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STOCHASTIC INEQUALITIES FOR A REDUNDANCY ENHANCEMENT TO A SERIES OR PARALLEL SYSTEM

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We consider the question of where to allocate a redundant spare in a series or parallel system of components in order to stochastically optimize the resulting performance of the system. Both active (or warm or parallel) and standby (or cold) redundancy are considered. We show that if the components are stochastically ordered in the usual sense, then an active redundancy allocation to the weakest (strongest) component is stochastically optimal for a series (parallel) system. The situation is more delicate for standby redundancy. If the components are ordered according to the likelihood ratio ordering, then it is stochastically optimal to make a standby redundancy to the strongest component in a parallel system. It is shown however that even for this stronger sense of component ordering, the stochastically optimal redundancy allocation in a series system is not necessarily to the weakest component.

1. Introduction

We let $T_1, \ldots, T_n$ be random variables representing the lifetimes of n components which make up a series or parallel system. We will assume the lifetimes are independent, and that they are stochastically ordered (usually increasing) in some sense. There are two types of redundancy enhancements to the system that we consider: (1) an **active** (also called a **warm** or **parallel**) redundancy, and (2) a **standby** (also called a **cold**) redundancy. An active redundant spare is one which is actively working in parallel with one of the components in the system, while a standby spare is one which only begins to operate when the component for which it is 'standing by' ceases to operate. In any case the system performance as a whole is enhanced by a redundancy, and we will be interested in placing the redundant spare in the system so as to stochastically maximize its resulting lifetime. Consideration of an active redundancy leads one to study the maximum of random variables while that of a standby redundancy leads to the study of convolutions.

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For either type of redundancy enhancement we will consider the situation where the available spare component is (1) **common** or (2) **like.** By a **common** spare we will mean there exists a component with independent lifetime T which can be placed in redundancy with any of the components in the system. We will be in a position to consider using 'like' spares if there are spare components with respective lifetimes $T_1', \ldots, T_n'$ where $T_i \overset{d}{=} T_i'$ (equal in distribution) for $i = 1, \ldots, n$. The redundant spare for the ith component will then have lifetime T_i'.

The lifetime of a parallel system with components $T_1, \ldots, T_n$ is given by

$$\tau_P(T_1, \ldots, T_n) = \max\{T_1, \ldots, T_n\}$$

while the lifetime of the series system with the same components would be

$$\tau_S(T_1, \ldots, T_n) = \min\{T_1, \ldots, T_n\}.$$

Stochastic results for active redundancy enhancement by either a 'common' or 'like' spare are reasonably straightforward when the lifetimes $T_1, \ldots, T_n$ are stochastically increasing in the usual sense ($T_1 \overset{st}{\leq} \cdots \overset{st}{\leq} T_n$). (Remember that $X \overset{st}{\leq} Y \Leftrightarrow F_X(t) \geq F_Y(t)$ for all t where F_X and F_Y are respectively the distribution functions of X and Y.)

In section 2 it is noted that for a common spare with independent lifetime T, $\tau_S(T_1, \ldots, \max(T_i, T), \ldots, T_n)$ is stochastically decreasing in i, while the distribution of $\tau_P(T_1, \ldots, \max(T_i, T), \ldots, T_n)$ is independent of i. In the situation where independent like spares $T_1', \ldots, T_n'$ are available, one may show that

$$\tau_S(T_1, \ldots, \max(T_i, T_i'), \ldots, T_n)$$

is stochastically decreasing in i, while

$$\tau_P(T_1, \ldots, \max(T_i, T_i'), \ldots, T_n)$$

is stochastically increasing in i.

The topic of standby redundancy is treated in section 3. Here we make use of a result of Brown and Solomon (1973) (see also Ross (1983)) concerning the likelihood ratio ordering. Their result states that if X, Y are independent nonnegative random variables with respective densities f and g such that $X \overset{lr}{\leq} Y$ (i.e. $g(x)/f(x) \uparrow$ in x), then for any function $h(x, y)$ with the property that $h(x, y) \geq h(y, x)$ whenever $x < y$, it follows that

$$h(X, Y) \overset{st}{\geq} h(Y, X).$$

This result implies that if $X_1, \ldots, X_n$ are independent where $X_1 \overset{lr}{\leq} \cdots \overset{lr}{\leq} X_n$ and h is any arrangement increasing function of n variables, then

$$h(X_1, \ldots, X_n)$$

is 'stochastically' arrangement increasing. By an arrangement increasing function of n variables we shall mean a function which increases in value as the order of the components approaches the situation where the coordinates are increasing. (For more on arrangement increasing functions see Marshall and Olkin (1979).) Ross's result is useful in making stochastic statements with respect to standby redundancy. For the situation when a common spare T is available, one may show that if $T_1 \overset{lr}{\leq} \cdots \overset{lr}{\leq} T_n$, then

$$\tau_P(T_1, \ldots, T_i + T, \ldots, T_n)$$

is stochastically increasing in i, while

$$\tau_S(T_1, \ldots, T_i + T, \ldots, T_n)$$

is stochastically decreasing in i. When a 'like' spare situation is relevant, one may show that

$$\tau_P(T_1, \ldots, T_i + T_i', \ldots, T_n)$$

is stochastically increasing in i, but a similar result for the series system is not valid.

We give an example of Gamma distributed random variables $T_1, \ldots, T_n$ where $T_1 \overset{lr}{\leq} \cdots \overset{lr}{\leq} T_n$ does not imply that

$$\tau_S(T_1, \ldots, T_i + T_i', \ldots, T_n)$$

is stochastically decreasing in i. Although other examples of a positive nature are given, it remains to find sufficient conditions on the stochastic ordering of components in order to insure that it is stochastically optimal to allocate a like spare component to the weakest component in a series system.

The results in this paper summarize much of the recent work in the area of stochastic ordering and redundancy allocation to series and parallel systems.

2. Stochastic Order for an Active Redundancy

Initially we consider an active redundancy allocation of a 'common' component with life distribution T. A k out of n system is a system of n components which functions if k or more of the components function. A parallel system is a 1 out of n system while a series system is an n out of n system. Boland, El–Neweihi and Proschan (1992) show that when the components in a k out of n system are stochastically ordered so that $T_1 \overset{st}{\leq} \cdots \overset{st}{\leq} T_n$, then it is always stochastically optimal to improve the weakest component with a common active redundancy when given the choice. In particular they obtain:

THEOREM 2.1 *Let $T_1, \ldots, T_n$, T be independent lifetimes where $T_1 \overset{st}{\leq} \cdots \overset{st}{\leq} T_n$. Then*

a) *Series System: $\tau_S(T_1, \ldots, \max(T_i, T), \ldots, T_n)$ is stochastically decreasing in i, and*

b) *Parallel System: $\tau_P(T_1, \ldots, \max(T_i, T), \ldots, T_n)$ is (clearly) independent of i.*

Now let us suppose a k out of n system is composed of independent components with lifetimes $T_1, \ldots, T_n$ where $T_1 \overset{st}{\leq} \cdots \overset{st}{\leq} T_n$. Moreover let us assume that a set of independent 'like' spares with lifetimes $T_1', \ldots, T_n'$ is available ($T_i \overset{d}{=} T_i'$ for $i = 1, \ldots, n$) for active redundancy with the original system. In the case where only one active redundancy allocation is permitted, a natural question is to determine where this might be done in order to give the greatest improvement to the system. Unfortunately there is no general stochastic result for arbitrary k out of n systems. Boland, El–Neweihi and Proschan (1992) give an example of a 2 out of 3 system where the answer might be with either component 1, 2 or 3 depending on the specific distributions of T_1, T_2 and T_3 and the point in time being considered. For series and parallel systems however we have the following result:

THEOREM 2.2 *Let $T_1, \ldots, T_n, T_1', \ldots, T_n'$ be independent lifetimes where $T_i \overset{d}{=} T_i'$ for $i = 1, \ldots, n$ and $T_1 \overset{st}{\leq} \cdots \overset{st}{\leq} T_n$. Then*

a) *Series System*

$$\tau_S(T_1, \ldots, \max(T_i, T_i'), \ldots, T_n)$$

is stochastically decreasing in i, while

b) *Parallel System*

$$\tau_P(T_1, \ldots, \max(T_i, T_i'), \ldots, T_n)$$

is stochastically increasing in i.

PROOF Let F_i be the distribution function for T_i and T_i'. For any $t \geq 0$ and $i = 1, \ldots, n-1$ we have that $F_i(t) \geq F_{i+1}(t)$. Hence

$$(1 - F_i^2(t))\overline{F}_{i+1}(t) = (1 - F_i^2(t))(1 - F_{i+1}(t)) \geq \overline{F}_i(t)(1 - F_{i+1}^2(t)),$$

and it follows that

$$\begin{aligned} &\text{Prob}[\tau_S(T_1, \ldots, \max(T_i, T_i'), \ldots, T_n) > t] \\ &\quad \geq \text{Prob}[\tau_S(T_1, \ldots, \max(T_{i+1}, T_{i+1}'), \ldots, T_n) > t]. \end{aligned}$$

This proves the result for a series system and similarly the parallel result follows.

Before passing to the next section on the mathematically more delicate problem of standby redundancy, we note that for active redundancy considerations (when the components are stochastically ordered) we can now say the weakest component is the most important **stochastically** in a series system while it is the strongest component in a parallel system. Further results in active redundancy may be found in Boland, El–Neweihi and Proschan (1988) and Xie and Shen (1989) and (1991).

3. Stochastic Order for a Standby Redundancy

In this section we consider the allocation of a **standby** redundant spare (where the available spare is either **common** or **like** in the sense previously described) in a series or parallel system. As in the previous section, one would expect in particular that when the components are stochastically ordered and independent, then an allocation to the weakest component in a series system and the strongest in a parallel system will be stochastically optimal. Some results of this type are in fact true, but only on the assumption that the components are stochastically ordered according to the likelihood ratio ordering (which is a stronger ordering than the usual stochastic ordering). We will also see however that examples exist where the components in a series system are increasing in the likelihood ratio sense, but it is not stochastically optimal to make a 'like' standby redundancy with the weakest component.

If X and Y are independent with respective densities f and g, we say X is less than Y in the likelihood ratio sense ($X \overset{lr}{\leq} Y$) if

$$g(x)/f(x)$$

is increasing over the common support of X and Y. It is well known that $X \overset{lr}{\leq} Y \Rightarrow X \overset{st}{\leq} Y$, but not conversely (see Ross (1983)). Brown and Solomon (1973) proved that if $X \overset{lr}{\leq} Y$ and h is a function with the property that

$$h(x,y) \geq h(y,x) \quad \text{whenever} \quad x \leq y,$$

then $h(X,Y) \overset{st}{\geq} h(Y,X)$.

We now use this result of Brown and Solomon to prove the following Theorem of Boland, El–Neweihi and Proschan (1992) concerning standby redundancy of a **common** spare in a series or parallel system.

THEOREM 3.1 *Let $T_1, \ldots, T_n$, T be independent lifetimes where $T \overset{lr}{\leq} \cdots \overset{lr}{\leq} T_n$. Then*

a) *Series System: $\tau_s(T_1, \ldots, T_i + T, \ldots, T_n)$ is stochastically decreasing in i, and*

b) *Parallel System: $\tau_p(T_1, \ldots, T_i + T, \ldots, T_n)$ is stochastically increasing in i.*

PROOF For any $t \geq 0$, we define the functions

$$\begin{aligned} h_1(t_1, t_2) &= \min(t_1 + t, t_2) \quad \text{and} \\ h_2(t_1, t_2) &= \max(t_1, t_2 + t). \end{aligned}$$

Then clearly for any $t_1 \leq t_2$, $h_i(t_1, t_2) \geq h_i(t_2, t_1)$ for $i = 1, 2$. It follows by conditioning on the values of T and using the result of Ross that

$$\begin{aligned} \tau_S(T_1 + T, T_2) &\overset{st}{\geq} \tau_S(T_1, T_2 + T) \qquad \text{and} \\ \tau_P(T_1 + T, T_2) &\overset{st}{\leq} \tau_P(T_1, T_2 + T). \end{aligned}$$

The result for general $n > 2$ follows by independence.

Examples may be easily constructed (see Boland, El–Neweihi, Proschan (1992)) to show the results of Theorem 3.1 are not valid when the likelihood ratio ordering of the components is relaxed to the ordinary stochastic ordering.

We now consider the question of making one standby redundancy allocation when 'like' spares are available. The parallel case is easy, as the following Theorem (Boland, El–Neweihi, and Proschan (1992) or Shaked and Shanthikumar (1990)) demonstrates.

THEOREM 3.2 *Let $T_1, \ldots, T_n, T_1', \ldots, T_n'$ be independent lifetimes where $T_i \overset{d}{=} T_i'$ for $i = 1, \ldots, n$, and $T_1 \overset{lr}{\leq} \cdots \overset{lr}{\leq} T_n$. Then for a parallel system*

$$\tau_P(T_1, \ldots, T_i + T_i', \ldots, T_n)$$

is stochastically increasing in i.

PROOF

$$\begin{aligned} &\tau_P(T_1, \ldots, T_i + T_i', T_{i+1}, \ldots, T_n) \\ &\quad \overset{st}{\leq} \tau_P(T_1, \ldots, T_i, T_{i+1} + T_i', \ldots, T_n) \quad \text{(by Theorem 3.1b)} \\ &\quad \overset{st}{\leq} \tau_P(T_1, \ldots, T_i, T_{i+1} + T_{i+1}', \ldots, T_n) \quad (\text{since } 0 \leq T_i' \overset{st}{\leq} T_{i+1}'). \end{aligned}$$

Shaked and Shanthikumar (1990) implicitly prove that if $T_1, \ldots, T_n$, $S_1, \ldots, S_n$ are independent lifetimes where $T_1 \overset{lr}{\leq} \cdots \overset{lr}{\leq} T_n$ and $S_1 \overset{lr}{\leq} \cdots \overset{lr}{\leq} S_n$, then

$$\tau_P(T_1, \ldots, T_i + S_i, \ldots, T_n)$$

is stochastically increasing in i [see their Lemma 3.2]. This in particular implies the results of Theorem 3.1b) and Theorem 3.2 above. This interesting paper of Shaked and Shanthikumar presents many other results about resource allocations to parallel and series systems.

What about the corresponding result of Theorem 3.2 for series systems. Natvig (1985) showed that if $T_1, \ldots, T_n, T_1', \ldots, T_n'$ are independent Gamma random variables with the same shape parameter m, and where $T_i \overset{d}{=} T_i'$, then $T_1 \overset{st}{\leq} \cdots \overset{st}{\leq} T_n$ implies that

$$\tau_S(T_1, \ldots, T_i + T_i', \ldots, T_n)$$

is stochastically decreasing in i. Boland, Proschan and Tong (1990) show that if each T_i is uniformly distributed on $[0, \theta_i)$ where $\theta_1 \leq \cdots \leq \theta_n$, then a similar result holds. Other examples of a parametric nature also exist. However Boland, Proschan and Tong (1990) also gave the following example:

EXAMPLE 3.1 Let T_1, T_2, T_1', T_2' be independent random variables where $T_1 \overset{d}{=} T_1' \equiv \Gamma(m, \theta)$, and $T_2 = T_2' = \Gamma(m+1, \theta)$, for some $m \geq 1$ and $\theta > 0$. Then $T_1 \overset{lr}{\leq} T_2$ but

$$\tau_S(T_1 + T_1', T_2) \overset{st}{\not\geq} \tau_S(T_1, T_2 + T_2').$$

This is demonstrated by showing that the difference between the reliability of the first and second systems is positive for small t and negative for large t.

We note in conclusion that the likelihood ratio ordering is not strong enough to imply that when independent components are ordered in this sense in a series system the stochastically optimum choice for a like redundant spare allocation is with the weakest component. One might ask then, is there a stronger stochastic ordering for which this intuitively plausible result is true?

References

BOLAND, P.J., EL–NEWEIHI, E. AND PROSCHAN, F. (1988). Active redundancy allocation in coherent systems. *Prob. Eng. Inform. Sci.* **2** 343–353.

BOLAND, P. J., EL–NEWEIHI, E. AND PROSCHAN, F. (1991). Redundancy importance and allocation of spares in coherent systems. *J. Statist. Plann. Inf.* **29** 55–66.

BOLAND, P. J., EL–NEWEIHI, E. AND PROSCHAN, F. (1992). Stochastic order for redundancy allocation in series and parallel systems. *Adv. Appl. Prob.* **24** 161–171.

BOLAND, P. J., PROSCHAN, F. AND TONG, Y. L. (1990). Standby redundancy policies for series systems. Tech. Report, University College, Dublin.

BROWN, M. AND SOLOMON, H. (1973). Optimal issuing policies under stochastic field lives. *J. Appl. Prob.* **10** 761–768.

MARSHALL, A. W. AND OLKIN, I. (1979). *Inequalities: Theory of Majorization and its Applications.* Academic Press, New York.

NATVIG, B. (1985). New light on measures of importance of system components. *Scand. J. Statist.* **12** 43–54.

ROSS, S. M. (1983). *Stochastic Processes.* Wiley, New York.

SHAKED, M. AND SHANTHIKUMAR, J. G. (1990). Optimal allocation of resources to nodes of parallel and series systems. Tech. Report, University of Arizona.

SHEN, K. AND XIE, M. (1991). The effectiveness of adding standby redundancy at system and component levels. *IEEE Trans. Reliab.* **40** (1) 53–55.

XIE, M. AND SHEN, K. (1989). On ranking of system components with respect to different improvement actions. *Microelectron. Reliab.* **29** (2) 159–164.

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