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EXTREME ORDER STATISTICS FOR A SEQUENCE OF DEPENDENT RANDOM VARIABLES[1]

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Let $X_1, \ldots, X_n$ be a sequence of dependent random variables from a continuous density function. Denote by $X_{(1)} = \min(X_1, \ldots, X_n)$ and $X_{(n)} = \max(X_1, \ldots, X_n)$ the extreme order statistics. In this article Bonferroni–type inequalities and product–type approximations of order $k \geq 1$ are derived for the distribution and the moments of extreme order statistics for a sequence of stationary random variables. These results are particularized to m–spacings from a uniform distribution and moving sums of size m for independent normal random variables. These inequalities and approximations are compared with approximations and asymptotic results that have been previously derived.

From the numerical results it is evident that there is merit in studying higher order Bonferroni–type inequalities and product–type approximations. The product–type approximations appear to be the most accurate approximations for the distribution and the moments of extreme order statistics.

1. Introduction

Let $X_1, \ldots, X_n$ be a sequence of dependent random variables from a continuous density function. Denote the extreme order statistics by

$$X_{(1)} = \min(X_1, \ldots, X_n) \quad \text{and} \quad X_{(n)} = \max(X_1, \ldots, X_n).$$

The distribution and the moments of extreme order statistics have been studied extensively in the iid case (David (1981) and Leadbetter, Lindgren and Rootzen (1983)). A major part of these studies focuses on the elegant asymptotic theory that has been developed for the iid case or in the dependent case for stationary sequences of random variables that satisfy the strong mixing condition, including the stationary m–dependent sequences (Leadbetter, Lindgren and Rootzen (1983) and Reiss (1989)).

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As the asymptotic results are not always accurate, the problem of approximating the distribution and the mean of extreme order statistics for dependent random variables has attracted many researchers. Arnold (1980, 1985), Arnold and Groeneveld (1979), Aven (1985), Gravey (1985) and Hoover (1989) have derived general bounds for the mean of the extreme order statistics. The use of the Boole–Bonferroni inequality and the Sidak inequality is outlined in David (1981, Section 5.3). Tong (1982, 1990, Chapter 6) has used the Sidak inequality to approximate the distribution and the mean of extreme order statistics for dependent normal random variables.

The methods that have been used in the articles listed above, only lightly utilize the dependence structure inherent in the distribution of $X_1, \ldots, X_n$. In this article I will consider only stationary sequences of dependent random variables. For this case, Bonferroni–type inequalities and product–type approximations of order $k \geq 1$ will be discussed in Section 2. The performance of Bonferroni–type and product–type inequalities for positively and negatively dependent random variables is being investigated in Glaz, Kuo and Yiannoutsos (1991).

In Section 3 these inequalities and approximations are applied to the distribution and the mean of the smallest m–spacing from a uniform distribution. A comparison with approximations and asymptotic results that have been previously derived will be presented in Tables 1 and 2 in Section 3.

In Section 4 the distribution of the maximum of a moving sum of m independent and identically distributed normal random variables is considered. Bonferroni–type inequalities and product–type approximations are derived and compared in Tables 3 and 4 with the Poisson approximation using the Chen–Stein method.

2. Product–Type and Bonferroni–Type Approximations and Inequalities

The first occurrence of using a *product–type inequality* to approximate a multivariate cumulative distribution function is recorded in Kimball's (1951) article. The following result is proved there:

THEOREM 2.1 *Let Y be a random variable with the density function $f(y)$ and let $g_i(y)$, $i = 1, \ldots, n$ be nonnegative monotone functions of the same type. Then,*

$$E\left[\prod_{i=1}^{n} g_i(Y)\right] \geq \prod_{i=1}^{n} E[g_i(Y)]. \tag{2.1}$$

An immediate consequence of this result is the following:

COROLLARY 2.2 *Let Y be a random variable with the density function $f(y)$ and $X_i = g_i(Y)$, where $g_i(y)$ are nonnegative monotone functions of the same type, $i = 1, \ldots, n$. Then for any sequence of nonnegative constants $c_1, \ldots, c_n$*

$$P[X_1 \leq c_1, \ldots, X_n \leq c_n] \geq \prod_{i=1}^{n} P[X_i \leq c_i] \tag{2.2}$$

and

$$P[X_1 > c_1, \ldots, X_n > c_n] \geq \prod_{i=1}^{n} P[X_i > c_i]. \tag{2.3}$$

To extend the product–type inequalities (2.2)–(2.3) to a larger class of distributions, Esary, Proschan and Walkup (1967) introduced the following concept of positive dependence.

DEFINITION 2.1 The random vector $\mathbf{X} = (X_1, \ldots, X_n)'$ is *associated* if for all coordinatewise nondecreasing functions f and g

$$\mathrm{Cov}[f(\mathbf{X}), g(\mathbf{X})] = E[f(\mathbf{X})g(\mathbf{X})] - E[f(\mathbf{X})]E[g(\mathbf{X})] \geq 0.$$

If $\mathbf{X}$ is associated we will say that the random variables $X_1, \ldots, X_n$ are associated.

THEOREM 2.3 (Esary, Proschan and Walkup (1967)). *If $\mathbf{X} = (X_1, \ldots, X_n)'$ is associated then for all $c_1, \ldots, c_n$ inequalities* (2.2) *and* (2.3) *hold.*

REMARK Inequalities (2.2) and (2.3) are often called, in the context of simultaneous confidence intervals, Sidak inequalities (Sidak (1967, 1971)).

Inequalities (2.2)–(2.3) are referred to as *first order product–type inequalities,* since only one dimensional marginal distributions have been used. We say that an approximation or an inequality for the probability of an intersection or union of n events is of *order* k, if j dimensional marginal distributions are used in computing it, where $j \leq k$. While the first order product–type approximations and inequalities have the advantage of ease of computation, they are often quite inaccurate (Glaz and Johnson (1984)), the reason being that the dependence structure inherent in the random process is exploited only to a minimal degree. Therefore, Glaz and Johnson (1984, 1986) proposed to study product–type inequalities and approximations of degree $k \geq 2$. The following representation has motivated the study of these inequalities and approximations.

Let $A_j = (X_j \in I_j)$, $j = 1, \ldots, n$. Then for $k \geq 2$,

$$P\left(\bigcap_{j=1}^{n} A_j\right) = P\left(\bigcap_{j=1}^{k} A_j\right) \prod_{j=k+1}^{n} P\left(A_j \mid A_{j-1} \cap \cdots \cap A_{j-k+1} \bigcap_{s=1}^{j-k} A_s\right). \tag{2.4}$$

If one is able to evaluate marginal probabilities up to dimension k, then it is of interest to study when

$$\gamma_k = P\left(\bigcap_{j=1}^{k} A_j\right) \prod_{j=k+1}^{n} P\left(A_j \mid A_{j-1} \cap \cdots \cap A_{j-k+1}\right) \tag{2.5}$$

is an accurate (upper or lower) bound or an accurate approximation.

If the X_j's are stationary and $I_j = I$ then the above equation simplifies to:

$$\begin{aligned}\gamma_k = {} & P(X_j \in I; j = 1, \ldots, k) \\ & [P(X_k \in I \mid X_j \in I; j = 1, \ldots, k-1)]^{n-k}.\end{aligned} \tag{2.6}$$

A thorough discussion of the higher order product–type inequalities, conditions for their validity and the applications to various areas in probability and statistics are presented in Block, Costigan and Sampson (1988a, 1988b) and Glaz (1990a, 1991b).

In some applications (Glaz (1983, 1989), Glaz and Johnson (1986), Glaz and Naus (1991), Kenyon (1990) and Ravishanker, Wu and Glaz (1991)) the dependence structure of the distribution does not support the product–type inequalities for $k > 1$. Instead one can sometimes assert that

$$\lim_{j \to \infty} P(X_j \in I_j \mid X_i \in I_i; i = 1, \ldots, j-1) = \theta, \tag{2.7}$$

where $0 < \theta < 1$ (Glaz (1989) and Glaz and Johnson (1986)), a property referred to in the statistical literature as *quasi–stationarity* (Darroch and Seneta (1965) and Tweedie (1974)). In this case, if X_j are stationary and $I_j = I$, one can use γ_k given in equation (2.6) as an approximation for $P(X_j \in I; j = 1, \ldots, n)$.

In Sections 3 and 4 the performance of product–type approximations will be evaluated for the distribution of extreme order statistics of two stationary sequences of random variables. In both cases the concept of quasi–stationarity will be utilized to support the use of these approximations.

The classical *Bonferroni inequalities* for the probability of a union of n events have been introduced in Bonferroni (1937). Let $A_1, \ldots, A_n$ be a sequence of events and define the event $A = \bigcup_{i=1}^{n} A_i$. Then for $2 \leq k \leq n$,

$$\sum_{j=1}^{k} (-1)^{j-1} S_j \leq P(A) \leq \sum_{j=1}^{k-1} (-1)^{j-1} S_j$$

where k is an even integer and for $j = 1, \ldots, n$

$$S_j = \sum_{1 \leq i_1 < \cdots < i_j \leq n} P\left(\bigcap_{m=1}^{j} A_{i_m}\right). \tag{2.8}$$

The first order Bonferroni upper bound is referred to as Boole's inequality and was introduced earlier in Boole (1854). Boole's inequality has been used in approximating the tail distribution of extreme order statistics (David (1981, Section 5.3)).

The classical Bonferroni inequalities for order $k > 1$ are computationally complex and can be quite inaccurate (Prékopa (1988)). Therefore, attempts have been made to improve them. The improved inequalities are referred to as Bonferroni–type. In this article I will concentrate mainly on one special class of Bonferroni–type inequalities proposed by Hunter (1976) and Worsley (1982). Let $A_j = (X_j \in I_j)$, $j = 1, \ldots, n$. The basic idea of this approach is to express

$$\begin{aligned} A = A_1 \quad &\cup \quad (A_2 \cap A_1^c) \cup (A_3 \cap A_2^c \cap A_1^c) \\ &\cup \quad \cdots \cup (A_n \cap A_{n-1}^c \cap \cdots \cap A_1^c) \end{aligned} \tag{2.9}$$

and obtain the following inequality

$$\begin{aligned} P(A) &\le P(A_1) + \sum_{j=2}^{n} P(A_j \cap A_{j-1}^c) \\ &= \sum_{j=1}^{n} P(A_j) - \sum_{j=2}^{n} P(A_j \cap A_{j-1}), \end{aligned} \tag{2.10}$$

which is a special case of a more general second order inequality discussed in Hunter (1976) and Worsley (1982).

Recently, Hoover (1990) extended this class of Bonferroni–type inequalities to order $k \geq 3$. Consider again the identity (2.9). Then,

$$P(A) \le S_1 - \sum_{i=1}^{n-1} p_{i,i+1} - \sum_{j=2}^{k-1} \sum_{i=1}^{n-j} p^*_{i,i+1,\ldots,i+j} \tag{2.11}$$

where S_1 is defined in equation (2.8),

$$p_{i,i+1} = P(A_i \cap A_{i+1})$$

and

$$p^*_{i,i+1,\ldots,i+j} = P(A_i \cap A_{i+1}^c \cap \cdots \cap A_{i+j-1}^c \cap A_{i+j}).$$

The inequality (2.11) is an improvement over the inequality (2.10) and is a member of a more general class of inequalities discussed in Hoover (1990).

In Sections 3 and 4, X_j are stationary and $I_j = I$ for $j = 1, \ldots, n$ and hence $A_1, \ldots, A_n$ are stationary events. It is tedious but routine to show that inequality (2.11) implies for the problem at hand:

$$(2.12) \qquad P\left(\bigcap_{j=1}^{n} A_j\right) \geq (n-k+1)P\left(\bigcap_{j=1}^{k} A_j\right) - (n-k)P\left(\bigcap_{j=1}^{k-1} A_j\right) = \beta_k.$$

The advantage in using the Bonferroni–type inequalities (2.12) is that they are valid without any assumptions on the distribution of the random vector $\mathbf{X}$. On the other hand, they are usually inaccurate and quite often give a negative value. The product–type inequalities or approximations always produce a value between 0 and 1. The following result supports the use of product–type approximations and inequalities.

THEOREM 2.4 (Glaz (1990a)). *Let* $\mathbf{X} = (X_1, \ldots, X_n)'$ *be a random vector and* $A_j = (X_j \in I_j)$, $j = 1, \ldots, n$. *If* γ_k *and* β_k *are given by equations* (2.6) *and* (2.12) *respectively, then*

$$\gamma_k > \beta_k.$$

REMARK Other interesting approaches to obtain improved Bonferroni–type inequalities are discussed in Hoppe and Seneta (1990), Prékopa (1988), Seneta (1988) and Tomescu (1986).

In Sections 3 and 4 of this article I will illustrate the performance of the Bonferroni–type inequalities and the product–type approximations that were discussed above for the problem of evaluating the distribution and the mean of extreme order statistics for two sequences of stationary dependent random variables.

3. Extreme Spacings

Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics of iid observations from the uniform distribution on the interval (0,1]. Consider the m–spacings defined by

$$(3.1) \qquad X_{(m+i-1)} - X_{(i)}, \qquad i = 1, \ldots, n-m+1.$$

The distribution of the smallest of the m–spacings,

$$(3.2) \qquad M_n^{(m)} = \min_{1 \leq i \leq n-m+1} \left\{ X_{(m+i-1)} - X_{(i)} \right\},$$

has been studied extensively (Barton and David (1956), Berman and Eagleson (1983, 1985), Cressie (1977a, 1977b, 1980, 1984), Darling (1953), Glaz

(1989, 1991a), Huntington and Naus (1975), Naus (1965, 1966, 1982), Neff and Naus (1980), Newell (1963), Pyke (1965), and Wallenstein and Neff (1987)). The distribution of $M_n^{(m)}$ is closely related to the distribution of *scan statistic*,

$$N_d = \sup_{0 \leq x \leq 1-d} N_{x,x+d}, \tag{3.3}$$

where $N_{x,x+d}$ is the number of observations that are in the scanning interval $(x, x+d]$ and $0 \leq d \leq 1$. $M_n^{(m)}$ is the smallest interval containing m points. The following relation is true:

$$P\{N_d \geq m\} = P\{M_n^{(m)} \leq d\}. \tag{3.4}$$

The exact distribution of $M_n^{(m)}$ is derived for $m = 2$ in Barton and David (1956) and Darling (1953), and for $m > 2$, under certain restriction in Naus (1965, 1966a, 1966b). A thorough discussion about the exact result for the case $m > 2$ (without any restrictions) is presented in Neff and Naus (1980), who tabulated $P\{M_n^{(m)} \leq d\}$ for $0 < d < .5$ and $3 \leq m < n \leq 20$. The formulas for evaluating the distribution of $M_n^{(m)}$ are complicated and computationally impractical for large value of n, moderate value of m and small value of d. Therefore, there has been an interest in evaluating asymptotic results: Berman and Eagleson (1983), Cressie (1977a, 1980), and McClure (1976). One can also employ the Chen–Stein method (Arratia, Goldstein and Gordon (1989, 1990), Chen (1975) and Stein, (1972, 1986, Chapter VIII)) and obtain Poisson approximation for the distribution of $M_n^{(m)}$.

In what follows an m^{th} order product–type approximation and an m^{th} order Bonferroni–type inequality are derived for $P\{M_n^{(m)} \leq d\}$. Similar results can be obtained for the distribution of the largest of the m–spacings.

For $3 \leq m \leq n/2$, $0 < d < .5$ and $0 \leq i \leq n-m+1$ define the events

$$A_i = \{X_{(m+i-1)} - X_{(i)} \leq d\},$$

where $X_{(0)} = 0$. It follows that

$$P\left\{M_n^{(m)} \leq d\right\} = P\left\{\bigcup_{i=1}^{n-m+1} A_i\right\} = 1 - P\left\{\bigcap_{i=1}^{n-m+1} A_i^c\right\}.$$

The following notation will be used throughout this article:

$$Q_1^* = P\{A_0\}, Q_k^* = P\left\{A_0 \cap \left(\bigcap_{j=1}^{k-1} A_j^c\right)\right\}, 2 \leq k \leq n-m+1 \tag{3.5}$$

and

$$(3.6)\qquad Q_k = P\left\{\bigcap_{j=0}^{k-1} A_j^c\right\}, \qquad 1 \le k \le n-m+1.$$

For $i = 1,\ldots,n-m$, $Q_{i+1} = Q_i - Q_{i+1}^*$ and, therefore, for $i \ge 2$, $Q_i = Q_1 - \sum_{j=2}^{i} Q_j^*$. It follows from Glaz (1991a, Section 2) that for $1 \le k \le n-m$

$$(3.7)\qquad P\left\{M_n^{(m)} \le d\right\} \le 1 - Q_k + (n-m+1-k)Q_k^*.$$

Equation (3.7) is the Bonferroni–type inequality given in equation (2.11). The following result is used in evaluating the inequality (3.7) for $3 \le k \le m$:

THEOREM 3.1 (Glaz (1991a)). *Let $X_{(1)},\ldots,X_{(n)}$ be the order statistics of iid observations from the uniform distribution on the interval* $(0,1]$. *Then for $3 \le k \le m \le n/2$ and $0 < d < .5$*

$$(3.8)\qquad \begin{aligned} Q_k^* &= b(m-1;n,d) - b(m;n,d) \\ &+ \sum_{j=k}^{n-m+1} (-1)^j \prod_{i=1}^{k-2}\left[1 - \frac{j(j-1)}{i(i+1)}\right] b(m+j-1;,n,d), \end{aligned}$$

where

$$b(j;n,d) = \binom{n}{j} d^j (1-d)^{n-j}.$$

The above result is also useful in evaluating the following product–type approximation for $P\{M_n^{(m)} \le d\}$. For $1 \le i \le n-m$, write

$$(3.9)\qquad P\{M_n^{(m)} \le d\} = 1 - Q_{n-m+1} = 1 - Q_i \prod_{j=i+1}^{n-m+1} (Q_j/Q_{j-1}),$$

which can be approximated by

$$(3.10)\qquad P^{(k)}\{M_n^{(m)} \le d\} = 1 - Q_k(Q_k/Q_{k-1})^{n-m+1-k},$$

where $1 \le k \le n-m$. Approximation (3.10), referred to as the k^{th} order product–type approximation, has been studied in Glaz (1989, 1991a). The product–type approximation (3.10) for $k = m$ can be viewed as an $(m-1)$ order Markov like approximation, where the terms Q_k/Q_{k-1}, for $m+1 \le k \le n-m+1$, are approximated by Q_m/Q_{m-1}. This approximation is supported by the asymptotic result stating that as $n \to \infty$ and $k \to \infty$ and $nd = O(1)$, $Q_k/Q_{k-1} \to \theta$, where $0 < \theta < 1$ is a constant (Glaz (1989, Theorem 3.1)).

In Table 1 the performance of the product–type approximation, the Bonferroni–type inequality and the asymptotic approximations mentioned above are compared for $n = 500$ and selected values of m, and d. From the numerical results it is evident that the product–type approximations are the most accurate ones. The Poisson asymptotic approximation using the Chen–Stein method, denoted by CS, outperforms all the other asymptotic approximations.

					Asymptotic Approximations		
d	m	Simulation	$P^{(m)}\{M_n^{(m)} < d\}$ (3.10)	$UB(m)$ (3.7)	Berman and Eagleson (1983)	Cressie (1977a)	CS
.001	4	.960	.997	>1	.999	1.000	.999
	5	.507	.506	.700	.587	.728	.577
	6	.069	.068	.071	.083	.122	.080
.005	8	.977	.971	>1	.999	1.000	.999
	9	.680	.670	>1	.883	1.000	.869
	10	.271	.267	.309	.440	.995	.417
	12	.019	.018	.018	.031	.258	.028
.01	12	.935	.916	>1	.999	1.000	.999
	13	.665	.645	>1	.938	1.000	.921
	14	.341	.331	.397	.643	1.000	.604
	15	.135	.135	.144	.300	1.000	.270
	16	.048	.048	.049	.109	.999	.095
.05	38	.618	.582	.807	.999	1.000	.997
	40	.314	.304	.351	.947	1.000	.880
	44	.046	.045	.046	.293	1.000	.194
	46	.014	.014	.014	.099	1.000	.058

Table 1. Comparison of Seven Approximations to $P\{M_n^{(m)} < d\}$ for $n = 500$.

Note: This simulation is based on 20,000 trials.

We now turn to the problem of evaluating $E[M_n^{(m)}]$. Exact results are available for $m = 2$ (Parzen (1960)) and for $n/2 < m \leq n$ (Naus (1966)). For $2 < m \leq n/2$ the following approximation is evaluated in Glaz (1991a). First, note that in this case $P\{M_n^{(m)} > x\} = 0$ for $x > .5$ and therefore

$$E[M_n^{(m)}] = \int_0^{.5} \bar{F}(x)dx, \tag{3.11}$$

where

$$\bar{F}(x) = P\{M_n^{(m)} > x\}.$$

Using the extended Simpson's rule for $2N$ points (Davis and Polonsky (1972, p. 886)) to evaluate numerically the integral in equation (3.11) we get that

$$E[M_n^{(m)}] \approx \frac{1}{12N}\left[1 + 4\sum_{i=1}^{N} \bar{F}(x_{2i-1}) + \sum_{i=1}^{N-1} \bar{F}(x_{2i})\right], \tag{3.12}$$

where $x_i = i/4N$. The numerical procedure is set up as follows. Start with $N = 25$ (50 points) and proceed to double the number of points in the interval $[0, .5]$ until the difference between successive approximations for $E[M_n^{(m)}]$ is less than 10^{-6}. In Table 2 below the approximation (3.12) is evaluated for $n = 100$ and selected values of m. These approximations are compared with simulated values and inequalities derived by Arnold and Groeneveld (1979) and Aven (1985, Corollary 2.1). The approximation that one can obtain from the Bonferroni–type inequalities will not be presented here as they have not produced accurate results. Related Bonferroni–type inequalities for expected values of order statistics have been studied in Hoover (1989).

m	Simulation $E[M_n^{(m)}]$	$E[M_n^{(m)}]$ (3.12)	LB Arnold &	UB Groeneveld (1979)	LB Aven (1985, Corr. 2.1)
10	.039	.039	—	.089	—
20	.120	.122	—	.188	—
30	.213	.215	—	.287	—
40	.312	.313	—	.386	—
50	.415	.416	—	.485	—

Table 2. A Comparison of Five Approximations for $E[M_n^{(m)}]$, $n = 100$.

Note: $E[M_n^{(m)}]$ was estimated from a simulation with 20,000 trials. — denotes negative values for the lower bounds.

From Table 2 it is evident that the product–type approximation provides an accurate approximation for the expected length of the smallest m^{th} order spacing. For large m the Arnold and Groeneveld (1979) upper bound appear to be quite good.

4. Moving Sums of Normal Random Variables

Let $Z_1, \ldots, Z_n$ be iid standard normal random variables. Define the sequence of moving sums of order m:

$$X_i = Z_i + \cdots + Z_{i+m-1}, \qquad i = 1, \ldots, n-m+1.$$

Approximations for the distribution of $X_{(n)} = \max(X_1, \ldots, X_n)$ have been discussed in Lai (1974), Bauer and Hackl (1980), Glaz and Johnson (1986) and Glaz (1990a). In this section I will evaluate the product–type and the Bonferroni–type approximations for the distribution of $X_{(n)}$ and compare them with the Poisson approximation (Aldous (1989, p. 50) and Holst and Janson (1990)). If the Z_i's have a discrete distribution the problem of approximating the distribution of the extreme order statistics of moving sums has been discussed in Glaz (1983), Glaz and Naus (1991), Naus (1982) and Samuel–Cahn (1983).

To evaluate the product–type approximation γ_k in equation (2.6) and the Bonferroni–type inequality β_k in equation (2.12) for

$$P\{X_{(n)} \le a\} = P(X_1 \le a, \ldots, X_n \le a)$$

we have to evaluate the $k-1$ and k dimensional multivariate normal probabilities. In this paper I will use the algorithm developed in Schervish (1984). The inequalities and the approximations will be computed for $1 \leq k \leq 5$ only since for $k = 6$ the evaluation of the multivariate probabilities becomes time consuming. The use of product–type approximations for approximating the distribution of extreme order statistics for moving sums of normal random variables is supported by the quasi–stationarity property for moving sums of iid random variables in Glaz and Johnson (1986). For moving sums of iid discrete random variables the quasi–stationarity has been established in Samuel–Cahn (1983).

The Poisson approximation using the Chen–Stein method (Aldous (1989, p. 48–52)) is given by

$$\lim_{n\to\infty} P\{X_{(n)} \leq a\} = \exp\{-nx\}, \tag{4.1}$$

where a is large and

$$x = P\{X_1 + \cdots + X_m > a\}. \tag{4.2}$$

In Tables 3 and 4 the performance of the product–type approximations, Bonferroni–type inequalities and the Poisson asymptotic approximation using the Chen–Stein method (denoted in the tables by CS) is compared for $m = 10$ and $a = 10$.

n	P^*	γ_5	β_5	γ_4	β_4	γ_3	β_3	γ_1	β_1	CS
200	.926	.926	.923	.925	.923	.923	.920	.861	.850	.861
400	.856	.855	.844	.854	.842	.849	.836	.736	.694	.736
600	.792	.789	.764	.788	.762	.781	.752	.629	.537	.630
800	.732	.729	.684	.726	.681	.718	.669	.538	.381	.538
1000	.675	.673	.605	.670	.601	.660	.585	.460	.224	.460
2000	.455	.451	.206	.448	.198	.434	.168	.210	—	.210
3000	.305	.303	—	.299	—	.286	—	.096	—	.096
4000	.202	.203	—	.200	—	.188	—	.044	—	.044
5000	.133	.136	—	.133	—	.128	—	.020	—	.020

Table 3. A Comparison of Ten Approximations for $P[X_{(n)} \leq 10]$.

Note: P^* was estimated from a simulation with 10,000 trials. — denotes a negative value for the approximations.

n	P^*	γ_5	γ_4	γ_3	γ_2	CS
6000	.090	.091	.089	.081	.069	.0092
7000	.060	.061	060	.054	.045	.0042
8000	.041	.041	.040	.035	.029	.0019
9000	.028	.028	.027	.023	.018	.0009
10000	.020	.019	.018	.015	.012	.0004

Table 4. A Comparison of Six Approximations for $P[X_{(n)} \leq 10]$.

Note: P^* was estimated from a simulation with 10,000 trials.

From Tables 3 and 4 it is evident that the product–type approximations are more accurate than the Bonferroni–type inequalities and the Poisson approximation. The product–type approximation γ_5 is remarkably accurate throughout the entire range. It is also interesting to note that the Poisson approximation (4.1) is equal to the value of

$$\gamma_1 = (1 - x)^n$$

where x is defined in equation (4.2). It is routine to verify that the Poisson approximation always exceeds γ_1, but for a value of x that is close to 0 (which is the case in this example that was chosen to enhance the performance of the Poisson approximation) both approximations are equivalent.

At this point, I would like to note that Prékopa (1988) has used a linear programming approach to derive optimal k^{th} order Bonferroni–type inequalities for the probability of a union or intersection of n events. The difficulty in applying this approach is the need to evaluate the terms S_j given in equation (2.8) for large j. For a special case of the problem considered in this section ($m = 2$) Glaz (1990b) compares the performance of the product–type approximation with these Bonferroni–type inequalities. Again, the product–type approximation appears to be a more accurate approximation.

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