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SOME REMARKS ON A NOTION OF POSITIVE DEPENDENCE, ASSOCIATION, AND UNBIASED TESTING[1]

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A new notion of positive dependence is studied. The new notion implies association of k random variables but is weaker than the notion of conditionally increasing in sequence.

1. Introduction

Tong (1990) discusses various notions of positive dependence of a collection of k random variables $(X_1, X_2, \ldots, X_k)$. Among the notions are multivariate totally positive of order 2 (MTP_2), conditionally increasing in sequence (CIS), and (positively) associated (A). In Proposition 5.1.2 on page 95 of Tong (1990), it is noted that

$$MTP_2 \Rightarrow CIS \Rightarrow A. \tag{1.1}$$

For applications it can be important to know whether a set of k random variables is MTP_2 or CIS, sometimes because the property implies A and sometimes because of other probability statements or inequalities that can be achieved. In Cohen, Kemperman, and Sackrowitz (CKS) (1992) another notion of positive dependence was introduced. This new notion, which we will call weak conditionally increasing in sequence ($WCIS$) is implied by CIS but implies A. Thus $WCIS$ is weaker than CIS but yet still $WCIS$ implies A. CKS (1992) used $WCIS$ to establish a class of unbiased tests for testing whether the natural parameters of k exponential family PF_2 distributions lie on a line against the alternative that the parameters are convex, i.e., their weighted second order differences are nonnegative.

Association is frequently a method of establishing unbiasedness of classes of tests. See for example, Perlman and Olkin (1980), Cohen and Sackrowitz (1987), and Cohen, Perlman and Sackrowitz (1991). CKS (1992) prove unbiasedness by establishing A via $WCIS$. In that study both MTP_2 and CIS fail to hold for the relevant variables.

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In this note we formally define $WCIS$ and show that

$$MTP_2 \Rightarrow CIS \Rightarrow WCIS \Rightarrow A. \tag{1.2}$$

An example is given where a random vector $\mathbf{X} = (X_1, \ldots, X_k)'$ is $WCIS$ but not CIS and another example is given in which $\mathbf{X}$ is A but not $WCIS$.

For a $k \times 1$ normal random vector with covariance matrix Σ, we indicate necessary and sufficient conditions for it to have the CIS property or $WCIS$ property. These conditions are useful when it is of importance to establish the CIS or $WCIS$ property for its own use as opposed to establishing it for the purpose of establishing A. If one were seeking to establish A for a normal random vector, the known necessary and sufficient condition that $\Sigma \geq 0$ would be used. This is a result of Pitt (1982).

In the next section, we give the results on $WCIS$.

2. Weak Conditionally Increasing in Sequence ($WCIS$)

Let $\mathbf{X}^{k \times 1}$ be a random vector with density function $f_{\mathbf{X}}(\mathbf{x})$.

DEFINITION 2.1 The random vector $\mathbf{X}$ and its density are said to be MTP_2 if

$$f_{\mathbf{X}}(\mathbf{x} \vee \mathbf{y}) f_{\mathbf{X}}(\mathbf{x} \wedge \mathbf{y}) \geq f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{y}), \quad \text{for all } \mathbf{x}, \mathbf{y}, \tag{2.1}$$

where $\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \ldots, \max(x_k, y_k))$ and $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \ldots, \min(x_k, y_k))$.

DEFINITION 2.2 The random vector $\mathbf{Y}$ is stochastically greater than or equal to the random vector $\mathbf{X}$ ($\mathbf{X} \leq^P \mathbf{Y}$) if

$$Eh(\mathbf{X}) \leq Eh(\mathbf{Y}), \tag{2.2}$$

for all nondecreasing functions h for which the expectations in (2.2) exist. (h is nondecreasing if it is nondecreasing in x_i while $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$ are fixed, $i = 1, 2, \ldots, k$.)

DEFINITION 2.3 The random variables in $\mathbf{X} = (X_1, \ldots, X_k)'$ are said to be conditionally increasing in sequence (CIS) if for $j = 1, \ldots, k-1$

$$\begin{aligned} &[X_{j+1} | (X_1 = x_1, X_2 = x_2, \ldots, X_j = x_j)] \\ &\quad \leq^P [X_{j+1} | (X_1 = x_1^*, \ldots, X_j = x_j^*)], \end{aligned} \tag{2.3}$$

for $x_1 \leq x_1^*, x_2 \leq x_2^*, \ldots, x_j \leq x_j^*$. (See Tong (1990, p. 92).)

DEFINITION 2.4 The random variables in $\mathbf{X}$ are said to be weak conditionally increasing in sequence ($WCIS$) if for $j = 1, \ldots, k-1$,

$$\begin{aligned} &[(X_{j+1}, \ldots, X_k) | X_1 = x_1, \ldots, X_j = x_j] \\ \text{(2.4)} \qquad &\leq^P [(X_{j+1}, \ldots, X_k) | X_1 = x_1, \ldots, X_j = x_j^*], \end{aligned}$$

for $x_j \leq x_j^*$.

In other words, $WCIS$ means that for each $j = 1, \ldots, k$, the random vector $(X_{j+1}, \ldots, X_k)$ given $X_1, \ldots, X_j$ need only be stochastically nondecreasing in $X_j = x_j$.

DEFINITION 2.5 The random variables in $\mathbf{X} = (X_1, \ldots, X_k)'$ are said to be (positively) associated (A) if

$$Eh_1(\mathbf{X})h_2(\mathbf{X}) \geq Eh_1(\mathbf{X})Eh_2(\mathbf{X}) \tag{2.5}$$

holds for all nondecreasing functions h_1, h_2 for which the expectations in (2.5) exist.

The relation (2.5) is sometimes called the multivariate correlation inequality.

Now we prove

PROPOSITION 2.6 *The following implications are true:*

$$MTP_2 \overset{(a)}{\Rightarrow} CIS \overset{(b)}{\Rightarrow} WCIS \overset{(c)}{\Rightarrow} A.$$

Furthermore, all implications are strict for $k \geq 3$. (Implication (a) requires $f_{\mathbf{X}}(\mathbf{x}) > 0$.)

PROOF The implication (a) is proven in Tong (1990, p. 95). The implication (c) is proven in CKS (1992), Theorem 2.5. To prove the implication (b), we need to show that (2.3) implies (2.4). Fix j and let $h(x_{j+1}, \ldots, x_k)$ be a nondecreasing function. For each $m = j, \ldots, k$, define

$$h_{(m)}(X_1, \ldots, X_m) = E[h(X_{j+1}, \ldots, X_k) | X_1, \ldots, X_m].$$

We make the following three observations:

$$h_{(j)}(X_1, \ldots, X_j) = E[h(X_{j+1}, \ldots, X_k) | X_1, \ldots, X_j], \tag{2.6}$$

$$\begin{aligned} h_{(m)}(X_1, \ldots, X_m) &= E[h(X_{j+1}, \ldots, X_k) | X_1, \ldots, X_m] \\ &= E\{E[h(X_{j+1}, \ldots, X_k) | X_1, \ldots, X_{m+1}] | X_1, \ldots, X_m\} \\ \text{(2.7)} \qquad &= E[h_{(m+1)}(X_1, \ldots, X_{m+1}) | X_1, \ldots, X_m], \\ &\qquad \text{for } m = j, \ldots, k-1, \end{aligned}$$

$$h_{(k)}(X_1,\ldots,X_k)=h(X_{j+1},\ldots,X_k). \tag{2.8}$$

We claim that, due to (2.7), if $h_{(m+1)}$ is nondecreasing as a function of $X_1,\ldots,X_{m+1}$, then $h_{(m)}$ is nondecreasing as a function of $X_1,\ldots,X_m$. To see this, let $x_i \le x_i^*$, $i=1,2,\ldots,m$. Then by (2.7),

$$\begin{aligned} h_{(m)}(x_1,\ldots,x_m) &= E[h_{(m+1)}(X_1,\ldots,X_{m+1})|X_1=x_1,\ldots,X_m=x_m] \\ &= E[h_{(m+1)}(x_1,\ldots,x_m,X_{m+1})|X_1=x_1,\ldots,X_m=x_m] \\ &\le E[h_{(m+1)}(x_1^*,\ldots,x_m^*,X_{m+1})|X_1=x_1,\ldots,X_m=x_m] \end{aligned} \tag{2.9}$$

whenever $h_{(m+1)}$ is nondecreasing in all its arguments. Furthermore, the *CIS* property implies that the last expression in (2.9) is less than or equal to

$$E[h_{(m+1)}(x_1^*,\ldots,x_m^*,X_{m+1})|X_1=x_1^*,\ldots,X_m=x_m^*]=h_{(m)}(x_1^*,\ldots,x_m^*).$$

Thus, $h_{(m+1)}$ nondecreasing implies $h_{(m)}$ nondecreasing for all $m=j,\ldots,k-1$. From (2.8) we see that $h_{(k)}$ is nondecreasing as it is equal to h. Therefore, $h_{(j)}$ is nondecreasing in $X_1,\ldots,X_j$. This completes the proof of implication (b) since (2.4) only requires that $h_{(j)}$ be nondecreasing in x_j.

To show that all implications are strict for $k\ge 3$, it suffices to give a counterexample for each case. Tong (1990, p. 96) gives an example for implication (a).

(b) Let $\mathbf{X}^{3\times1}\sim N(\mathbf{0},\Sigma)$ where

$$\Sigma=\begin{pmatrix}14 & 8 & 3\\ 8 & 5 & 2\\ 3 & 2 & 1\end{pmatrix}.$$

We claim $\mathbf{X}$ is *WCIS* but not *CIS*. Note first that normal random vectors $\mathbf{U}^{m\times1}$ and $\mathbf{V}^{m\times1}$ with the same covariance matrix are such that $\mathbf{U}\le^P\mathbf{V}$ if and only if $EU_i\le EV_i$, $i=1,2,\ldots,m$. Therefore, to show $\mathbf{X}$ is *WCIS*, we must show $E(X_2,X_3|X_1=x_1)$ is nondecreasing in x_1 and $E(X_3|X_1=x_1,X_2=x_2)$ is nondecreasing in x_2.

Now $E(X_2,X_3|X_1=x_1)=(\frac{8}{14},\frac{3}{14})'x_1$ which is increasing in x_1 and

$$E(X_3|X_1=x_1,X_2=x_2)=-(\frac{1}{6})x_1+(\frac{2}{3})x_2, \tag{2.10}$$

which is increasing in x_2. Note also from (2.10) that $E(X_3|X_1=x_1,X_2=x_2)$ is decreasing in x_1 which proves that $\mathbf{X}$ is not *CIS*.

(c) Let $\mathbf{X}^{3\times1}\sim N(0,\Sigma)$, where

$$\Sigma=\begin{pmatrix}5 & 8 & 2\\ 8 & 14 & 3\\ 2 & 3 & 1\end{pmatrix}.$$

Here $E(X_3|X_1 = x_1, X_2 = x_2) = (\frac{2}{3})x_1 - (\frac{1}{6})x_2$ which is decreasing in x_2. Thus this $\mathbf{X}$ is not $WCIS$. Yet clearly $\mathbf{X}$ is A by virtue of Pitt (1982). □

REMARK 2.7 The implication (c) should be interpreted as follows: If there exists a permutation $\{j_1, \ldots, j_k\}$ of $\{1, 2, \ldots, k\}$ such that $(X_{j_1}, \ldots, X_{j_k})$ are $WCIS$, then $(X_1, \ldots, X_k)$ are associated.

REMARK 2.8 The proof of implication (b) given above also demonstrates that CIS given in (2.3) for $j = 1, \ldots, k-1$ is equivalent to

$$[(X_{j+1}, \ldots, X_k)|X_1 = x_1, \ldots, X_j = x_j] \\ \leq^P [(X_{j+1}, \ldots, X_k)|X_1 = x_1^*, \ldots, X_j = x_j^*]. \tag{2.11}$$

REMARK 2.9 Let $\mathbf{X}^{k\times 1} \sim N(\underline{\mu}, \Sigma)$. Let $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where Σ_{22} is of order $p \times p$. Then a necessary and sufficient condition for $\mathbf{X}$ to be CIS is that $\Sigma_{21}\Sigma_{11}^{-1} \geq 0$ for $p = 1, \ldots, k-1$. A necessary and sufficient condition for $\mathbf{X}$ to be $WCIS$ is that the last column of $\Sigma_{21}\Sigma_{11}^{-1}$ have all nonnegative elements for $p = 1, \ldots, k-1$. The statement follows by noting that

$$E\mathbf{X}^{(2)}|\mathbf{X}^{(1)} = \underline{\mu}^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{x}^{(1)} - \underline{\mu}^{(1)}),$$

where $\mathbf{X}^{(1)} = (X_1, \ldots, X_p)'$, $\mathbf{X}^{(2)} = (X_{p+1}, \ldots, X_k)'$.

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