Stochastic Orders and Decision under Risk
IMS Lecture Notes – Monograph Series (1991)

REPAIR POLICIES AND STOCHASTIC ORDER

BY THOMAS H. SAVITS*

University of Pittsburgh

This paper reviews the role of stochastic order as it relates to the study of maintained systems in reliability theory. The classical univariate comparison results for the age and block replacement policies are presented. Extensions to stochastic comparison of processes and recent generalizations of age and block policies are also discussed.

1. Introduction. The purpose of this paper is to survey the role of stochastic order as it relates to the study of maintained systems in reliability theory. Some of the earliest treatments of maintenance considerations date back to Khintchine (1932), Lotka (1939), Campbell (1941), and others. An excellent presentation of the historical background is contained in the book of Barlow and Proschan (1965).

Maintenance policies are followed so as to reduce the number of system failures. Typically, as a unit ages it tends to break down more frequently. Since these *unplanned* failures can be costly, it may at some point in time be more cost effective to simply replace the unit before it fails with a new unit. Any such strategy of *planned* replacement is called a maintenance policy. Two policies that have received considerable attention in the literature are known as age and block replacement policies. These will be reviewed in Section 2.

The (cost) effectiveness of a maintenance policy has been quantified in various ways. One simple measure is in terms of the expected number of failures (and/or repairs). This information would be sufficient provided the replacement costs are identical and constant. For more complex cost structures, however a better measure is provided through the notion of stochastic order. Early comparisons were between random variables; the modern viewpoint is to compare processes.

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* Supported by AFOSR Grants No. AFOSR-84-0113, AFOSR-89-0370 and NSA MDA-904-90-H-4036.

AMS 1980 Subject Classification: Primary 62N05; Secondary 60K10

Key words and phrases: Stochastic comparison, age maintenance, block maintenance, minimal repair, NBU, IFR.

The outline of this paper is as follows. In Section 2 we review the classical (marginal) comparison results. The framework for stochastic comparison of counting processes is given in Section 3. We extend the classical notions of age and block replacement policies in Section 4, and list some new results in Section 5. An illustrative example is presented in the final section.

Throughout this article, the life distribution F of a new unit is assumed to be a continuous function; its survival probability is denoted by $\bar{F}(t) = 1-F(t)$. We shall also use the term "repair" in the wide sense: a replacement can be interpreted as a *complete repair*. Generally, the letter "N" is used to denote a process which counts the number of unplanned repairs. Superscripts "A" and "B" refer to an age policy (even in the extended sense of a repair policy introduced in Section 4) and block policy, respectively. If we want to emphasize the dependence on the life distribution F, a subscript will then be used.

Lastly, we mention that, because of space considerations, only comparison results for unplanned repair/replacements were considered here. A more complete cost analysis would also include a comparison of planned replacements. These comparisons have been considered in the literature (e.g., see Barlow and Proschan (1965, 1981) and Block, Langberg and Savits (1989, 1990).

Some additional related material may also be found in Blumenthal, Greenwood and Herbach (1976), Deng (1985), Langberg (1988), O'Brien (1975), Shaked and Shanthikumar (1989), and Whitt (1981).

2. Review of Classical Results. For the classical situation we suppose that a new unit on line has survival distribution $\bar{F}$. There are three general strategies we want to consider: (i) no maintenance policy, i.e., replacement at failure only; (ii) a block replacement maintenance policy; and (iii) an age replacement maintenance policy.

In the first case, we simply wait for a unit to fail and then replace it with a new identical unit. Since there is no maintenance policy, all replacements are unplanned. We denote the number of such unplanned replacements in $[0, t]$ by $N(t)$. Thus $\{N(t), t \geq 0\}$ is a renewal process generated by F.

For a block replacement maintenance policy, complete replacements are performed at failure (unplanned replacements) and also at absolute times $T, 2T, \cdots$ (planned replacements). The number of unplanned replacements in $[0, t]$ is denoted by $N^B(T; t)$.

In the case of an age replacement maintenance policy, we have complete replacements at failure (unplanned replacements) or at age T (planned replacements), whichever comes first. The process $N^A(T; t)$ counts the number of unplanned replacements in $[0, t]$.

The basic question of interest then is how these strategies compare. It is intuitively clear that if the performance of a unit deteriorates as it ages, a planned replacement policy (e.g., age or block) would result in fewer unplanned replacements. This is indeed the situation.

We first need a notion of wear-out. In the reliability literature there are several nonparametric classes of life distributions that are used to describe this phenomenon. The two that we discuss here are the *new better than used* (NBU) *class* and the *increasing failure rate* (IFR) *class.* A life distribution F is said to be NBU if $\bar{F}(s+t) \leq \bar{F}(s)\bar{F}(t)$ for all $s, t \geq 0$; it is said to be IFR if $\bar{F}(s+t)/\bar{F}(s)$ is nonincreasing in $s \geq 0$ for each $t \geq 0$.

Intuitively, the first notion (NBU) says that the probability of surviving an additional t units of time is less than that of a new unit; the second notion (IFR) says that the probability of surviving an additional t units of time decreases as it ages. It should be noted that if F is IFR, then it is also NBU.

In order to compare the strategies, the standard notion of stochastic ordering is used, i.e., a random variable X is said to be stochastically smaller than a random variable Y if $\bar{F}(t) = P(X > t) \leq P(Y > t) = \bar{G}(t)$ for all t. In this case, we write $X \overset{\text{st}}{\leq} Y$, or alternatively, $F \overset{\text{st}}{\leq} G$.

The classical comparison results are listed in Theorem 2.1. A good reference for these results is Barlow and Proschan (1981).

(2.1) THEOREM. *Let T, $t \geq 0$ be fixed. If F is NBU, then*

(i) $N^B(T;t) \overset{\text{st}}{\leq} N(t)$;

(ii) $N^A(T;t) \overset{\text{st}}{\leq} N(t)$;

(iii) $N^B(T;t) \overset{\text{st}}{\leq} N^B(kT;t)$ *for* $k = 1, 2, \cdots$;

(iv) $N^A(T;t) \overset{\text{st}}{\leq} N^A(kT;t)$ *for* $k = 1, 2, \cdots$.

When F is IFR, we have

(v) $N^A(T_1;t) \overset{\text{st}}{\leq} N^A(T_2;t)$ *whenever* $T_1 \leq T_2$;

(vi) $N^B(T;t) \overset{\text{st}}{\leq} N^A(T;t)$.

(2.2) REMARK. The results (i)–(v) also characterize the life class of F. That is, for example, if (i) holds for all $T, t \geq 0$, then F is NBU.

The results (i) and (ii) of Theorem 2.1 show that under a wear-out assumption, a planned replacement maintenance policy reduces stochastically the number of unplanned replacements. The results (iii), (iv), and (v) compare what happens when the parameter T is changed, and (vi) compares age and block policies.

An important point to observe about the classical comparisons is that even though a stochastic process is used to record the number of unplanned replacements, only marginal comparisons are considered. This may be sufficient for making simple cost comparisons, however, a more complicated cost structure would generally require information about the entire process. This provides the motivation for the next section.

3. Comparison of Counting Processes. We first introduce some basic terminology. By a counting process we mean a stochastic process whose sample paths are nonnegative right-continuous step functions starting at zero at $t = 0$ and only increasing by jumps of size one. The totality of such paths is denoted by S.

If K and L are two counting processes, we say that K is stochastically smaller than L, written as $K \overset{\text{st}}{\leq} L$, if for every $n = 1, 2, \cdots$ and all choices $0 \leq t_1 < t_2 < \cdots < t_n$,

$$P\{(K(t_1), \cdots, K(t_n)) \in U\} \leq P\{(L(t_1), \cdots, L(t_n)) \in U\}$$

for every Borel upper set U in $\mathbf{R}^n$. A set U in $\mathbf{R}^n$ is called upper if whenever $\mathbf{x} = (x_1, \cdots, x_n) \in U$ and $\mathbf{y} = (y_1, \cdots, y_n)$ satisfies $y_i \geq x_i$, $i = 1, \cdots, n$, then $\mathbf{y} \in U$.

(3.1) Remark. A more general treatment of stochastic ordering is given in Kamae, Krengel and O'Brien (1977). Indeed, S can be viewed as a partially ordered Polish space. If λ_K and λ_L denote the corresponding probability measures induced on S by the counting processes K and L, then $K \overset{\text{st}}{\leq} L$ is equivalent to $\lambda_K \overset{\text{st}}{\leq} \lambda_L$.

A technique that is useful for stochastic comparison of processes consists of the following three steps. Let $K = \{K(t), t \geq 0\}$ and $L = \{L(t), t \geq 0\}$ be two counting processes.

Step 1: Identify "objects" $U_1, U_2, \cdots$ in K and $V_1, V_2, \cdots$ in L and a monotonic function $\Psi : \prod_{n=1}^{\infty} E_n \to S$ such that $K = \Psi(\langle U_n \rangle)$ and $L = \Psi(\langle V_n \rangle)$. Here E_n is an appropriate state space for U_n and V_n which is equipped with a suitable notion of upper sets. The notation $\langle U_n \rangle$ is used to denote the sequence on the product space $\prod_{n=1}^{\infty} E_n$ whose nth element is U_n. Similarly for $\langle V_n \rangle$.

Step 2: Show $\langle U_n \rangle \overset{\text{st}}{\leq} \langle V_n \rangle$ by showing that

$$U_1 \overset{\text{st}}{\leq} V_1$$

and

$$(U_{k+1} \mid U_1 = u_1, \cdots, U_k = u_k) \overset{\text{st}}{\leq} (V_{k+1} \mid V_1 = v_1, \cdots, V_k = v_k)$$

for all $u_1 \leq v_1, \cdots, u_k \leq v_k$ and $k \geq 1$. This is a main result in Kamae, Krengel and O'Brien (1977).

STEP 3: Conclude that $K \overset{\text{st}}{\leq} L$ if Ψ is nondecreasing or $K \overset{\text{st}}{\geq} L$ if Ψ is nonincreasing.

Perhaps the most crucial step is the first. This is the decomposition step. The goal is to decompose the counting processes into simpler pieces. The particular choice of these pieces (or "objects") depends upon the nature of the counting process and on one's ingenuity.

We now illustrate with some examples. In the first example, the "objects" are restrictions of the counting process to specified time intervals. For the second example, the "objects" are the interarrival times, and in the last case, we use the arrival times (also, see Section 6 for a more specific example).

EXAMPLE 1. Let $\{I_n\}$ be a partition of $[0, \infty)$. Then define $K_n = \{K(t); t \in I_n\}$ and Ψ by

$$\Psi(\langle K_n \rangle)(t) = K_j(t) \text{ if } t \in I_j.$$

EXAMPLE 2. If the interarrival times of K are denoted by $X_1, X_2, \cdots$, then Ψ is given by

$$\Psi(\langle X_n \rangle)(t) = \sum_{j=1}^{\infty} I_{[0,t]}(X_1 + \cdots + X_j).$$

EXAMPLE 3. This time we consider the arrival times $S_1, S_2, \cdots$ of K. Now

$$\Psi(\langle S_n \rangle)(t) = \sum_{j=1}^{\infty} I_{[0,t]}(S_j).$$

Note that Ψ is nondecreasing in Example 1, whereas it is nonincreasing in Examples 2 and 3.

(3.2) REMARKS. Let $\langle X_n \rangle$, $\langle S_n \rangle$, $\langle Y_n \rangle$, $\langle T_n \rangle$ denote the interarrival and arrival times for the counting processes K and L respectively.

(i) $\langle X_n \rangle \overset{\text{st}}{\leq} \langle Y_n \rangle$ implies $\langle S_n \rangle \overset{\text{st}}{\leq} \langle T_n \rangle$ since nondecreasing functions preserve stochastic order.

(ii) $\langle S_n \rangle \overset{\text{st}}{\leq} \langle T_n \rangle$ if and only if $K \overset{\text{st}}{\geq} L$. Hence, the stochastic ordering of interarrival times gives a stronger result than the stochastic ordering of the processes themselves. This observation may be useful if the cost structure is a nondecreasing function of the interarrival times but not of the arrival times.

4. Extending the Notions of Block and Age. The classical notions of block and age replacement policies are extended in Block, Langberg and Savits (1989, 1990). In addition to their mathematical interest, they also have technical utility.

We briefly describe these extensions in the case of an arbitrary counting process Q. Let $Q_i = \{Q_i(t), t \geq 0\}$ be independent copies of Q and $Z = \langle z_k \rangle$ a sequence of positive numbers.

The extension in the block policy case is straightforward. We define

$$Q^B(Z;t) = \begin{cases} Q_1(t) & \text{if } 0 \leq t < z_1, \\ \sum_{i=1}^{k} Q_i(z_i-) + Q_{k+1}(t - z_1 - \cdots - z_k) & \text{if } \sum_{i=1}^{k} z_i \leq t < \sum_{i=1}^{k+1} z_i. \end{cases}$$

The process $Q^B(Z;t)$ counts the number of unplanned repairs in $[0,t]$. Intuitively, we have planned replacements at the times $z_1, z_1 + z_2, \cdots$; unplanned repairs between $\sum_{i=1}^{k} z_i$ and $\sum_{i=1}^{k+1} z_i$ are governed by the process Q_{k+1}.

The extension in the age policy case is of a different nature. We shall need to make use of the interarrival times $\{V_{ij}\}$ for the process Q_i. A typical sample path may be described as follows. Suppose that $V_{11} < z_1$, $V_{12} < z_2$, $V_{13} < z_3$ but $V_{14} \geq z_4$. Then for $0 \leq t < \zeta_1 = V_{11} + V_{12} + V_{13} + z_4$ we set $Q^A(Z;t) = Q_1(t)$. The times $V_{11}, V_{11} + V_{12}, V_{11} + V_{12} + V_{13}$ are interpreted as unplanned repair times, whereas the time ζ_1 corresponds to a planned replacement. Thus we have unplanned repairs as long as the times between repairs is not "too great" as measured by the z_k's. Next suppose that $V_{21} < z_5$ but $V_{22} \geq z_6$. Then for $0 \leq t < \zeta_2 = V_{21} + z_6$ we define $Q^A(Z; t+\zeta_1) = Q_1(\zeta_1-) + Q_2(t)$. Continue in this manner. The process $Q^A(Z;t)$ counts the number of unplanned repairs in $[0,t]$.

(4.1) REMARK. The planned replacement events above do not, in general, have an age interpretation, i.e., a planned replacement is governed by the time between repairs and not by the age of the unit. The only exception to this is when the process Q is a renewal process, as in the classical case. We thus choose to call this extended notion a repair replacement policy instead of an age replacement policy.

5. Applications. We now restrict our attention to two special cases. For the first case we take the counting process Q in Section 4 to be a renewal process N generated by F. Then, for $Z = \langle z_k \rangle$, we define the processes $N^A(Z)$ and $N^B(Z)$ by $N^A(Z;t) = Q^A(Z;t)$ and $N^B(Z;t) = Q^B(Z;t)$. If all $z_k = T$, we obtain the classical age and block counting processe $N^A(T;t)$ and $N^B(T;t)$, respectively, of Section 2.

For the second case, we take Q to be a nonhomogeneous Poisson process N_m with mean function $E[N_m(t)] = -\ell n\ \bar{F}(t)$. The process N_m thus corresponds to a minimal repair process, i.e, at failure times the item is restored to its functioning state just prior to failure. Now set $N_m^A(Z;t) = Q^A(Z;t)$ and $N_m^B(Z;t) = Q^B(Z;t)$.

For the above six processes there are many possible comparisons that might be of interest. These are considered in great detail in Block, Langberg and Savits (1989, 1990). The following theorem lists a few of the results.

(5.1) THEOREM.

(i) *All classical results in Theorem 2.1 remain valid as stochastic comparison of processes.*

(ii) $N_{m,G}^B(Z) \overset{\text{st}}{\le} N_{m,F}^A(Z)$ *for all* Z *if and only if* $F \overset{\text{st}}{\le} G$.

(iii) $N_{m,G}^A(Z) \overset{\text{st}}{\le} N_{m,F}^A(Z)$ *for all* Z *if and only if* $F \overset{\text{st}}{\le} G$.

(iv) $N^B(Z) \overset{\text{st}}{\le} N_m$ *for all* Z *if and only if* F *is NBU.*

(v) $N^A(Z) \overset{\text{st}}{\le} N_m$ *for all* Z *if and only if* F *is NBU.*

(vi) $N_m^A(Z) \overset{\text{st}}{\le} N_m$ *for all* Z *if* F *is IFR.*

(5.2) REMARK. Most of the above results were obtained through comparison of interarrival times. Thus those results are actually stronger than indicated. However, for (vi), we could only find a proof based on arrival times.

6. Illustrative Example. We close this article with a simple example which illustrates some of the technical aspects. We consider a stochastic comparison between two minimal repair processes based on different distributions, namely $N_{m,F}$ and $N_{m,G}$. Our goal is to find conditions such that $N_{m,F} \overset{\text{st}}{\ge} N_{m,G}$.

Method 1. Our first attempt is to consider the interarrival times $\langle X_n\rangle$, $\langle Y_n\rangle$ of $N_{m,F}, N_{m,G}$ respectively. By definition of a minimal repair process, we have

$$P(X_{k+1} > t \mid X_1 = x_1, \cdots, X_k = x_k) = \frac{\bar{F}(t + x_1 + \cdots + x_k)}{\bar{F}(x_1 + \cdots + x_k)},$$

and a similar expression for $(Y_{k+1} \mid Y_1 = y_1, \cdots, Y_k = y_k)$ in terms of G. Thus, if we want to conclude that $\langle X_n\rangle \overset{\text{st}}{\le} \langle Y_n\rangle$ (and hence $N_{m,F} \overset{\text{st}}{\ge} N_{m,G}$) using the method outlined in Step 2 of Section 3, we must require that

$$\bar{F}_d(t) \equiv \frac{\bar{F}(t+d)}{\bar{F}(d)} \leq \frac{\bar{G}(t+e)}{\bar{G}(e)} \equiv \bar{G}_e(t) \tag{6.1}$$

for all $t \geq 0$ and all $0 \leq d \leq e$.

Method 2. The next approach is to use the arrival times $\langle S_n \rangle$ and $\langle T_n \rangle$ instead. Now, we can write

$$P(S_{k+1} > | S_1 = s_1, \cdots, S_k = s_k) = \frac{\bar{F}(t \vee s_k)}{\bar{F}(s_k)}$$

and a similar expression for $(T_{k+1} \mid T_1 = t_1, \cdots, T_k = t_k)$ in terms of G. Here $a \vee b$ means $\max\{a, b\}$. It is not hard to show this time that in order for $\langle S_n \rangle \overset{\text{st}}{\leq} \langle T_n \rangle$, we only need to require the condition

$$\bar{F}_d(t) \leq \bar{G}_d(t) \tag{6.2}$$

for all $d, t \geq 0$.

Although condition (6.2) is weaker than condition (6.1) it is not the best possible result.

Method 3. For this approach we make use of a special fact about nonhomogeneous Poisson processes. Let M be a Poisson process with mean rate one. Then $N_{m,F}(t) = M[-ln\ \bar{F}(t)]$ and $N_{m,G}(t) = M[-ln\ \bar{G}(t)]$. Since a Poisson process has nondecreasing sample paths, it follows that $N_{m,F}(t) \geq N_{m,G}(t)$ pathwise provided $-ln\ \bar{F}(t) \geq -ln\ \bar{G}(t)$, i.e., $N_{m,F} \overset{\text{st}}{\geq} N_{m,G}$ if

$$F \overset{\text{st}}{\leq} G. \tag{6.3}$$

Moreover, this condition is also necessary.

Acknowledgement: The author would like to thank the referees for their helpful suggestions.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260