ON THE ASYMPTOTIC BEHAVIOR OF SOME STATISTICS BASED ON MORPHOLOGICAL OPERATIONS

Marc Moore, Sylvain Archambault
Département de Mathématiques Appliquées
École Polytechnique de Montréal
Montréal, QC
Canada, H3C 3A7

ABSTRACT

Some operations defined in mathematical morphology (e.g. erosion, dilation, opening, closing) can be used in the definition of useful statistics to be computed from an observed image. Images generated by a stochastic mechanism, and observed on a window, are considered and two statistics are defined. The uniform almost sure convergence of these statistics is studied in the situation where the size of the window increases, and also in the situation where many independent copies of the image are observed on a fixed window. The convergence in law to a normal distribution is also considered. Two examples are presented.

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1. Introduction

Mathematical morphology is a powerful tool to study images (see Serra (1982)). Among the basic concepts of this theory we find a class of four basic operations defined on sets: the erosion, the dilation, the opening and the closing. These operations and their properties constitute what is sometimes called the Serra's calculus. The utility of these operations to summarize binary images is clearly shown by Ripley (1986).

In this paper we consider binary (black, white) images on R^2 generated by a stochastic mechanism (process). The image I will be identified with its black part i.e. I is the random set made of the black points in the image; we assume the set I is closed. Let W be a window on which the image I is observed. Two forms of statistics will be considered:

$$S_W(r) = \frac{\text{mes}\{x : I \bmod T_r \supset \{x\}, x \in W \circ T_r\}}{\text{mes}[W \circ T_r]}$$

and

$$U_W(r) = \frac{\text{mes}\{x : I \bmod T_r \supset \{x\}, x \in W \circ T_r\}}{\text{mes}\{x : I \supset \{x\}, x \in W \circ T_r\}}$$

where $\bmod T_r$ denotes a fixed basic morphological operation using a structuring element T of "size" r, and $W \circ T_r$ is the subset of points x in W for which it can be verified if the random set $I \bmod T_r$ includes x or not, it will be a sequence of one or two erosions of W (see Ripley (1986)). For example, if $\bmod T_r$ is the opening by T_r then

$$S_W(r) = \frac{\text{mes}\{x : (I \ominus \widetilde{T}_r) \oplus T_r \supset \{x\}, x \in W \ominus (T_r \oplus \widetilde{T}_r)\}}{\text{mes}[W \ominus (T_r \oplus \widetilde{T}_r)]}$$

where $\ominus$ and $\oplus$ are respectively the Minkowski subtraction and addition, and $\widetilde{T}_r = \{-x : x \in T_r\}$.

Considered as functions of r these statistics define curves that can be used to study or compare different images. This is well illustrated by Ripley (1986).

For certain stochastic processes $\mathcal{P}$ the statistics $S_W(r)$ can be considered as an estimate of the probability that an image I generated by $\mathcal{P}$ covers a given point (the origin 0 say) after having been transformed by $\bmod T_r$ i.e. $P[I \bmod T_r \supset \{0\}] = F(r)$. In some applications it is necessary to consider also conditional probabilities. For example if T_r is a disk of radius r, $P[(I \ominus \widetilde{T}_r) \oplus T_r \supset \{0\} | I \supset \{0\}] = G(r)$ is known as the "répartition granulométrique" in metallurgy. The function $1 - G(r)$ can be estimated by the statistic $U_W(r)$ with T_r being the disk of radius r and $\bmod T_r$ being the opening by T_r.

The sample functions $S_W(r)$ and $U_W(r)$ can be used to test if an observed image has been generated by a given stochastic process $\mathcal{P}$ (goodness-of-fit test). To this end the sample function (e.g. $S_W(r)$) is compared to the "theoretical function" under $\mathcal{P}$ (e.g. $F(r)$). It is then necessary to know the theoretical functions for different processes $\mathcal{P}$. In general It is not possible to obtain these functions analytically. We present some asymptotic results useful to approximate these functions by simulations.

In Section 2 some remarks are made about the first two moments of the statistics considered, and about the mathematical foundations. Section 3 is devoted to the almost sure convergence of the statistics. Two situations are considered, the one where the window is enlarged and the one where a large number of independent images are observed on a fixed window. It is verified that for many stochastic processes the limits are the same. So to determine the theoretical functions by simulation we may use the average of a large number of independent images generated by the process and observed on a fixed window. The convergence in law of the statistics $S_W(r)$ is studied in Section 4. Finally, in Section 5 we mention two examples of image generating stochastic processes for which all the assumptions made are satisfied.

Some of the considerations made here are related to some of those made by Baddeley (1980). He considers statistics similar to $S_W(r)$ but where the denominator is simply $\text{mes}[W]$. The mixing conditions imposed by Baddeley seems more difficult to verify than those imposed here.

2. The first two moments of the statistics

We consider first the statistics $S_W(r)$. The theoretical functions we consider are simply $E[S_W(r)]$. To give a precise meaning to this expectation the mathematical structure has to be made more precise.

We have already assumed that an image I is a closed subset of R^2, let $\mathcal{I}$ be the class of such images (sets). The class $\mathcal{I}$ is equipped with the hit and miss topology, and let $\mathcal{S}$ be the σ-algebra on $\mathcal{I}$ generated by this topology (Serra (1982) pp. 75 and 545). A given process $\mathcal{P}$ generates a probability measure P on $(\mathcal{I}, \mathcal{S})$. The structuring elements we consider are non empty compact subsets of R^2. For such structuring elements, and for the four basic operations, the transformed image $I \bmod T_r$, $I \in \mathcal{I}$, is also an element of $\mathcal{I}$ (Serra (1982), pp. 546). The processes we consider are assumed to be stationary in the following sense.

Definition. An image generating stochastic process $\mathcal{P}$ is first order stationary relatively to the morphological operation $\bmod T_r$ if

$$P[\{I : I \bmod T_r \supset \{x\}\}] = P[I \bmod T_r \supset \{x\}]$$

is independent of $x \in R^2$ (so x can be taken to be the origin 0).

Note that from a result due to Matheron (see Serra (1982), Theorem XIII-3 pp. 547) the event $\{I : I \bmod T_r \supset \{x\}\}$ belongs to $\mathcal{S}$ for every $x \in R^2$. This result also permits to apply the Fubini theorem in a classical way (Robbins (1944)) to prove the following Proposition.

Proposition 1. *If the image generating process $\mathcal{P}$ is first order stationary relatively to $\bmod T_r$, then*

$$E[S_W(r)] = P[I \bmod T_r \supset \{0\}] \tag{1}$$

and

$$\text{Var}\,[S_W(r)] = \frac{1}{(\text{mes}[W \circ T_r])^2} \int_{W \circ T_r} \int_{W \circ T_r} \{P[\{I : I \bmod T_r \supset \{x\}, I \bmod T_r \supset \{y\}\}] - (P[I \bmod T_r \supset \{0\}])^2\} dx\, dy. \tag{2}$$

To the process $\mathcal{P}$ we associate the function

$$F(r) = P[I \bmod T_r \supset \{0\}].$$

From relation (2) the following proposition is easily obtained.

Proposition 2. *If the image generating process $\mathcal{P}$ is first order stationary relatively to* $\mathrm{mod}\, T_r$ *and if there exist a function* $\varphi_r(x,y) : R^2 \times R^2 \to R$ *such that*

$$|P[\{I : I \,\mathrm{mod}\, T_r \supset \{x\},\ \ I \,\mathrm{mod}\, T_r \supset \{y\}\}] - F^2(r)| \leq \varphi_r(x,y) \tag{i}$$

and

$$\lim_{W \to R^2} \int_W \int_W \varphi_r(x,y) dx\, dy / \mathrm{mes}[W] = 0 \tag{ii}$$

($W \to R^2$ means that W increases toward the whole plane), then $\langle S_W(r)\rangle$ converges in probability ($W \to R^2$) to $F(r)$. Much stronger convergence results will be given in the next section.

For the statistics $U_W(r)$, due to their ratio form, it is more difficult to obtain expressions for their expectation and variance. However, if the assumptions of Proposition 2 are satisfied $\langle U_W(r)\rangle$ will converge in probability ($W \to R^2$) to $F(r)/F(0)$; so this ratio is consistently estimated by $U_W(r)$ but the estimation will in general be biased. In practice we are interested in functions of the form $F(r)/F(0)$ when the transformation is decreasing (erosion and opening), in that case

$$Q(r) = P[I \,\mathrm{mod}\, T_r \supset \{0\} | I \supset \{0\}] = F(r)/F(0).$$

When the transformation is increasing (dilation and closing) we could rather be interested in

$$Q(r) = P[I \,\mathrm{mod}\, T_r \supset \{0\} | I \not\supset \{0\}] = \frac{F(r) - F(0)}{1 - F(0)}.$$

In any cases consistent estimations of $F(r)$ and $F(0)$ lead to a consistent estimation of $Q(r)$.

3. Strong and uniform convergence

We consider first the convergence with respect to an increasing sequence of windows and a fixed r. Let $W_1 \subset W_2 \subset \ldots \subset W_n \subset \ldots$ be such that $\lim_{n\to\infty} W_n = R^2$.

Proposition 3. *If*

(i) the image generating process is first order stationary relatively to $\mathrm{mod}\, T_r$,

(ii) the function $\varphi_r(x,y)$ mentioned in Proposition 2 is such that

$$\frac{1}{(\mathrm{mes}[W_n \circ T_r])^2} \int_{W_n \circ T_r} \int_{W_n \circ T_r} \varphi_r(x,y) dx\, dy \leq \frac{C}{(\mathrm{mes}[W_n \circ T_r])^q}$$

where C and q are positive constants,

(iii) the form of the W_n's is such that

$$\lim_{n \to \infty} \frac{\mathrm{mes}[W_n \circ T_r]}{n^2} = K < \infty,$$

then

$$\langle S_{W_n}(r)\rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} F(r).$$

Proof. The argument is a standard one, we mention only the main steps. Let

$$Y_n = \frac{\text{mes}\{x : I \bmod T_r \supset \{x\}, x \in W_n \circ T_r\} - F(r)\,\text{mes}[W_n \circ T_r]}{\text{mes}[W_n \circ T_r]}$$

clearly $E[Y_n] = 0$ and $E[Y_n^2] \leq \frac{C}{(\text{mes}[W_n \circ T_r])^q}$.

Let s be an integer larger than $1/q$ and define the sequence $\langle Y_n' \rangle$ where $Y_n' = Y_{n^s}$. For a fixed $\varepsilon > 0$ consider the events

$$A_n^{(\varepsilon)} = \{|Y_n'| \geq \varepsilon\} \quad n = 1, 2, \ldots.$$

From the Chebyshev's inequality and conditions (ii) and (iii) we have

$$\sum_{n=1}^{\infty} P[A_n^{(\varepsilon)}] < \infty$$

and so from the Borel-Cantelli lemma

$$P[A_n^{(\varepsilon)} i.o.] = 0.$$

Since this is valid for every $\varepsilon > 0$ we have

$$\langle Y_n' \rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0 \text{ that is } \langle Y_{n^s} \rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0.$$

Let

$$\begin{aligned}
U_n &= \max_{n^s \leq m < (n+1)^s} \left| Y_{n^s} - \frac{\text{mes}[W_m \circ T_r]}{\text{mes}[W_{n^s} \circ T_r]} Y_m \right|, \\
V_n &= \max_{n^s \leq m < (n+1)^s} \left| Y_m - \frac{\text{mes}[W_m \circ T_r]}{\text{mes}[W_{n^s} \circ T_r]} Y_m \right|, \\
T_n &= \max_{n^s \leq m < (n+1)^s} |Y_{n^s} - Y_m|,
\end{aligned}$$

clearly if $\langle U_n \rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0$ and $\langle V_n \rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0$, since $\langle Y_{n^s} \rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0$, we have $\langle Y_n \rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0$.

To prove that $\langle U_n \rangle \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0$ let $\varepsilon > 0$ be fixed and define the events

$$B_n^{(\varepsilon)} = \{|U_n| \geq \varepsilon\} \quad n = 1, 2, \ldots.$$

We have

$$P[B_n^{(\varepsilon)}] \leq E[U_n^2]/\varepsilon^2$$

and using condition (iii) it can be verified that

$$E[U_n^2] \leq d/n^2$$

where d is a positive constant. So, again from the Borel-Cantelli lemma,

$$P[|U_n| \geq \varepsilon \ \ i.o.] = 0$$

and the $\langle U_n \rangle \underset{n\to\infty}{\overset{a.s.}{\to}} 0$. A similar argument permits to show that $\langle V_n \rangle \underset{n\to\infty}{\overset{a.s.}{\to}} 0$. Q.E.D.

Consider now the situation where the window W is fixed, r is fixed and we have access to independent copies $I_1, I_2, \ldots, I_m, \ldots$ of images generated by the process $\mathcal{P}$, each being observed on W (we assume that W is large enough so that $\text{mes}[W \circ T_r] > 0$). Let

$$\overline{S}_W^{(m)}(r) = \sum_{j=1}^{m} S_W^{(j)}(r)/m$$

where $S_W^{(j)}(r) = \frac{\text{mes}\{x : I_j \bmod T_r \supset \{x\}, x \in W \circ T_r\}}{\text{mes}[W \circ T_r]}$.

Since the random variables $S_W^{(j)}(r)$ are i.i.d. and take their values between zero and one, we have

$$\langle \overline{S}_W^{(m)}(r) \rangle \underset{m\to\infty}{\overset{a.s.}{\to}} E[S_W^{(j)}(r)] = F(r)$$

(a.s. is with respect to the product measure).

This fact and the Proposition 3 indicate that to estimate $F(r)$ we may use one image observed on a very large window or, what is easier in practice, a large number of independent images generated from $\mathcal{P}$ observed on a fixed window.

We now examine if the strong convergence for a given r, guaranteed by Proposition 3, is uniform with respect to r i.e. if

$$\sup_{r \geq 0} |S_{W_n}(r) - F(r)| \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

We consider first the random variables

$$S'_{W_n}(r) = \frac{\text{mes}\{x : I \bmod T_r \supset \{x\}, x \in W_n\}}{\text{mes}[W_n]}.$$

These random variables do not exactly define statistics because it may happen that for some points $x \in W_n$ it is not possible to determine if $I \bmod T_r \supset \{x\}$, so $S'_{W_n}(r)$ may not be computable from the data. However, since I is defined on the whole plane, $S'_{W_n}(r)$ is well defined.

Proposition 4. *If*

(i) the assumptions of Proposition 3 with $W_n \circ T_r$ replaced by W_n are satisfied for each $r \geq 0$,

(ii) $r_1 < r_2$ implies that $T_{r_1} \subset T_{r_2}$ and all points interior to T_{r_2} can be covered by a translation of T_{r_1} remaining in T_{r_2},

(iii) $\text{mes}[T_r]$ is continuous with respect to r, then

$$\sup_{r \geq 0} |S'_{W_n}(r) - F(r)| \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

Proof. The assumption (ii) implies that for every $I \in \mathcal{I}$, $S'_{W_n}(r)$ is non increasing in r if the operation is the erosion or the opening and non decreasing in r if the operation is the dilation or the closing.

The assumption (iii) insures that $S'_{W_n}(r)$ is continuous to the right or continuous to the left.

These facts permit to transpose directly the proof of the Glivenko-Cantelli theorem (Loève (1963), pp 20) to obtain the desired result. Q.E.D.

From this result for $S'_{W_n}(r)$ we have the following one for the statistic $S_{W_n}(r)$.

Proposition 5. *Suppose the sequence of windows $\langle W_n \rangle$ is such that for every n and r there exists a $n(r) > n$ for which*

(i) $W_{n(r)} \circ T_n = W'_n \supset W_n$,

(ii) the frontier of W'_n and the frontier of W_n have a common part of positive length,

(iii) the W'_n satisfy the assumptions made about the sequence of windows,

(iv) for each n and r there exists a $m(n,r)$ such that $W_n \circ T_r = W'_{m(n,r)}$,

(v) $r_1 < r_2$ implies that $n(r_1) < n(r_2)$,

then under the assumptions of Proposition 4 with W'_n replacing W_n (i.e. $W_n \circ T_r$ is replaced by W_n in the assumptions (ii) and (iii) of Proposition 3), for every $R > 0$

$$\sup_{0 \le r \le R} |S_{W_n}(r) - F(r)| \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0.$$

Proof. From the assumption about the W_n, for every n and r there exists a $n(r)$ such that $S'_{W'_n}(r) = S_{W_{n(r)}}(r)$ for every $I \in \mathcal{I}$. Since we assume that Proposition 4 is valid when W'_n replaces W_n we have that for every $\varepsilon > 0$ there exists a M_ε such that

$$P[\{I : \sup_{0 \le r \le R} |S'_{W'_\nu}(r) - F(r)| < \varepsilon \text{ for every } \nu > M_\varepsilon\}] = 1.$$

Let $N = M_\varepsilon(R)$ (as defined in the assumption about the W_n's), for each $n > N$ and $r \in [0, R]$ we have $m(n,r) > M_\varepsilon$ and

$$\sup_{0 \le r \le R} |S_{W_n}(r) - F(r)| = \sup_{0 \le r \le R} |S'_{W'_{m(n,r)}}(r) - F(r)|$$

for every $I \in \mathcal{I}$.

So

$$\{I : \sup_{0 \le r \le R} |S_{W_n}(r) - F(r)| < \varepsilon \text{ for each } n > N\} \supset$$
$$\{I : \sup_{0 \le r \le R} |S'_{W'_\nu}(r) - F(r)| < \varepsilon \text{ for each } \nu > M_\varepsilon\}$$

and since this last event has probability one we obtain the desired result. Q.E.D.

Let $I_1, \ldots, I_m \ldots$ be independent images from $\mathcal{P}$ observed on a fixed window W. For a given $R > 0$ (R and W being such that $m[W \circ T_R] > 0$) consider

$$\overline{S}^{(m)}_{W,R}(r) = \frac{1}{m} \sum_{j=1}^{m} \frac{\text{mes}\{x : I_j \bmod T_r \supset \{x\}, x \in W \circ T_R\}}{\text{mes}[W \circ T_r]}.$$

Under the assumptions of Proposition 4 regarding the structuring elements T_r we have

$$\sup_{0 \le r \le R} |\overline{S}^{(m)}_{W,R}(r) - F(r)| \underset{n \to \infty}{\overset{a.s.}{\to}} 0.$$

So it also possible to obtain, from many independent copies of the image, a uniformly good approximation to $F(r)$ on $[0, R]$.

Concerning the statistics $U_W(r)$, under the assumptions of Proposition 3, for every fixed $r, \langle U_{W_n}(r) \rangle$ converges almost surely to $F(r)/F(0)$.

Also we easily see that under the assumptions of Proposition 5

$$\sup_{0 \le r \le R} |U_{W_n}(r) - F(r)/F(0)| \underset{n \to \infty}{\overset{a.s.}{\to}} 0.$$

When independent images from $\mathcal{P}$ are observed on a fixed window we have

$$\sup_{0 \le r \le R} \left| \frac{\overline{S}^{(m)}_{W,R}(r)}{\overline{S}^{(m)}_{W,R}(0)} - \frac{F(r)}{F(0)} \right| \underset{m \to \infty}{\overset{a.s.}{\to}} 0$$

if the structuring elements satisfy the assumptions made in Proposition 4.

4. Asymptotic normality

Our goal here is to establish the asymptotic normality of $(\text{mes}[W_n \circ T_r])^{1/2}[S_{W_n}(r) - F(r)]$ for each fixed r. To simplify the presentation we assume that each W_n is a rectangle which is a union of unit squares. Let $\{C_{ij} : (i,j) \in Z^2\}$ be the partition of the plane by unit squares $C_{ij}, (i,j) \in Z^2$ identifying the lower left-hand corner of the square. We assume also that

$$W_n \circ T_r = \bigcup_{\substack{\ell_n \le i \le \ell'_n \\ h_n \le j \le h'_n}} C_{ij}$$

and denote by $|W_n \circ T_r|$ the number of squares required to cover $W_n \circ T_r$.

We define the random variables

$$X_{ij} = \text{mes}\{x : I \bmod T_r \supset \{x\}, \ \ x \in C_{ij}\}, \ \ (i,j) \in Z^2,$$

clearly $|X_{ij}| \le 1$ and

$$S_{W_n}(r) = \sum_{i=\ell_n}^{\ell'_n} \sum_{j=h_n}^{h'_n} X_{ij}/|W_n \circ T_r|. \tag{3}$$

From Proposition 1, if the process $\mathcal{P}$ is first order stationary relatively to $\bmod T_r$ then $E[X_{ij}] = F(r)$. In order to obtain the asymptotic normality we formulate the following assumptions.

H_1: The process $\mathcal{P}$ is first order stationary relatively to $\bmod T_r$ and moreover is such that the stochastic process $\{X_{ij};\ (i,j) \in Z^2\}$ is stationary i.e. the joint laws of the random variables X_{ij} are invariant under translation (shift invariant).

H_2: If $Z \subset Z^2, \mathcal{A}_z$ will denote the σ-algebra generated by the X_{ij} for which $(i,j) \in Z$ and $|Z|$ will denote the number of elements in Z. Given two subsets of Z^2, Z_1 and Z_2, the distance between Z_1 and Z_2 is defined by

$$d(Z_1, Z_2) = \inf_{\substack{(i_1,j_1)\in Z_1 \\ (i_2,j_2)\in Z_2}} \max\{|i_1 - i_2|, |j_1 - j_2|\}. \tag{4}$$

Let

$$\alpha_{k\ell}(m) = \sup\{|P[A \cap B] - P[A]P[B]| : A \in \mathcal{A}_{z_1}, B \in \mathcal{A}_{z_2}, |Z_1| = k, |Z_2| = \ell, d(Z_1, Z_2) \geq m\}$$

where m is a positive integer and k, ℓ are positive integers or may be infinite. The mixing coefficients $\alpha_{k\ell}(m)$ are assumed to satisfy the following conditions:

(i) for each k, ℓ such that $k + \ell \leq 4$, $\sum_{m=1}^{\infty} m\alpha_{k\ell}(m) < \infty$,

(ii) $\lim_{m\to\infty} m^2 \alpha_{1\infty}(m) = 0$,

(iii) for some $\delta > 0$, $\sum_{m=1}^{\infty} m[\alpha_{11}(m)]^{\delta/(2+\delta)} < \infty$.

Under the assumptions H_1 and H_2 we have

$$\sum_{(i,j)\in Z^2} |\operatorname{Cov}(X_{00}, X_{ij})| < \infty,$$

let

$$\sigma_r^2 = \sum_{(i,j)\in Z^2} \operatorname{Cov}(X_{00}, X_{ij}).$$

H_3: The process $\mathcal{P}$ is such that $\sigma_r^2 > 0$.

Proposition 6. *Under assumptions H_1, H_2 and H_3 we have*

$$(\operatorname{mes}[W_n \circ T_r])^{1/2}[S_{W_n}(r) - F(r)] \underset{n\to\infty}{\longrightarrow} N[0, \sigma_r^2].$$

Proof. From (3) we have

$$(\operatorname{mes}[W_n \circ T_r])^{1/2}[S_{W_n}(r) - F(r)] = |W_n \circ T_r|^{1/2} \left[\frac{\sum_{i=\ell_n}^{\ell'_n} \sum_{j=h_n}^{h'_n} X_{ij}}{|W_n \circ T_r|} - F(r) \right].$$

The assumptions made guarantee that the random variables X_{ij} and the regions $W_n \circ T_r$ satisfy the assumptions made by Bolthausen (1982), so the result follows from his theorem. Q.E.D.

Remarks.

1. Baddeley (1980) obtains a similar result but under a different type of mixing condition. In many cases we found the conditions given here easier to verify.

2. The conditions made about the form of W_n could be relaxed. Baddeley (1980) shows how this can be done.

3. We verify that

$$\sigma_r^2 = \int_{C_{00}} \left[\int_{R^2} \{P[\{I : I \operatorname{mod} T_r \supset \{0\}, I \operatorname{mod} T_r \supset \{y\}\}] - F^2(r)\} dy \right] dx.$$

So, if the process $\mathcal{P}$ is such that the inside integral has a value which is independent of $x \in C_{00}$, we have

$$\sigma_r^2 = F(r) \int_{R^2} \{P[I \operatorname{mod} T_r \supset \{y\} | I \operatorname{mod} T_r \supset \{0\}] - F(r)\} dy.$$

5. Examples

A. The Poisson line process

First the plane is partitioned into convex polygons (cells) by lines randomly chosen according to a Poisson process. More precisely, a line is described by (p, θ), $p \geq 0$, $0 \leq \theta < 2\pi$, the polar coordinates of the foot of the perpendicular from the origin to the line, and points (p, θ) are chosen according to an homogeneous planar Poisson process of intensity λ. When the plane has been so divided a color, black or white, is assigned to each polygon. The color is randomly chosen with probabilities p and $1 - p$. The choice is independent from cell to cell and always made with the same probabilities. Figure 1 shows a realization of such a process for which λ is such that an average of 60 lines cross the unit square and $p = 1/2$. The stationarity, isotropy and other properties of this process are well known (Switzer (1965), Ahuja and Schachter (1983)). It is then easy to verify that this process satisfy the assumption H_1 of Proposition 6 (this for the four basic morphological operations and every $r > 0$), clearly $P[I \supset \{0\}] = p$.

Given two points x, y at a distance d, the probability that they are in the same cell generated by the Poisson line process is $e^{-2\lambda d}$, then

$$P[\{I : I \supset \{x\}, I \supset \{y\}\}] - p^2 = p(1-p)e^{-2\lambda d}$$

and so the assumptions of Proposition 2 and assumption (ii) of Proposition 3 are satisfied for $r = 0$. These assumptions are also verified for $r > 0$ (see below).

The main difficulty is in the verification of the assumption H_2 of Proposition 6. We begin by considering the image itself (i.e. $r = 0$). We start with the mixing coefficient $\alpha_{1\infty}(m)$. Consider two sets of sites (i, j), the first one containing only one site, taken to be the origin, the second one, Z, containing an infinity of sites but such that for every

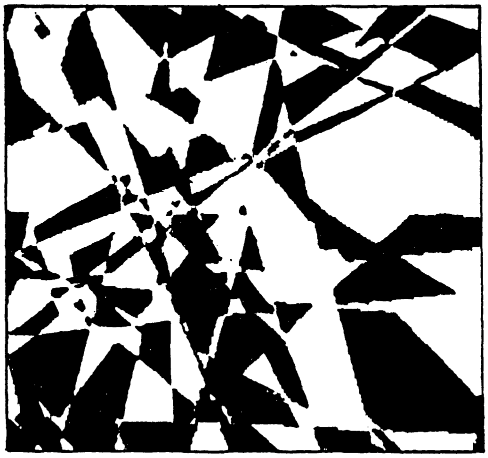

Figure 1. Image generated by a Poisson line process.

$(i,j) \in Z, d((0,0),(i,j)) = \max\{|i|,|j|\} \geq m$. Let A be an event generated by X_{00} and B an event generated by the X_{ij}'s, $(i,j) \in Z$. We want to obtain an upper bound for $\sup_{A,B} |P[A|B] - P[A]|$.

We define the following eight regions of the plane (see Figure 2)

$$\begin{aligned} R_1 &= [1,\infty) \times [1,\infty) & R_5 &= [1,\infty) \times [0,1) \\ R_2 &= (-\infty,0) \times [1,\infty) & R_6 &= [0,1) \times [1,\infty) \\ R_3 &= (-\infty,0) \times (-\infty,0) & R_7 &= (-\infty,0) \times [0,1) \\ R_4 &= [1,\infty) \times (-\infty,0) & R_8 &= [0,1) \times (-\infty,0). \end{aligned}$$

Let

$$d_1 = ||(1,1),(i_1^*,j_1^*)|| = \inf\{||(1,1,),(i,j)|| : (i,j) \in Z \cap R_1\},$$

$||a,b||$ denoting the Euclidean distance between the points a and b. The part of the disk of radius d_1 centered at (1, 1) and situated in R_1 does not contain any point of Z. If the process generating the lines gives at least one line crossing the two segments $[(1,1),(1+d_1,1)]$ and $[(1,1),(1,1+d_1)]$, then C_{00} will be separated from all the C_{ij} for which $(i,j) \in Z \cap R_1$ and the colors in C_{00} will then be independent from those in these C_{ij}. The set of (p,θ) corresponding to the lines crossing these two segments is

$$E_1 = \{(p,\theta) : 0 < \theta \leq \pi/4 \text{ and } 0 \leq p \leq d_1 \sin\theta, \text{ or } \pi/4 < \theta < \pi/2 \text{ and } 0 \leq p \leq d_1 \cos\theta\}$$

and the area of that set is $d_1[2-\sqrt{2}]$.

Similarly for the regions R_2, R_3 and R_4 we define the distances d_k and the sets $E_k, k = 2,3,4$.

Consider now the region R_5 and let

$$d_5 = \inf\{||(1,0),(i,j)|| : (i,j) \in Z \cap R_5\}.$$

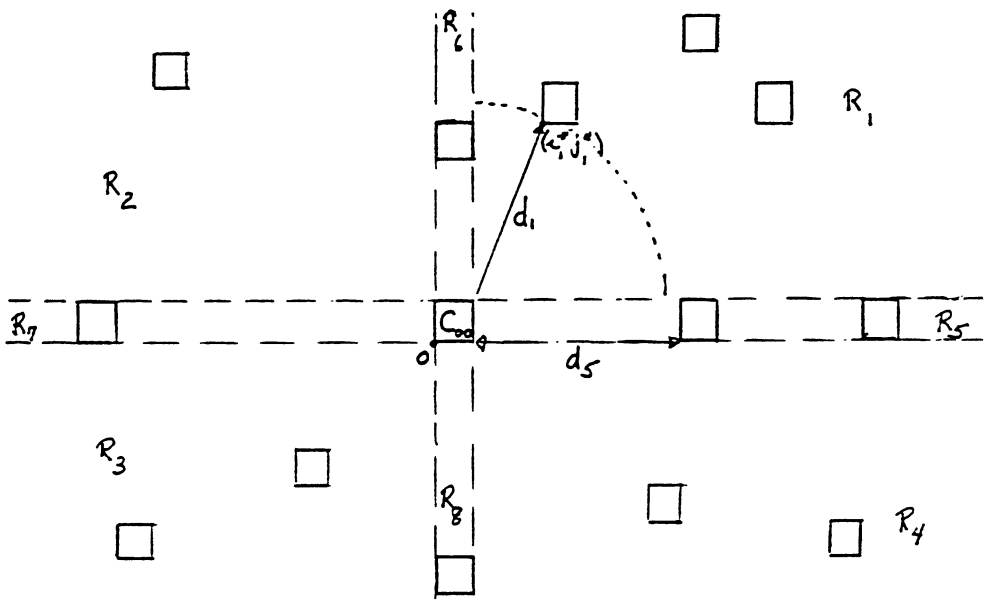

Figure 2.

Denote by E_5 the set of (p, θ) corresponding to the lines separating C_{00} from all the $C_{ij}, (i,j) \in Z \cap R_5$. It can be verified that the area of E_5 is $2[(1+d_5^2)^{1/2} - 1]$. For the regions R_6, R_7 and R_8 the distances d_k and the sets $E_k, k = 6,7,8$, are similarly defined.

We also denote by $E_i, i = 1, \ldots, 8$ the events: "the Poisson line process generates at least one line for which $(p, \theta) \in E_i$". Let F be the event $E_1 \cap \ldots \cap E_8$, we have

$$P[A|B] = P[A|B \cap F] + P[F^c|B]\{P[A|B \cap F^c] - P[A|B \cap F]\}.$$

If the event F is realized the cells intersecting C_{00} do not intersect any of the C_{ij}, $(i,j) \in Z$. Then, since the color assignation is independent from cell to cell, X_{oo} is independent from each of the $X_{ij}, (i,j) \in Z$. Also the fact that F is realized does not give information about the black part of C_{00}, so $P[A|B \cap F] = P[A]$ and

$$|P[A|B] - P[A]| \leq P[F^c|B] \leq \sum_{k=1}^{8} P[E_k^c|B].$$

It is easy to see that $P[E_k^c|B] \leq P[E_k^c]$ i.e. the knowledge of B may indicate that some lines indeed isolate C_{00} from the C_{ij}, $(i,j) \in R_k \cap Z$. Then

$$\sup_{A,B} |P[A|B] - P[A]| \leq \sum_{k=1}^{8} P[E_k^c] = \sum_{k=1}^{4} e^{-\lambda d_k[2-\sqrt{2}]} + \sum_{k=5}^{8} e^{-2\lambda[(1+d_k^2)^{1/2}-1]},$$

and given the relation between the Euclidean distance and the distance define by (4), there exists a constant $\nu > 0$ such that

$$\alpha_{1\infty}(m) \leq 8e^{-\nu m}$$

and so we have the condition (ii) of H_2.

The same type of argument gives that for each k, ℓ such that $k + \ell \leq 4$

$$\alpha_{k\ell}(m) \leq Ke^{-\nu m} \quad K, \nu > 0.$$

so condition (i) of H_2 is satisfied, and condition (iii) of H_2 is satisfied for every $\delta > 0$.

From the Remark 3 following Proposition 6 we have

$$\sigma_0^2 = \int_{R^2} p(1-p)e^{-2\lambda\|0,x\|}dx > 0.$$

The assumption H_2 is also satisfied when $r > 0$. The proof in the case $r = 0$ is based on a conditioning such that when an event F is satisfied, what happen concerning the colors on a given set of squares is independent of what happen concerning the colors on another set of squares far apart. So, the same argument leads to similar mixing coefficients for any fixed $r > 0$.

From the Remark 3 following Proposition 6 we can verify that for each basic morphological operation we have $\sigma_r^2 > 0$.

In conclusion, all the results stated in Propositions 1 to 6 are valid for this image generating process. Figure 3 shows, for the process with λ and p as in Figure 1, the graph of an estimate of the function $F(r)$ corresponding to the erosion by a square structuring element. The estimate is the average from 250 independent simulations on a fixed window.

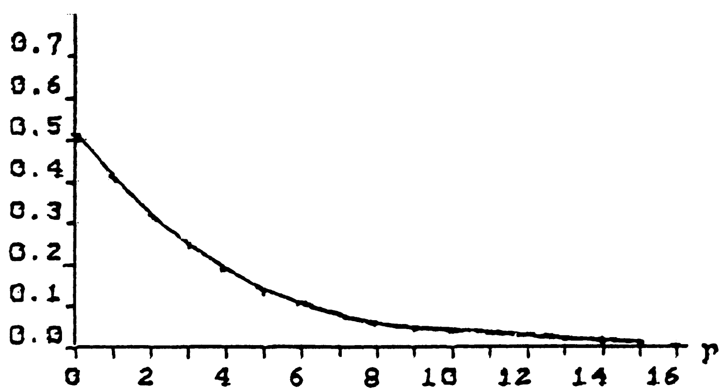

Figure 3. $F(r) = \mathrm{P}[I \ominus T_r \supset \{o\}]$, T_r a square, for an image generated as in Figure 1.

B. The Voronoï polygon process

Again here the image is generated in two steps. First the plane is partitioned into convex polygons (cells) and then a color is assigned to each cell. The coloring process is the same as in Example A. To obtain the cells we consider the realizations $\tau_1, \ldots$ of a planar homogeneous Poisson process with intensity λ, and for each τ_i we consider the cell given by the points $x \in R^2$ closest to τ_i than to any of the other τ_j's. This process is stationary and isotropic. Figure 4 shows a realization of this process, λ is such that an average of 50 points τ_i are generated on the square and $p = 1/2$.

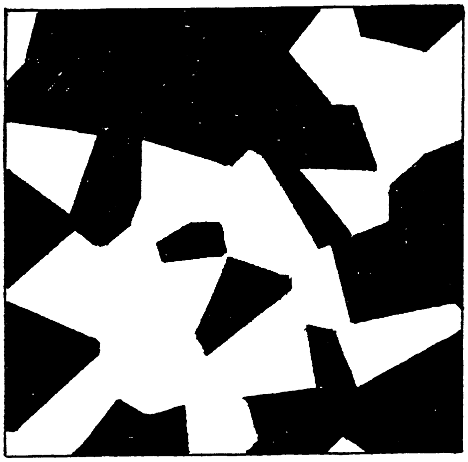

Figure 4. Image generated by a Voronoi polygon process.

It is easy to see that assumption H_1 of Proposition 6 is satisfied (for the four basic morphological operations and every $r > 0$). Given two points at a distance d we have

$$P[\{I : I \supset \{x\}, I \supset \{y\}\}] - p^2 = p(1-p)e^{-k(\lambda^{1/2}d)^\alpha}$$

where $k > 0$ and $\alpha > 1$ (Moore 1981), so the assumptions of Proposition 2 and assumption (ii) of Proposition 3 are satisfied for $r = 0$. These assumptions are also verified for $r > 0$.

Here again the main difficulty is with the assumption H_2 of Proposition 6. Since our argument is similar to the one used in Example A it will be sufficient to consider the case $r = 0$. Also since the argument for the $\alpha_{k\ell}(m), k + \ell \leq 4$ is easy when we know how to treat $\alpha_{1\infty}(m)$ we consider only that case.

The sets of sites, the regions $R_1 \ldots R_8$ and the distance d_1 are as in Example A. Consider the line with slope one crossing R_1 and let T be its intersection with the circle of radius d_1 centered at (1, 1) (see Figure 5). Let $(1, t_1)$ and $(t_1, 1)$ be the projections of T on the axes defining R_1. We consider two subregions of R_1.

(a) The subregion $R_1' = [1, \infty) \times [t_1, \infty)$.

Let D_1 be the part of the disk of radius $(t_1 - 1)/2$ centered at $(1, t_1)$ and located in R_1'. Let C_1 be the part of the disk of radius $(t_1 - 1)/2$ and centered at (1, 1) and located in $(-\infty, 1) \times (-\infty, 1)$ (see Figure 5). We can verify that for every $y \in R_1'$

$$\sup_{u \in D_1} ||y, u|| \leq \inf_{v \in C_1} ||y, v||.$$

(b) The subregion $R_1'' = [t_1, \infty) \times [1, \infty)$.

Let D_2 be the part of the circle of radius $(t_1 - 1)/2$ centered at $(t_1, 1)$ and located in R_1''. For each $y \in R_1''$

$$\sup_{u \in D_2} ||y, u|| \leq \inf_{v \in C_1} ||y, v||.$$

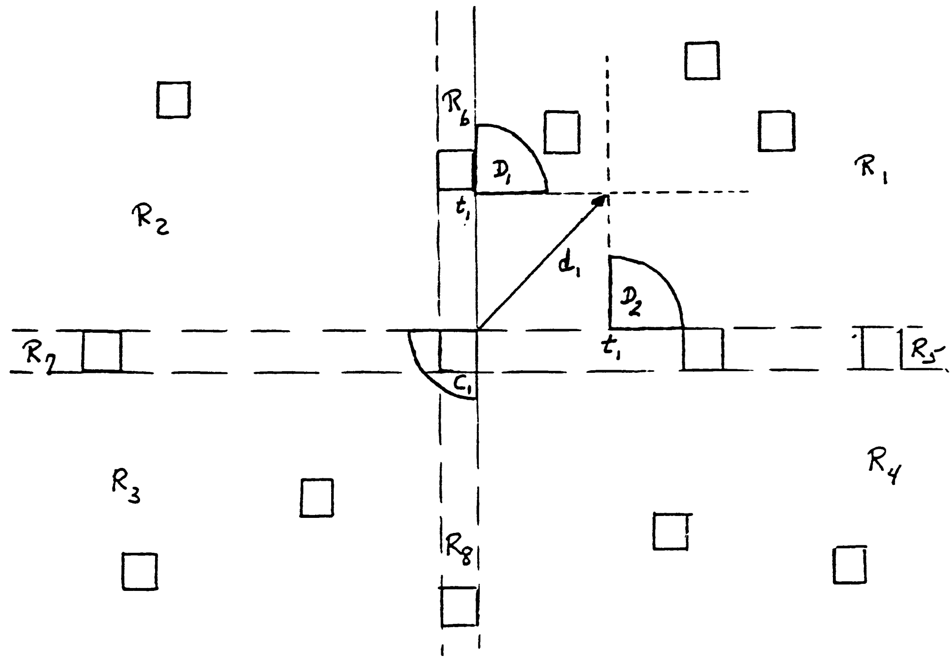

Figure 5.

Also for each pair of points $x, y \in C_1$ we have

$$||x, y|| \leq \inf_{u \in D_1} ||x, u|| \quad \text{and} \quad ||x, y|| \leq \inf_{u \in D_2} ||x, u||.$$

If d_1 is large enough C_{00} will be included in C_1.

From these facts, if the Poisson process generates at least one point in C_1, at least one point in D_1 and at least one point in D_2, then each point in the $C_{ij}, (i,j) \in Z \cap R_1$, and each point in C_{00} will belong to different cells i.e. C_{00} and the $C_{ij}, (i,j) \in Z \cap R_1$ will be separated. Let E_1 be the event "at least one point is generated in C_1" and $G_{1\ell}$ be the events "at least one Poisson point is generated in $D_{\ell''}(\ell = 1, 2)$. Similarly the events E_k and $G_{k\ell}, k = 2, 3, 4, \ell = 1, 2$ are defined for the regions R_2, R_3 and R_4.

For the region R_5 the distance d_5 is defined as in Example A. Let C_5 be the circle of radius $d_5/4$ centered at $(1, 1/2)$ and D_5 be the circle of radius $d_5/4$ centered at $(d_5 + 1, 1/2)$. We can verify that for each $x \in C_{00}$

$$\sup_{u \in C_5} ||x, u|| \leq \inf_{v \in D_5} ||x, v||,$$

also for each $(i,j) \in Z \cap R_5$ and each $y \in C_{ij}$

$$\sup_{v \in D_5} ||y, v|| \leq \inf_{u \in C_5} ||y, u||.$$

So, if the Poisson process generates at least one point in C_5 and at least one point in D_5 then C_{00} will be separated from all the $C_{ij}, (i,j) \in Z \cap R_5$. Let E_5 be the event "at least one point is generated in C_5" and G_5 be the event "at least one point is generated in D_5". Similarly the events E_k and $G_k, k = 6,7,8$ are defined for the regions R_6, R_7 and R_8.

Let F be the event

$$\left(\bigcap_{k=1}^{8} E_i\right) \cap \left(\bigcap_{k=1}^{4}\bigcap_{\ell=1}^{2} G_{k\ell}\right) \cap \left(\bigcap_{k=5}^{8} G_k\right).$$

As in Example A,

$$\sup_{A,B} |P[A|B] - P[A]| \leq \sum_{k=1}^{8} P[E_k^c] + \sum_{k=1}^{4}\sum_{\ell=1}^{2} P[G_{k\ell}^c] + \sum_{k=5}^{8} P[G_k^c]$$

and here we have

$$P[E_k^c] = e^{-\lambda\pi d_k^2/8} k = 1,\ldots,4 \quad P[G_{k\ell}^c] = e^{-\lambda\pi d_k^2/8} \quad k = 1,\ldots,4; \quad \ell = 1,2$$

$$P[E_k^c] = e^{-\lambda\pi d_k^2/16} = P[G_k^c] \quad k = 5,\ldots,8.$$

These facts and the relation between the Euclidean distance and the distance defined by (4) ensure the existence of positive constants K_1 and K_2 such that

$$\alpha_{1\infty}(m) \leq K_1 \quad e^{-K_2 m^2}$$

and so the condition (ii) in H_2 is satisfied.

From the Remark 3 following Proposition 6 it can be verified that $\sigma_r^2 > 0$ for each $r \geq 0$.

Figure 6 shows for the process with λ and p as in Figure 4, the graph of an estimate of the function $F(r)$ corresponding to the opening by a square structuring element. The estimate is the average from 250 independent simulations on a fixed window.

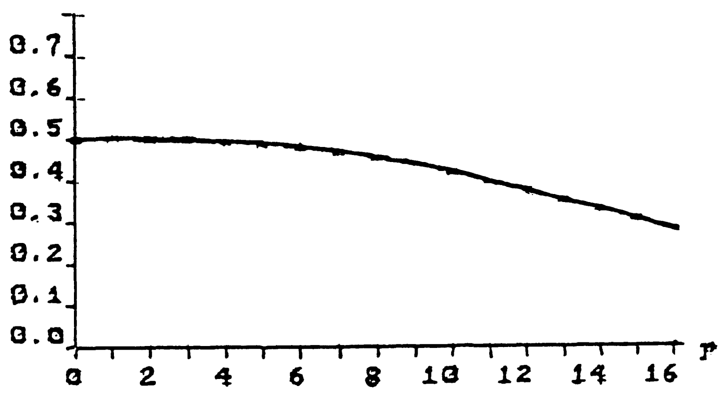

Figure 6. $F(r) = \mathrm{P}[(I \ominus \tilde{T}_r) \oplus T_r \supset \{0\}]$, T_r a square, for an image generated as in Figure 4.

Remark

These two examples are based on the Poisson process. It is possible to give examples for which this process is not involved. For example, image generating processes based on spatial moving average processes (Moore 1988) can be defined. Since these processes exhibit only a finite range dependence the assumptions made here are satisfied.

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