(t; \theta)$ is increasing and logconcave in θ will imply that $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in SI - SchurCV(st)$. This is a stronger conclusion than that in theorem 2.15.

3. Stochastic Transposition Increasingness

In this section we will define stochastic transposition increasingness and give sufficient conditions on $X_i(r_i), i = 1, \dots, m$ and h under which $Y(\mathbf{r}) := h(X_1(r_1), \dots, X_m(r_m))$ is stochastically transposition increasing with respect to $\mathbf{r}$. For this we will need the following definition of transposition increasing functions (which is slightly different from that given in Hollander, Proschan and Sethuraman (1981), and Marshall and Olkin (1979)).

DEFINITION 3.1 Let $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{x}$ be a permutation of $\mathbf{y}$. Then $\mathbf{x}$ is more *arranged* than $\mathbf{y}$ if $\mathbf{x}$ can be obtained from $\mathbf{y}$ by a finite number of successive pairwise interchanges of two coordinates at a time such that each interchange results in a decreasing order for the interchanged elements. We denote this $\mathbf{x} \geq_a \mathbf{y}$. (e.g. $(1,5,4,3) \geq_a (1,5,3,4)$; $(1,5,3,4) \geq_a (1,4,3,5)$ and $(1,5,4,3) \geq_a (1,4,3,5)$.)

Note that the above definition of the arrangement ordering ($\geq_a$) allows one to simplify the analysis of transposition increasingness, often making it sufficient to prove the desired result just for the two dimensional case.

DEFINITION 3.2 A function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ is *transposition increasing* [*decreasing*] if $\mathbf{x} \geq_a \mathbf{y}$ implies $\phi(\mathbf{x}) \geq [\leq] \phi(\mathbf{y})$.

DEFINITION 3.3 *(Stochastic Transposition Increasingness)*: A real valued random variable $Z(\mathbf{x})$ parametrized by $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^m$ is *stochastically transposition increasing* in the sense of the usual stochastic [increasing convex, increasing concave] ordering if for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}, \mathbf{x} \geq_a \mathbf{y}$ implies $Z(\mathbf{x}) \geq_{st} [\geq_{icx}, \geq_{icv}] Z(\mathbf{y})$. We denote this $\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{X}\} \in S - TI(st)[S - TI(icx), S - TI(icv)]$.

If $-Z(\mathbf{x})$ is stochastically transposition increasing then we say that $Z(\mathbf{x})$ is stochastically transposition decreasing (and denote $S - TD(st)$ etc.)

As an immediate consequence of the definition of stochastic transposition increasingness one has the following partial characterization of the optimal solution to the allocation problem (1.1).

THEOREM 3.4 *Let $Y(\mathbf{r}) = h(X_1(r_1), \ldots, X_m(r_m))$. Suppose $\{Y(\mathbf{r}), \mathbf{r} \in \mathcal{S}\} \in S - TI(st)[S - TI(icx), S - TI(icv)]$. Then*
(i) for any increasing [increasing and convex, increasing and concave] function g and $\mathbf{r}, \mathbf{s} \in \mathcal{S}$ one has,

$$\mathbf{r} \geq_a \mathbf{s} \Rightarrow Eg \circ h(X_1(r_1), \ldots, X_m(r_m)) \geq Eg \circ h(X_1(s_1), \ldots, X_m(s_m))$$

(ii) If $\mathcal{S} = \mathcal{S}_0 := \{\mathbf{r} : \sum_{i=1}^m r_i = R, r_i \geq 0\}$, then an optimal solution $\mathbf{r}^*$ to (1.1) will satisfy $r_1^* \leq \cdots \leq r_m^*$.

Note that when we have a discrete resource of R units there can exist a large number of feasible solutions for (1.1). For example with $R = 10$ and $m = 10$ there are total of 92,378 different possible solutions (i.e. $|\mathcal{S}_0| = 92{,}378$). The number of solutions that satisfy the characterization given above is only 42 (see Table 1 of Shanthikumar and Yao (1988)). From the above discussion it is clear that it is worthwhile to search for sufficient conditions on $X_i(r_i), i = 1, \ldots, m$ and h for $Y(\mathbf{r})$ to be stochastically transposition increasing. We shall do this next in this section (and examples where such conditions are naturally satisfied are given in Section 4.) Before that we note the following easily verified lemma.

LEMMA 3.5 *Let $F_Z(t; \mathbf{x}) = P\{Z(\mathbf{x}) \leq t\}$ and $\bar{F}_Z(t; \mathbf{x}) = P\{Z(\mathbf{x}) > t\}$ be respectively, the cumulative and survival functions of $Z(\mathbf{x})$. Then $\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{X}\} \in S - TI(st)[S - TI(icx), S - TI(icv)] \Leftrightarrow \bar{F}_Z(t; \mathbf{x})[\int_t^\infty \bar{F}_Z(s; \mathbf{x})ds, -\int_{-\infty}^t F_Z(s; \mathbf{x})ds]$ is transposition increasing in $\mathbf{x}$.*

In the remainder of this section we will assume that $\{X_i(\theta), \theta \in \Theta\}, i = 1, \ldots, m$ are m mutually independent collections of random variables.

THEOREM 3.6 *Let $\gamma_i(\cdot; \theta)$ be the hazard rate function of the absolutely continuous positive random variable $X_i(\theta), \theta \in \Theta, i = 1, \ldots, m$. Suppose for each $t \in \mathbb{R}_+$, we have $\gamma_i(t; \theta)$ componentwise monotone and submodular in $(i, \theta) \in \{1, \ldots, m\} \times \Theta$. Then for any increasing and symmetric function h and $Y(\mathbf{x}) = h(X_1(x_1), \ldots, X_m(x_m))$ one has $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in S - TD(st)$. That is for any $\mathbf{x}, \mathbf{y} \in \Theta^m$, increasing symmetric function h and increasing* *function g*,

$$\text{(3.1)} \qquad \mathbf{y} \geq_a \mathbf{x} \rightarrow Eg \circ h(X_1(y_1), \ldots, X_m(y_m)) \\ \leq Eg \circ h(X_1(x_1), \ldots, X_m(x_m)).$$

PROOF We will first establish the theorem for the case $m = 2$ and $\gamma_i(\cdot;\theta)$ decreasing in i and θ. The other case where $\gamma_i(\cdot;\theta)$ is increasing in i and θ can be similarly proved. Suppose, without a loss of generality, $x_1 = y_2 \leq y_1 = x_2$. Since for each $t \in \mathbb{R}_+$, $\gamma_i(t;\theta)$ is componentwise decreasing and submodular in $(i, \theta) \in \{1, \ldots, m\} \times \Theta$ one has

$$\text{(3.2)} \quad \gamma_1(t;x_1) \geq \gamma_1(t;y_1) \geq \gamma_2(t;x_2); \quad \gamma_1(t;x_1) \geq \gamma_2(t;y_2) \geq \gamma_2(t;x_2)$$

and

$$\text{(3.3)} \qquad \gamma_1(t;x_1) + \gamma_2(t;x_2) \leq \gamma_1(t;y_1) + \gamma_2(t;y_2).$$

Therefore there exist $\hat{\gamma}_1(t;y_1)$ and $\hat{\gamma}_2(t;y_2)$ such that

$$\text{(3.4)} \quad \gamma_2(t;x_2) \leq \hat{\gamma}_1(t;y_1) \leq \gamma_1(t;y_1); \quad \gamma_2(t;x_2) \leq \hat{\gamma}_2(t;y_2) \leq \gamma_2(t;y_2),$$

and

$$\text{(3.5)} \qquad \gamma_1(t;x_1) + \gamma_2(t;x_2) = \hat{\gamma}_1(t;y_1) + \hat{\gamma}_2(t;y_2).$$

Let $Z_i, \hat{Z}_i, i = 1, 2$ be four mutually independent random variables such that $Z_i =^{st} X_i(x_i), i = 1, 2$ and $\hat{Z}_i$ has a hazard rate function $\hat{\gamma}_i(\cdot;y_i), i = 1, 2$. Then from (3.4) one sees that $\hat{Z}_i$ is larger than $X_i(y_i)$ in the hazard rate ordering, $i = 1, 2$. That is

$$\text{(3.6)} \qquad \hat{Z}_i \geq_{hr} X_i(y_i), i = 1, 2.$$

Now using a derivation same as that employed in the proof of Theorem 2.9 it can be shown that

$$\text{(3.7)} \quad (\min\{Z_1, Z_2\}, \max\{Z_1, Z_2\}) \geq_{st} (\min\{\hat{Z}_1, \hat{Z}_2\}, \max\{\hat{Z}_1, \hat{Z}_2\}).$$

Then from (3.6) one has

$$(\min\{\hat{Z}_1, \hat{Z}_2\}, \max\{\hat{Z}_1, \hat{Z}_2\}) \geq_{st} \\ (\min\{X_1(y_1), X_2(y_2)\}, \max\{X_1(y_1), X_2(y_2)\})$$

and hence for any increasing symmetric function h

$$h(X_1(x_1), X_2(x_2)) \geq_{st} h(X_1(y_1), X_2(y_2)).$$

That is we have established the theorem for $m = 2$. Extension to the general case can be routinely carried out using the property of arrangement ordering $\geq_a$ as given in Definition 3.1 (e.g. see Marshall and Olkin (1979), Shanthikumar (1987)).

For the above result we need the strong condition of submodularity of the hazard rate of $X_i(\theta)$. We will next present a result that is weaker than Theorem 3.6, but requires only the supermodularity [submodularity] of the survival function of $X_i(\theta)$.

THEOREM 3.7 *Let $\bar{F}_i(\cdot;\theta)$ be the survival function of $X_i(\theta), \theta \in \Theta, i = 1,\ldots,m$. Suppose for each fixed $t \in \mathbb{R}, \bar{F}_i(t;\theta)$ is componentwise increasing and supermodular [submodular] in $(i,\theta) \in \{1,\ldots,m\} \times \Theta$. Then for any increasing symmetric submodular [supermodular] function h and $Y(\mathbf{x}) = h(X_1(x_1),\ldots,X_m(x_m))$ one has $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in S{-}TD(icv)[S{-}TI(icx)]$. That is for any $\mathbf{x},\mathbf{y} \in \Theta^m$, increasing and symmetric submodular [supermodular] function h and increasing and concave [convex] function g,*

$$\begin{aligned} (3.8) \qquad \mathbf{y} \quad \geq_a \quad & \mathbf{x} \rightarrow Eg \circ h(X_1(y_1),\ldots,X_m(y_m)) \\ \leq [\geq] \quad & Eg \circ h(X_1(x_1),\ldots,X_m(x_m)). \end{aligned}$$

PROOF We will first establish the theorem for the case $m = 2$. Suppose, without a loss of generality, $x_1 = y_2 \leq y_1 = x_2$. Since for each $t \in \mathbb{R}_+$, $\bar{F}_i(t;\theta)$ is componentwise increasing and supermodular [submodular] in $(i,\theta) \in \{1,\ldots,m\} \times \Theta$ one has

$$(3.9) \qquad \bar{F}_1(t;x_1) \leq [\bar{F}_1(t;y_1), \bar{F}_2(t;y_2)] \leq \bar{F}_2(t;x_2)$$

and

$$(3.10) \qquad \bar{F}_1(t;x_1) + \bar{F}_2(t;x_2) \geq [\leq] \bar{F}_1(t;y_1)] + \bar{F}_2(t;y_2).$$

Therefore there exist $\bar{F}_1^*(t;y_1)$ and $\bar{F}_2^*(t;y_2)$ such that

$$(3.11) \qquad \bar{F}_1^*(t;y_1) \geq [\leq] \bar{F}_1(t;y_1); \quad \bar{F}_2^*(t;y_2) \geq [\leq] \bar{F}_2(t;y_2),$$

and

$$(3.12) \qquad \bar{F}_1(t;x_1) + \bar{F}_2(t;x_2) = \bar{F}_1^*(t;y_1) + \bar{F}_2^*(t;y_2).$$

Let $Z_i, \hat{Z}_i, i = 1,2$ be four mutually independent random variables such that $Z_i =^{st} X_i(x_i), i = 1,2$ and $\hat{Z}_i$ has the survival function $\bar{F}_i^*(\cdot;y_i), i = 1,2$. Then from (3.11) one sees that $\hat{Z}_i$ is larger [smaller] than $X_i(y_i)$ in the usual stochastic ordering, $i = 1,2$. That is

$$(3.13) \qquad \hat{Z}_i \geq_{st} [\leq_{st}] X_i(y_i), i = 1,2.$$

Now using a derivation similar to that employed in the proof of Theorem 3.4 of Shaked and Shanthikumar (1990a), it can be shown that there exist four random variables $\{\hat{X}_i; \hat{Y}_i, i = 1, 2\}$ defined on a common probability space such that

$$\begin{aligned} \hat{X}_i &=^{st} X(x_i); \hat{Y}_i =^{st} \hat{Z}_i, i = 1,2 \\ \hat{X}_1 &= \min\{\hat{Y}_1, \hat{Y}_2\} \\ \hat{X}_2 &= \max\{\hat{Y}_1, \hat{Y}_2\}. \end{aligned} \tag{3.14}$$

Now using a derivation same as that employed in the proof of Theorem 2.12 it can be shown that for any increasing symmetric submodular [supermodular] function ϕ,

$$\phi(\hat{X}_1^{(1)}, \hat{X}_2^{(2)}) + \phi(\hat{X}_1^{(2)}, \hat{X}_2^{(1)}) \geq [\leq] \phi(\hat{Y}_1^{(1)}, \hat{Y}_2^{(2)}) + \phi(\hat{Y}_1^{(2)}, \hat{Y}_2^{(1)}).$$

Therefore from (3.13) one sees that

$$E\phi(X_1(x_1), X_2(x_2)) \geq [\leq] E\phi(X_1(y_1), X_2(y_2)). \tag{3.15}$$

Observing that if g is an increasing concave [convex] function and h is an increasing submodular [supermodular] function then $\phi = g \circ h$ is a submodular [supermodular] function the proof of the theorem for the case $m = 2$ is complete. Extension to the general case can be routinely carried out using the property of arrangement ordering presented in Definition 3.1.

Combining Lemma 2.14 with the above Theorem 3.7 (Equation 3.15) one obtains

THEOREM 3.8 *Let $\bar{F}_i(\cdot; \theta)$ be the survival function of $X_i(\theta), \theta \in \Theta, i = 1, \ldots, m$. Suppose for each fixed $t \in \mathbb{R}, \bar{F}_i(t; \theta)$ is componentwise increasing and supermodular [submodular] in $(i, \theta) \in \{1, \ldots, m\} \times \Theta$. Then if we define $Y(\mathbf{x}) = \max\{X_1(x_1), \ldots, X_m(x_m)\}$ [$= \min\{X_1(x_1), \ldots, X_m(x_m)\}$] one has $\{Y(\mathbf{x}), \mathbf{x} \in \Theta^m\} \in S-TD(st)[S-TI(st)]$. That is for any $\mathbf{x}, \mathbf{y} \in \Theta^m$ and increasing function g,*

$$\begin{aligned} \mathbf{y} \geq_a \mathbf{x} \quad \to \quad & Eg(\max\{X_1(y_1), \ldots, X_m(y_m)\}) \\ \leq \quad & Eg(\max\{X_1(x_1), \ldots, X_m(x_m)\}) \end{aligned}$$

$$\begin{aligned} [\mathbf{y} \geq_a \mathbf{x} \quad \to \quad & Eg(\min\{X_1(y_1), \ldots, X_m(y_m)\}) \\ \geq \quad & Eg(\min\{X_1(x_1), \ldots, X_m(x_m)\})]. \end{aligned}$$

4. Applications

Allocation problems of the kind (1.1) described in Section 1 arise in many different areas (e.g. see Jean-Marie and Gun (1990), Shanthikumar (1988), Shanthikumar and Stecke (1986), Shanthikumar and Yao (1988), Yao and Shanthikumar (1987) and the papers referenced in there). In these papers each problem is analyzed and solved in its specific context. The results presented in Sections 2 and 3 now provide a unified way to solve many of these allocation problems. In this section we will present several applications of the results derived in Sections 2 and 3 to problems arising in telecommunication, manufacturing and reliability/performability systems.

4.1 *Parallel Queues with Resequencing*

Consider a single stage queueing system consisting of m parallel servers. The n-th customer arrives at time A_n and requires a service of length $B_n^{(i)}$ if serviced by the i- th server, $n = 1, 2, \ldots$. Customers on its arrival are assigned to one of the m parallel servers according to some assignment rule. Suppose the n-th customer is assigned the server $U_n, n = 1, 2, \ldots$. Each of the m servers is assumed to have a buffer with an unlimited capacity to store the waiting customers. Customers leaving this single stage are stored in a resequencing area where each customer is allowed to leave as soon as only after all the customers arrived to the system before it are released from the resequencing area. Queueing systems of this kind serve as models of telecommunication systems (e.g. see Baccelli, Makowski and Shwartz (1989), Gun (1989), Harrus and Plateau (1982), Jean-Marie (1987), Jean-Marie and Gun (1990)), of distributed database systems (e.g. see Kamoun, Kleinrock and Muntz (1981)) and of flexible assembly systems (e.g. see Buzacott (1990), Buzacott and Shanthikumar (1992)).

Let $V_i(t)$ be the workload at server i at time $t, t \in \mathbb{R}_+; i = 1, \ldots, m$. Then

$$(4.1) \qquad \begin{aligned} V_{U_n}(A_n) &= V_{U_n}(A_n-) + B_n^{(U_n)}, n = 1, 2, \ldots, \\ \frac{d}{dt}V_i(t) &= I\{V_i(t) > 0\}, t \in \mathbb{R}_+. \end{aligned}$$

Define the maximum workload at time t by

$$(4.2) \qquad \hat{V}(t) = \max\{V_1(t), \ldots, V_m(t)\}, t \in \mathbb{R}_+.$$

Then it is very easy to see that the sojourn time S_n of the n-th customer through the single stage and the sojourn time T_n of the n-th customer through the system (including the time spent, if any, at the resequencing area) are given by

$$(4.3) \qquad S_n = V_{U_n}(A_n) \text{ and } T_n = \hat{V}(A_n), n = 1, 2, \ldots$$

Define $W_n = \hat{V}(A_n-), n = 1, 2, \ldots$ A typical portion of a sample path of $\{\hat{V}(t), t \in \mathbb{R}_+\}$ is shown in Figure 1.

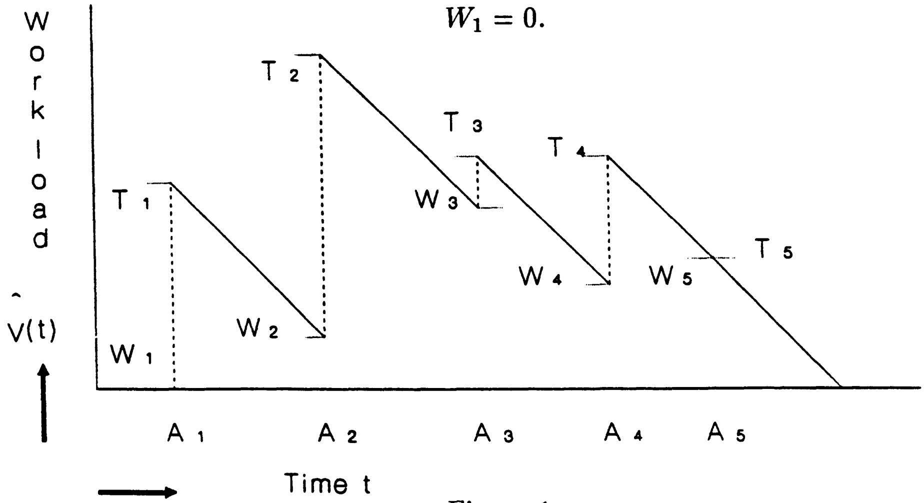

Figure 1

Assuming their existence let F_T, F_W and $F_{\hat{V}}$ be the stationary distribution of T_n, W_n and $\hat{V}(t)$. If the limit $\lambda = \lim_{n\to\infty}\{n/A_n\}$, and the density $f_{\hat{V}}$ of $F_{\hat{V}}$ exist, then equating the rates of up- and down-crossings over level x (e.g. see Cohen (1977), Shanthikumar (1980)) one gets

$$\lambda(\bar{F}_T(x) - \bar{F}_W(x)) = f_{\hat{V}}(x), x > 0. \tag{4.4}$$

Therefore

$$\bar{F}_T(x) = \bar{F}_W(x) + \frac{1}{\lambda} f_{\hat{V}}(x), x > 0. \tag{4.5}$$

Note that a similar relationship for a special case is derived in Jean-Marie and Gun (1990). Particularly they assume that (i) $\{A_n - A_{n-1}, n = 1,2,\ldots\}$, $\{B_n^{(i)}, n = 1,2,\ldots\}, i = 1,\ldots,m$ and $\{U_n, n = 1,2,\ldots\}$ are all mutually independent sequences of *i.i.d.* random variables, that (ii) the arrival process is Poisson and that (iii) $\{U_n, n = 1,2,\ldots\}$ is a sequence of multinomial trials with $P\{U_n = i\} = p_i, i = 1,\ldots,m$. Now we will make the same set of assumptions and look at the allocation of the probabilities $(p_1,\ldots,p_m)$ that will stochastically minimize the stationary system sojourn time $T(\mathbf{p})$. Since Poisson arrivals see time averages (see Wolff (1982)) it is clear that when these stationary distributions exist (i.e. when $0 \le \lambda p_i < \mu_i = 1/EB_n^{(i)}, i = 1,\ldots,m$), $F_W(x) = F_{\hat{V}}(x), x \in \mathbb{R}_+$. Hence (4.5) reduces (after integrating) to

$$\int_t^\infty \bar{F}_T(x)dx = \int_t^\infty \bar{F}_W(x)dx + \frac{1}{\lambda}\bar{F}_W(t), t > 0. \tag{4.6}$$

Let $W(\mathbf{p})$ be a generic random variable with distribution function F_W. Then from (4.6) and Lemma 2.6 one sees that if $\{W(\mathbf{p}), \mathbf{p} \in \mathcal{X}\} \in S -$

$SchurCX(st)$ then $\{T(\mathbf{p}), \mathbf{p} \in \mathcal{X}\} \in S - SchurCX(icx)$, where $\mathcal{X}$ is the set of values of $\mathbf{p}$ for which the stationary distributions exist. Since the stationary workloads at the m servers are independent because of Poisson arrival process and multinomial splitting one sees that

$$W(\mathbf{p}) = \max\{X_1(p_1), \ldots, X_m(p_m)\}, \tag{4.7}$$

where $X_i(p_i)$ is the stationary workload in an M/G/1 queueing system with arrival rate λp_i and service times $\{B_n^{(i)}, n = 1, 2, \ldots\}$ and $\{X_i(p_i)\}, i = 1, \ldots, m$ are independent sequences. From Theorem 4.17 of Shaked and Shanthikumar (1989) (also see Gun (1989)) it is known that $\{X_i(p_i), 0 \leq p_i < \mu_i/\lambda\} \in SICX(st)$. Then from (4.6) and Theorem 2.15 one sees that if $\{B_n^{(i)}, i = 1, \ldots, m\}$ are identical, then $\{T(\mathbf{p}), \mathbf{p} \in \mathcal{X}\} \in S - SchurCX(icx)$. Therefore balancing the allocation probabilities will minimize T in the increasing convex ordering. This conclusion was first derived by Jean-Marie and Gun (1990). The stochastic Schur convexity result for T is, however, new.

4.2 *Flexible Assembly Line*

Consider an m-stage serial assembly line. Parts arrive at this assembly line according to a Poisson process with rate λ. The nominal processing times (i.e. if only one standard worker is used) at stage i are *i.i.d.* exponential random variables with mean $1/\mu_i, i = 1, \ldots, m$. Suppose we have a total of R workers available and that we can assign them among the m stages of the assembly line. Problem of this kind arise naturally in many settings in the manufacturing systems (e.g. see Buzacott and Shanthikumar (1992), Shanthikumar and Yao (1988)). We will consider two typical problems that arise in this context (see Chapter 6 of Buzacott and Shanthikumar (1992)).

PROBLEM 1 Suppose the average workload assigned per stage is the same for all stages (i.e. $\mu_i = \mu, i = 1, \ldots, m$). The number of workers available is more than the number of stages (i.e. $R > m$). If we allocate r workers to the same stage the collective processing rate is $c(r), r = 1, \ldots R$. In an ideal case we would expect $c(n) = n$, but for the sake of generality, we will assume that $c(n)$ is increasing and concave in n. Let $T(\mathbf{r})$ be the stationary sojourn time of an arbitrary part through the assembly line if we allocate r_i workers to stage $i, i = 1, \ldots, m$. Let $X_i(n)$ be an exponential random variable with constant hazard rate $\gamma_i(n) = c(n)\mu - \lambda > 0$. Also assume that $\{X_i(n)\}, i = 1, \ldots, m$ are mutually independent. Then (e.g. see Jackson

(1963))

$$T(\mathbf{r}) = \sum_{i=1}^{m} X_i(r_i), \quad r_i : c(r_i) > \lambda/\mu. \tag{4.8}$$

Since $\{X_i(n), n : c(n) > \lambda/\mu\} \in SDCX(hr)$ from Theorem 2.9 one sees that $\{T(\mathbf{r}), r_i : c(r_i) > \lambda/\mu, i = 1, \ldots, m\} \in SD - SchurCX(st)$. Therefore a balanced worker allocation will stochastically minimize the total sojourn time T in the usual stochastic ordering.

PROBLEM 2 Suppose the stages are numbered such that the average workload assigned to stage i is larger than that assigned to stage $i+1$ (i.e. $1/\mu_i \geq 1/\mu_{i+1}$) ,$i = 1, \ldots, m$. The number of workers available is more than the number of stages (i.e. $R > m$). If we allocate r workers to the stage i the collective processing rate is $c_i(r), r = 1, \ldots R$. Note that $c_i(1) = \mu_i$. Let $T(\mathbf{r})$ be the stationary sojourn time of an arbitrary part through the assembly line if we allocate r_i workers to stage $i, i = 1, \ldots, m$. Let $X_i(n)$ be an exponential random variable with constant hazard rate $\gamma_i(n) = c_i(n) - \lambda > 0$. Also assume that $\{X_i(n)\}, i = 1, \ldots, m$ are mutually independent. Then (e.g. see Jackson (1963))

$$T(\mathbf{r}) = \sum_{i=1}^{m} X_i(r_i), \quad r_i : c_i(r_i) > \lambda, i = 1, \ldots, m. \tag{4.9}$$

Suppose $c_i(n)$ is componentwise increasing and submodular in (i, n). Then one finds that $\gamma_i(n)$ is componentwise increasing and submodular. Then from Theorem 3.6 one sees that $\{T(\mathbf{r}), r_i : c_i(r_i) > \lambda, i = 1, \ldots, m\} \in S - TD(st)$. Therefore allocating more workers to the stages with smaller indices (i.e., stages with more workload) will stochastically reduce the total sojourn time T in the usual stochastic ordering.

4.3 *Reliability/Performability System*

Consider a reliability/performability system (e.g., see Shaked and Shanthikumar (1990b)) consisting of m components. Suppose a total budget of R dollars is to be allocated among the m components. Suppose $X_i(r_i)$ is the lifetime of component i if r_i dollars is allocated for it. Suppose the performability function is $c(n)$ (i.e. when n components are alive the rate at which performance is accumulated is $c(n)$). For example $c(n)$ could be the production rate of a manufacturing system when n machines are working. We assume that $c(0) = 0$. The total performance as a function of the component lifetimes is

$$Y(\mathbf{r}) = h(X_1(r_1), \ldots, X_m(r_m)), \tag{4.10}$$

where

$$h(\mathbf{x}) = \int_0^\infty c(\sum_{i=1}^m I\{x_i > t\})dt, \mathbf{x} \in \mathbb{R}_+^m. \tag{4.11}$$

We will need the following characterization of h.

LEMMA 4.1 *Let h be defined as in* (4.11). *If c is an increasing and concave [convex] function, then h is an increasing submodular [supermodular] function.*

PROOF Observe that

$$\frac{d}{dx_j}h(\mathbf{x}) = c(1 + \sum_{i=1}^m I\{x_i > x_j\}) - c(\sum_{i=1}^m I\{x_i > x_j\}) \tag{4.12}$$

Observing that $c(1+n) - c(n)$ is decreasing [increasing] in n if c is concave [convex] and that $\sum_{i=1}^m I\{x_i > x_j\}$ is increasing in x_i it can be concluded that h is submodular [supermodular].

For example if $c(0) = 0, c(n) = 1, n = 1, 2, \ldots$ $[c(n) = 0, n = 0, 1, \ldots m - 1; c(n) = n + 1 - m, n = m, m + 1, \ldots]$, then $Y(\mathbf{r})$ will be the lifetime of a parallel [series] reliability system (e.g. see Barlow and Proschan (1975)). As expected since $c(n)$ is concave [convex] in n, the lifetime of a parallel [series] reliability system is submodular [supermodular] in the component lifetimes. Combining Theorem 2.13 with Lemma 4.1 one obtains

THEOREM 4.2 *Suppose the families $\{X_i(r_i)\}$ of the lifetimes of the components of the reliability/performability system are independent and identical. Then if $\{X_i(r_i)\} \in SICX(st)[SICV(st)]$ and the performability function is concave [convex], then $\{Y(\mathbf{r})\} \in SI - SchurCX(icv)$ [$SI - SchurCV(icx)$].*

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