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ORDERINGS ARISING FROM EXPECTED EXTREMES, WITH AN APPLICATION

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We bound the expected maximum order statistics $\{EX_{(n)}\}_{n=1}^{\infty}$ of a d.f. F_X both above and below. Our results have an interpretation in terms of stochastic orderings $\leq_e$ and $\leq_{we}$ defined as follows: $F_X \leq_e F_Y$ iff $EX_{(n)} \leq EY_{(n)}$ for all n, and $F_X \leq_{we} F_Y$ iff $EX_{(n)} \leq EY_{(n)}$ for n sufficiently large. We apply our results on $\leq_{we}$ to the end-to-end delay in a resequencing $M/G/\infty$ queue.

1. Introduction

If $X_1, \ldots, X_n$ are i.i.d. random variables with parent distribution F_X, let $X_{(n)}$ denote the maximum order statistic $\max(X_1, \ldots, X_n)$. We are interested in the case when F_X has nonnegative lower endpoint, and upper endpoint $+\infty$. In this case we wish to control the behavior of $X_{(n)}$ as $n \to \infty$; in particular, to bound it above and below in expectation or in related senses. The bounds should be as free of assumptions on the distribution F_X as possible.

Our original motivation for investigating this question was the study of stochastic models arising in computing (Downey and Maier (1990)). There the X_i are interpreted as time delays. (See Section 3 for a typical example, a resequencing $M/G/\infty$ queueing model.) But our results have a more general interpretation, in terms of stochastic inequalities. If a relation $\leq_e$ and its weak counterpart $\leq_{we}$ are defined on the class of finite-mean distributions of nonnegative r.v.'s by

$$(1) \qquad F_X \leq_e F_Y \iff EX_{(n)} \leq EY_{(n)}, \quad n \geq 1$$

$$(2) \qquad F_X \leq_{we} F_Y \iff EX_{(n)} \leq EY_{(n)}, \quad n \text{ suff. large}$$

then our results have implications for $\leq_e$ and $\leq_{we}$.

The orderings $\leq_e$ and $\leq_{we}$ are very natural, but seem never to have been studied before. Chan (1967) showed that a distribution is uniquely

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determined by its expected extreme order statistics, a result that has been considerably generalized (Huang (1987)). In fact F_X is uniquely determined by the sequence $\{EX_{(n)}\}_{n=N}^{\infty}$, for any $N \geq 1$. So both $\leq_e$ and $\leq_{we}$ are antisymmetric relations, and are therefore partial orders. We shall see that they are related to the increasing convex order $\leq_{icx}$.

Several different lines of research have yielded upper and lower bounds on $EX_{(n)}$. Arnold (1985) showed that if $EX^p < \infty$, then $EX^p = O(n^{1/p})$, $n \to \infty$. The precise statement is

$$EX_{(n)} \leq EX + \|X - EX\|_p n^{1/p} \tag{3}$$

with $\|Z\|_p$ signifying the L^p norm $(E|Z|^p)^{1/p}$; this result was rediscovered by Downey (1990). This is an example of a distribution-free result. Other results follow from the classical theory of the convergence in distribution of $X_{(n)}$, suitably normalized, as $n \to \infty$. It is well known that many distributions F_X lie in the domain of attraction $\mathcal{D}(\Lambda)$ of $\Lambda(t) \overset{\text{def}}{=} \exp(-e^{-t})$, the double exponential distribution. For them we have $(X_{(n)} - b_n)/a_n \Longrightarrow Y$ with Y distributed according to the law Λ, if a_n and b_n are appropriately chosen. Gnedenko (1943) showed that one may take $b_n = \bar{F}_X^{\leftarrow}(n^{-1})$ and $a_n = \bar{F}_X^{\leftarrow}(e^{-1}n^{-1}) - \bar{F}_X^{\leftarrow}(n^{-1})$; here $\bar{F}_X^{\leftarrow}$ is the right-continuous inverse of the complementary d.f. $\bar{F}_X \overset{\text{def}}{=} 1 - F_X$.

De Haan (1975) showed that if $F_X \in \mathcal{D}(\Lambda)$, convergence in distribution also obtains if a_n is chosen to equal $\mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))$. Here $\mu_X(t)$ signifies the mean residual life after time t, $\bar{F}_X(t)^{-1}\int_t^\infty \bar{F}_X(s)\,ds$. Pickands (1968) showed that moments converge as well. So if $F_X \in \mathcal{D}(\Lambda)$,

$$EX_{(n)} \sim \bar{F}_X^{\leftarrow}(n^{-1}) + \gamma\mu_X(\bar{F}_X^{\leftarrow}(n^{-1})), \qquad n \to \infty \tag{4}$$

since the Euler-Mascheroni constant γ is the first moment of the double exponential distribution. In general one expects that even if $F \notin \mathcal{D}(\Lambda)$, if F_X has a sufficiently thin and well-behaved tail then $X_{(n)}$ is not likely to differ from $\bar{F}_X(n^{-1})$ by much more than $\mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))$ in the $n \to \infty$ limit. However the question of which distributions F_X have the property that for all $\epsilon > 0$, there is an M such that

$$\limsup_{n\to\infty} P\left\{\left|\left(X_{(n)} - \bar{F}_X^{\leftarrow}(n^{-1})\right)/\mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))\right| > M\right\} \leq \epsilon \tag{5}$$

seems not to have been resolved. This property defines a larger class than $\mathcal{D}(\Lambda)$. Geometric distributions, for example, satisfy it but are not attracted to Λ.

It is known however (Gnedenko (1943)) that if $\bar{F}_X \in R_{-\infty}$, *i.e.*, the complementary d.f. is regularly varying with index $-\infty$, then

$$X_{(n)}/\bar{F}_X^{\leftarrow}(n^{-1}) \to 1 \tag{6}$$

in probability; the converse also holds. (In fact by the work of Lai and Robbins (1978) and Pickands (1968)) we may substitute for (6) the statement that for all $p > 0$, $E\left|X_{(n)}/\bar{F}_X^{\leftarrow}(n^{-1}) - 1\right|^p \to 0$.) Recall that $\bar{F} \in R_{-\infty}$ means that for all $c > 1$

$$\lim_{t\to\infty} \bar{F}(ct)/\bar{F}(t) = 0. \tag{7}$$

It is known (Resnick (1987)) that if F has upper endpoint $+\infty$, then $F \in \mathcal{D}(\Lambda) \Rightarrow \bar{F} \in R_{-\infty}$. So $\bar{F} \in R_{-\infty}$ is another natural weakening of the condition $F \in \mathcal{D}(\Lambda)$.

In general imposing such regularity conditions as the property (5), $F_X \in \mathcal{D}(\Lambda)$, or $\bar{F}_X \in R_{-\infty}$ will facilitate the control of the sequence $\{EX_{(n)}\}_{n=1}^{\infty}$. But as we sketch in the next section, the large-n asymptotics of this sequence can be usefully bounded in terms of $\bar{F}_X^{\leftarrow}(n^{-1})$ and $\mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))$ even if *no* regularity assumptions are imposed on F_X.

2. Recent Results

Suppose that F_X has upper endpoint t_X^* and is the distribution of an r.v. with finite mean. Since $X_{(n)}$ has distribution F_X^n, we have $\bar{F}_{X_{(n)}} = h_n(\bar{F}_X)$ with $h_n(u) \stackrel{\text{def}}{=} 1 - (1-u)^n$. So

$$EX_{(n)} = \int_0^\infty \bar{F}_{X_{(n)}}(t)\,dt = \int_0^\infty h_n(\bar{F}_X(t))\,dt. \tag{8}$$

It is natural to extend this statement to noninteger values of n; indeed, to all $n \in [0,\infty)$. With this definition $EX_{(n)}$, as a function of n, will be increasing and concave; in fact, its derivative is completely monotone in the sense of Widder (1971). For the remainder of this paper we allow n to take on noninteger values.

THEOREM 2.1 (Downey and Maier (1990)) *We have the following bounds on $EX_{(n)}$. For all $t \in [0,\infty)$ and $n \in [1,\infty)$*

$$EX_{(n)} \le t + n\int_t^\infty \bar{F}_X(s)\,ds \tag{9}$$

and for all $t \in [0, t_X^*)$

$$EX_{(n)} > (1-e^{-1})\left(t + n\int_t^\infty \bar{F}_X(s)\,ds\right) \tag{10}$$

in which $n \stackrel{\text{def}}{=} \bar{F}_X(t)^{-1}$. The same lower bound holds for arbitrary $n \in [1,\infty)$ if t is defined to equal $\bar{F}_X^{\leftarrow}(n^{-1})$.

REMARK The upper bound of the theorem is well known (Lai and Robbins (1978)); the lower bound follows from (8), and is a refinement of Chebyshev's inequality.

COROLLARY *In general*

$$(1-e^{-1})\left[\bar{F}_X^{\leftarrow}(n^{-1}) + \mu_X(\bar{F}_X^{\leftarrow}(n^{-1})-)\right] < EX_{(n)} \tag{11}$$
$$\leq \bar{F}_X^{\leftarrow}(n^{-1}) + \mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))$$

for all $n \geq 1$. *So if the distribution* F_X *is continuous,*

$$\frac{EX_{(n)}}{\bar{F}_X^{\leftarrow}(n^{-1}) + \mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))} \in (1-e^{-1}, 1] \tag{12}$$

for all $n \geq 1$.

PROOF (of Corollary). To obtain the upper bound we set $t = \bar{F}_X^{\leftarrow}(n^{-1})$. This implies $n \leq \bar{F}_X(t)^{-1}$, so the upper bound follows. It also implies $n \geq \bar{F}_X(t-)^{-1}$, so the lower bound follows as well. □

The corollary provides the desired distribution-free bound on $EX_{(n)}$ in terms of $\bar{F}_X^{\leftarrow}(n^{-1})$ and $\mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))$, or rather $\mu_X(\bar{F}_X^{\leftarrow}(n^{-1})-)$. Due to the presence of the $1-e^{-1}$ factor, for general distributions $X_{(n)}$ is allowed to differ in expectation from $\bar{F}^{\leftarrow}(n^{-1})$ by much more than $O(\mu_X(\bar{F}_X^{\leftarrow}(n^{-1})))$ in the large-n limit. The deviation may only be in the negative direction however. So for continuous distributions the inequality (5) may be replaced by

$$\limsup_{n\to\infty} P\left\{\left(X_{(n)} - \bar{F}_X^{\leftarrow}(n^{-1})\right)/\mu_X(\bar{F}_X^{\leftarrow}(n^{-1})) < -M\right\} \leq \epsilon \tag{13}$$

without any loss of generality.

Another consequence of Theorem 2.1 is the abovementioned relation between $\leq_e$ and $\leq_{icx}$. Recall that $F_X \leq_{icx} F_Y$ iff $\int_t^\infty \bar{F}_X\,ds \leq \int_t^\infty \bar{F}_Y\,ds$ for all $t \geq 0$. Equivalently, $Ef(X) \leq Ef(Y)$ for all increasing convex functions f on $[0,\infty)$. So $\leq_{icx}$ is a weaker ordering than $\leq_d$, the standard stochastic ordering.

THEOREM 2.2 (Downey and Maier (1990)) $\leq_e$ *and* $\leq_{icx}$ *are related as follows.*

1. $F_X \leq_{icx} F_Y \Rightarrow F_X \leq_e F_Y$.
2. $F_X \leq_e F_Y \Rightarrow F_X \leq_{icx} F_{\kappa Y}$ *for some universal constant* κ, *which may be taken to equal* $(1-e^{-1})^{-1}$.

PROOF (1) It is well known (Ross (1983)) that if $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$ are independent and $X_i \le_{icx} Y_i$ for all i, then

$$g(X_1,\dots,X_n) \le_{icx} g(Y_1,\dots,Y_n) \tag{14}$$

for all increasing convex functions g on $\mathbb{R}^n$. Since max is an increasing convex function of its arguments, $X_{(n)} \le_{icx} Y_{(n)}$. So $EX_{(n)} \le EY_{(n)}$.

(2) We shall prove the contrapositive of the claim. Assume that $F_X \not\le_{icx} F_Y$, *i.e.*, that $\int_t^\infty \bar F_X(s)\,ds > \int_t^\infty \bar F_Y(s)\,ds$ for some $t \in [0, t_X^*)$. The lower bound of Theorem 2.1, applied to F_X, says that

$$EX_{(n)} > (1-e)^{-1}\left(t + n\int_t^\infty \bar F_X(s)\,ds\right) \tag{15}$$

with $n \stackrel{\text{def}}{=} \bar F_X(t)^{-1}$. The upper bound of Theorem 2.1, applied to F_Y, says that

$$EY_{(n)} \le t + n\int_t^\infty \bar F_Y(s)\,ds. \tag{16}$$

Combining the bounds (15) and (16) yields $EX_{(n)} > (1-e^{-1})EY_{(n)}$. That is, if $\kappa \stackrel{\text{def}}{=} (1-e^{-1})^{-1}$ then $EX_{(n)} > E\kappa^{-1}Y_{(n)}$. So $F_X \not\le_e F_{\kappa^{-1}Y}$. □

Theorem 2.2 implies that if distributions which differ only by a change of scale are identified, $\le_e$ and $\le_{icx}$ become identical. This is a very curious result, and suggests that it may prove profitable to explore the ways in which stochastic orderings relate such 'scaling equivalence classes' of distributions. Scaling equivalence classes have been considered by Barlow and Proschan (1975).

For the queueing theory application of the next section we need a variant form of Theorem 2.2, which characterizes $\le_{we}$ rather than $\le_e$. Theorem 2.3, the proof of which is almost identical, relates $\le_{we}$ to the *weak* increasing convex ordering $\le_{wicx}$, defined as follows:

$$F_X \le_{wicx} F_Y \iff \int_t^\infty \bar F_X\,ds \le \int_t^\infty \bar F_Y\,ds, \quad t \text{ suff. large.} \tag{17}$$

Equivalently, $F_X \le_{wicx} F_Y$ if and only if $Ef(X) \le Ef(Y)$ for all increasing convex f supported sufficiently far away from zero. $\le_{wicx}$, unlike $\le_{icx}$, $\le_e$ and $\le_{we}$, is not a partial order: it is merely a pre-order.

THEOREM 2.3 *$\le_{we}$ and $\le_{wicx}$ are related as follows. For any $\gamma > 1$*

1. $F_X \le_{wicx} F_Y \Rightarrow F_X \le_{we} F_{\gamma Y}$.
2. $F_X \le_{we} F_Y \Rightarrow F_X \le_{wicx} F_{\gamma\kappa Y}$, *for κ the universal constant of* Theorem 2.2.

3. A Queueing Application

We now show how the above results yield useful bounds on a stochastic model introduced by Harrus and Plateau (1982) and pursued by Baccelli, Gelenbe and Plateau (1984). The model is based on an $M/G/\infty$ queue. Arrivals to the queue are Poisson; that is, interarrival times are distributed according to the law $EXP(\lambda)$, with λ some specified arrival rate. Since there are an infinite number of servers available, customers are processed immediately upon arrival; service time has some finite-mean distribution F_X. We write $\mu \stackrel{\text{def}}{=} (EX)^{-1}$ for the processing rate.

This $M/G/\infty$ queue will be recurrent, irrespective of the traffic intensity $\rho \stackrel{\text{def}}{=} \lambda/\mu$, and the stationary distribution of the number of busy servers will be Poisson with parameter ρ. However we require that for a customer to depart, all its predecessors must have departed. In other words the processing must not be allowed to alter the order of the arriving customers; they are released only in sequence. This introduces an additional *resequencing delay*: a customer's total delay time Y, the 'end-to-end' delay, will be the sum of the processing time X and (possibly) some additional holding time.

A formally stationary distribution for Y was worked out by Harrus and Plateau. Baccelli, Gelenbe and Plateau showed that if the queue begins empty, the distribution of the end-to-end delay of the jth customer does indeed converge, as $j \to \infty$, to the formula given by Harrus and Plateau. Their formula is equivalent to the following (Downey (1992a)):

$$F_Y(t) = \sum_{n=0}^{\infty} \frac{e^{-\rho}\rho^n}{n!} F_X(t) F_{X^*}^n(t) \tag{18}$$

in which F_{X^*} is the distribution of the equilibrium excess of the renewal process with renewal period distribution F_X. That is,

$$\bar{F}_{X^*}(t) = (EX)^{-1} \int_t^{\infty} \bar{F}_X(s)\, ds. \tag{19}$$

The interpretation of formula (18) is simple. If we condition on n servers being busy with previous arrivals when a new customer arrives, since the arrival time is random the time to completion of the kth server, $k = 1, \ldots, n$, will have distribution F_{X^*}. So the end-to-end delay of the new arrival will necessarily be $\max(X, X_1^*, \ldots, X_n^*)$, in which $X_1^*, \ldots, X_n^*$ are i.i.d. with parent distribution F_{X^*}. Since n is Poisson, removing the conditioning yields (18).

We wish to study how the end-to-end delay of this system, in the heavy traffic limit, depends on characteristics of the service time distribution other than its expectation. So we fix μ, and restrict ourselves to distributions with expectation μ^{-1}. We equip this class with a pre-order $\prec$ defined as follows:

if F_{X_1} and F_{X_2} are two service time distributions, we say that $F_{X_1} \prec F_{X_2}$ iff $EY_1 \le EY_2$ for all sufficiently large ρ. Here Y_1 and Y_2 are the corresponding end-to-end delay times, whose distributions are computed from F_{X_1} and F_{X_2} by (18).

It follows from (18) that

$$EY = \sum_{n=0}^{\infty} \frac{e^{-\rho}\rho^n}{n!} E\max(X, X_1^*, \ldots, X_n^*) \tag{20}$$

$$= \left(\sum_{n=0}^{\infty} \frac{e^{-\rho}\rho^n}{n!} EX_{(n)}^*\right) + \frac{1-e^{-\rho}}{\lambda}. \tag{21}$$

The expression (21) follows from (20) by noting that

$$E\max(X, X_1^*, \ldots, X_n^*) = EX_{(n)}^* + (n+1)^{-1}EX. \tag{22}$$

This is easily verified by integration by parts.

The formula (21) allows us to prove Theorem 3.1 below. The statement of the theorem relies on an ordering $\le_3$ and its weak counterpart $\le_{w3}$, defined as follows. We say that

$$F_{X_1} \le_3 F_{X_2} \iff \int_t^{\infty}\int_s^{\infty} \bar{F}_{X_1}(u)\,du\,ds \le \int_t^{\infty}\int_s^{\infty} \bar{F}_{X_2}(u)\,du\,ds, \quad t \ge 0 \tag{23}$$

Equivalently, $F_{X_1} \le_3 F_{X_2}$ iff $Ef(X_1) \le Ef(X_2)$ for all functions f on $[0,\infty)$ that are increasing, convex and have nonnegative third derivative. $\le_{w3}$ is the corresponding weak pre-order:

$$F_{X_1} \le_{w3} F_{X_2} \iff \int_t^{\infty}\int_s^{\infty} \bar{F}_{X_1}(u)\,du\,ds$$
$$\le \int_t^{\infty}\int_s^{\infty} \bar{F}_{X_2}(u)\,du\,ds, \quad t \text{ suff. large.} \tag{24}$$

Equivalently, $F_{X_1} \le_{w3} F_{X_2}$ iff $Ef(X_1) \le Ef(X_2)$ for all functions f on $[0,\infty)$ that are increasing, convex and have nonnegative third derivative, and are supported sufficiently far away from zero. The definitions (23) and (24) serve to define $\le_3$ and $\le_{w3}$ on the class of d.f.'s that have finite mean and variance.

THEOREM 3.1 *If F_{X_1} and F_{X_2} are two service time distributions with finite variance and the same (finite) mean, then for any $\gamma > 1$*

1. $F_{X_1} \le_{w3} F_{\gamma^{-1}X_2} \Rightarrow F_{X_1} \prec F_{\gamma X_2}$.
2. $F_{X_1} \prec F_{X_2} \Rightarrow F_{X_1} \le_{w3} F_{\kappa\gamma^2 X_2}$, *for κ the universal constant of* Theorem 2.2.

PROOF Since $EX_1 = EX_2$, the parameter ρ is the same for both service time distributions. Also, since $EX_1{}^2$, $EX_2{}^2 < \infty$ we have by examination that EX_1^*, $EX_2^* < \infty$. But if Z is any nonnegative r.v. with finite expectation, it is easily verified that

$$\sum_{n=0}^{\infty} \frac{e^{-\rho}\rho^n}{n!} EZ_{(n)} \sim EZ_{(\rho)}, \quad \rho \to \infty. \tag{25}$$

(This is a special case of an Abelian theorem for completely monotone functions (Downey (1992b)).) It follows by (21) that $EY_i \sim E(X_i^*)_{(\rho)}$, $\rho \to \infty$. Accordingly for any $\gamma > 1$

$$F_{X_1^*} \leq_{we} F_{X_2^*} \quad \Rightarrow \quad F_{X_1} \prec F_{\gamma X_2} \tag{26}$$

$$F_{X_1} \prec F_{X_2} \quad \Rightarrow \quad F_{X_1^*} \leq_{we} F_{\gamma X_2^*} \tag{27}$$

These two implications may be extended by applying Theorem 2.3; we get

$$F_{X_1^*} \leq_{wicx} F_{\gamma^{-1} X_2^*} \Rightarrow F_{X_1^*} \leq_{we} F_{X_2^*} \Rightarrow F_{X_1} \prec F_{\gamma X_2} \tag{28}$$

$$F_{X_1} \prec F_{X_2} \Rightarrow F_{X_1^*} \leq_{we} F_{\gamma X_2^*} \Rightarrow F_{X_1^*} \leq_{wicx} F_{\kappa\gamma^2 X_2^*} \tag{29}$$

By (19), the hypothesis of (28) may be written as

$$(\forall t \geq 0) \qquad \int_t^\infty \int_s^\infty \bar{F}_{X_1}(u)\, du\, ds \leq \gamma \int_t^\infty \int_s^\infty \bar{F}_{\gamma^{-1} X_2}(u)\, du\, ds, \tag{30}$$

and the conclusion of (29) as

$$(\forall t \geq 0) \qquad \int_t^\infty \int_s^\infty \bar{F}_{X_1}(u)\, du\, ds \leq \kappa^{-1}\gamma^{-2} \int_t^\infty \int_s^\infty \bar{F}_{\kappa\gamma^2 X_2}(u)\, du\, ds. \tag{31}$$

But (30) is implied by $F_{X_1} \leq_{w3} F_{\gamma^{-1} X_2}$, and similarly (31) implies $F_{X_1} \leq_{w3} F_{\kappa\gamma^2 X_2}$. So we are finished. □

Theorem 3.1 makes it clear that in analysing the effects of the service time distribution on the expected end-to-end delay in the heavy traffic limit, the ordering $\leq_{w3}$ on service time distributions will prove useful. It is difficult to see how this could have been deduced without the aid of Theorem 2.2.

It would of course be desirable to reduce the constant κ toward unity. Theorem 3.1 is a distribution-free result, and we expect substantial strengthening will be possible if regularity conditions are imposed on the service time distributions.

4. Conclusions

We have seen that for any finite-mean distribution F_X, $EX_{(n)}$ may be bounded for any n above and below in terms of $\bar{F}_X^{\leftarrow}(n^{-1})$ and the mean residual life $\mu_X(\bar{F}_X^{\leftarrow}(n^{-1}))$. $\mu_X(t)$ is expressible in terms of an integral of $\bar{F}_X(t)$, so it proved possible to relate $\leq_e$ to the 'integrated' stochastic ordering $\leq_{icx}$.

Our result Theorem 2.2, and its weak counterpart Theorem 2.3, are expressed in terms of a universal constant κ. It is not clear that our bounds, when the choice $\kappa = (1 - e^{-1})^{-1}$ of Section 2 are used, are tight. It would be desirable either to prove this or to compute the minimal value of κ, particularly from the point of view of applications such as that of Section 3. Moreover the classes of d.f.'s for which $X_{(n)} - \bar{F}_X^{\leftarrow}(n^{-1})$ is $O(\mu_X(\bar{F}_X^{\leftarrow}(n^{-1})))$, in expectation or in other senses, remain to be characterized.

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