Stochastic Orders and Decision under Risk
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STOCHASTIC ORDERINGS IN RELIABILITY

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In this paper we give a short survey of topics in reliability and show how stochastic ordering can be used in reliability theory: to describe aging, to compare performance processes, and to find optimal replacement policies.

1. Introduction. Examples of the use of stochastic orderings in reliability are numerous. Therefore, we do not attempt to give a survey of the field, but we only describe some main ideas on how to use stochastic orderings in reliability theory. It should not be unreasonable to expect that readers of this paper know the fundamentals of stochastic ordering as can be found e.g. in Mosler (1982) and Stoyan (1983), but perhaps they are not already familiar with the subject of reliability theory. Therefore we describe in Section 1.1 a well known system with maintenance through spares and repair in order to illustrate the typical topics and problems of reliability theory. After the discussion of some problems which show that stochastic ordering is a useful tool in reliability theory, we give in Section 1.2 a detailed description of the further contents of the paper; aging, comparison of processes, optimal maintenance through replacement. Section 1.3 gives the notation we will use for the different kinds of stochastic ordering in Section 2. After introducing the main concepts of aging, we give a summary of their generalization to the multicomponent case and to the use of more complicated information about the system. In Section 3 we give a detailed description of the performance process of our standard reliability system and state sufficient conditions that allow the stochastic comparison of standard reliability processes. We compare these conditions with the conditions given by Shaked and Shantikumar (1988) for the more special case of dependent coupled alternating renewal processes. In Section 4 we give an example to show how stochastic ordering is useful, if

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one uses Markov decision processes for finding optimal replacement policies for reliability systems.

In this survey we generally do not dwell on mathematical technical details. If not stated otherwise, we start from a probability space $(\Omega, \mathcal{F}, \Pr)$. We make the usual assumptions about the stochastic processes, e.g. that the state spaces are complete, separable and metric, that the sample paths are continuous from the right and have limits from the left. Partial orders $\leq$ on the state space are assumed to be complete and compatible with the linear structure on the state space if there is any. (Sometimes only reflexive and transitive relations $\leq$ are necessary.) If Ω is partially ordered, we suppose that for all $\omega' \in \Omega$ the sets $\{\omega \in \Omega : \omega \leq \omega'\}$ and $\{\omega \in \Omega : \omega' \leq \omega\}$ are measurable.

1.1. Topics in Reliability. We illustrate the terminology and some typical problems in reliability theory.

EXAMPLE. Standard Reliability System (SRS). (See e.g. Barlow and Proschan (1975), p. 204.) A system $\mathcal{S}$ consists of n units which can be "functioning" ("up") or "failed" ("down") and σ "service (or repair) facilities." If k of the units are up at time t and if $k \leq m$ (where $m \leq n$ is a given number), then all k functioning units are "working" ("operating"). If $m < k$, then $k-m$ of the functions units are "spares" ("in cold standby"). A unit in spare status cannot fail. If unit i is working, it has a positive random lifetime with given distribution function F_i. When a working unit fails, it is immediately replaced by a spare, if available; the failed unit is immediately sent to a repair facility. Repair begins immediately unless all σ repair facilities are occupied. Each unit i has positive random repair time with given distribution function G_i. If at time t the number $n-k$ of failed units is greater than σ, then $n-k-\sigma$ units have to "queue" for repair. All lifetimes and repair times are independent random variables.

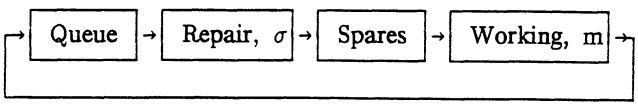

Figure 1. System operation of SRS.

Each unit can be in one of the states q (= queueing), r (= repair), s (= spare), w (= working). So we take $\mathcal{X} = \{q, r, s, w\}$ as state space for each unit and $\mathcal{X}^n$ as state space for the SRS. The state $X_i(t)$ of unit i at time t is a random variable taking values in $\mathcal{X}$, and the $\mathcal{X}^n$-valued random vector $X(t) = (X_1(t), \cdots, X_n(t))$ is sometimes called the "performance process" of the reliability system.

Let us define the system to be in down status at time t, if less than m units are working at time t (and to be in up status otherwise). Usually the

up status (resp. down status) is indicated by "1" (resp. "0"). The status $Z(t)$ of the system at time t is a random variable taking values in the set $\mathcal{Z}$ of possible states of the system; here $\mathcal{Z} = \{0,1\} = \{\text{down, up}\}$. Obviously $Z(t) = \phi(X(t))$, where ϕ is a function from $\mathcal{X}^n$ to $\mathcal{Z}$. In our case $\phi(x) = 0$, if the set $\{i : x_i = w\} =: \mathcal{W}(x)$ of working units contains less than m elements and $\phi(x) = 1$ otherwise. ϕ is called "structure function" of the system. If a state space contains only two elements, one speaks of the "binary case" or "two state case," otherwise of the "multistate case." In the two state case $\mathcal{Z} = \{0,1\}$ the space $\mathcal{X}^n$ is partitioned in two disjoint sets $\mathcal{U} := \{x : \phi(x) = 1\}$, $\mathcal{D} := \{x : \phi(x) = 0\}$. In other words, $X(t) \in \mathcal{D}$ (resp. $X(t) \in \mathcal{U}$) is equivalent to: the system is down (resp. up) at time t.

It is convenient to introduce a partial order "$\leq$" in $\mathcal{X} = \{q, r, s, w\}$: $q \leq s$, $q \leq w$, $r \leq s$, $r \leq w$ and in $\mathcal{Z} = \{0,1\}$: $0 \leq 1$ (down $\leq$ up). (Of course $a \leq a$ for each element a of a partially ordered set.)

We extend the partial ordering "$\leq$" from $\mathcal{X}$ to the product space $\mathcal{X}^n$ by component-wise ordering, i.e. by defining $x \leq y$ iff $x_i \leq y_i$ for all $i = 1, \cdots, n$.

Observe that the structure function ϕ is increasing with respect to these partial orderings in $\mathcal{X}^n$ and $\mathcal{Z}$: $\phi(x) \leq \phi(y)$ for all $x, y \in \mathcal{X}^n$ such that $x \leq y$ and ϕ is not constant. These are the two defining properties of the structure function of a coherent system. (As usual we use the same symbol "$\leq$" for different partial orderings if the meaning of the symbol is clear from the context.)

The subset $\mathcal{U}$ (resp. $\mathcal{D}$) of $\mathcal{X}^n$ defined above is increasing (resp. decreasing). (A set $\mathcal{A}$ is increasing (resp. decreasing) if $a \in \mathcal{A}$ and $a \leq b$ (resp. $b \leq a$) implies $b \in \mathcal{A}$).

The description of the standard system is not yet complete since we have given no rule for choosing the unit j which enters the repair facility first, if more than one unit is queueing for repair. (The analogue question arises for spares.) If we want to stick to the simple state space $\mathcal{X} = \{q, r, s, w\}$ we can choose any rule that depends only on the set of the units in the queue. (E.g. take $j := \min\{i : X_i(t) = q\}$.) If one wants to model the "first come first served" policy, our simple state space would not suffice. Here one could think of a "queueing facility" with $n - \sigma$ "cells" numbered from 1 to $n - \sigma$, (number 1 next to repair) and with $X_i(t) = q_k$ if at time t unit i is in cell k. That means that we have to replace the single state q by the set of $n - \sigma$ states $q_1, \cdots, q_{n-\sigma}$.

The following characteristics of a reliability system $\mathcal{S}$ are of interest.

- Point availability (or reliability) at epoch t_0: $\Pr(X(t_0) \in \mathcal{U})$.
- Interval availability in $[t_0, t_1]$: $\Pr(X(t) \in \mathcal{U} \text{ for all } t \in [t_0, t_1])$.

- Lifetime L of the system (supposing that it starts with $X(0) \in \mathcal{U}$): $L = \inf\{s \geq 0 : X(s) \in \mathcal{D}\}$.
- Survival function $\bar{F}_L$ of L (or of the system): $\bar{F}_L(t) = \Pr(L > t) = \Pr(X(s) \in \mathcal{U} \text{ for all } s \in [0,t])$. Then $F_L(t) := 1 - \bar{F}_L(t)$ is the distribution function of L.
- Mean time to First Failure of the system: $E(L)$, where E denotes the expectation.
- Survival function $\bar{F}_L(t \mid \vartheta)$ of the Residual Lifetime $\rho_\vartheta L$ of L past $\vartheta \in \mathbb{R}_+$:

$$\begin{aligned}\bar{F}_L(t \mid \vartheta) &= \Pr(L - \vartheta > t \mid L > \vartheta) \\ &= \Pr(X(s) \in \mathcal{U} \text{ for all } s \in [0, \vartheta + t] \mid X(s) \in \mathcal{U},\ 0 \leq s \leq \vartheta) \\ &=: \Pr(\rho_\vartheta) L > t) \text{ for all } \vartheta \text{ such that } \Pr(L > \vartheta) > 0).\end{aligned}$$

- Mean residual lifetime past ϑ: $E(\rho_\vartheta L) - \int_0^\infty \bar{F}_L(t \mid \vartheta) dt$.

Also limiting cases of interest, e.g. the limit of the point availability; $\lim_{t\to\infty} \Pr(X(t) \in \mathcal{U})$. (This limit is often easier to obtain than the point availability.)

Usually the distributions of the lifetimes and repair times of the units are considered as given input data. But sometimes one wishes to establish the type of lifetime distribution by using physical models. So physical models are used to describe how cracks initiate, propagate, and cause the breakdown of a unit (see e.g. Yao, et al. (1986)). Such physical processes are usually very complicated. Therefore we consider only a very simple but popular example.

EXAMPLE. Shock Models. Shock models are often used to explain how lifetime distributions arise. Let $(S_n, n \in \mathbb{N})$ denote a point process s.t. $0 < S_1 < S_2 < \cdots$ a.s. and let $N(t)$ denote the corresponding counting process $N(t) = \sum_n 1_{[0,t]}(S_n)$. At S_n, $n \in \mathbb{N}$, the unit suffers a shock which induces a damage $Y_n \geq 0$. The unit is up at time t, iff $X(t) := f(Y_1, \cdots, Y_{N(t)}, 0, \cdots) \leq c$, where f is a given increasing function from $\mathbb{R}_+^{\mathbf{N}}$ to $\mathbb{R}_+$ and $c \geq 0$.

Then the lifetime T of the unit has the distribution

$$\Pr(T > t) = \Pr(X(t) \leq c) = \sum_n \Pr(N(t) = n) \cdot \bar{P}_n,$$

where $\bar{P}_n = \Pr(X(t) \leq c \mid N(t) = n)$.

REMARK. Let F be the distribution of a positive random variable T. A nonnegative function h such that $F(t) = 1 - \exp\left(\int_0^t h(s)ds\right)$, $t \geq 0$, is called the hazard rate of F (or of T) or more specifically failure rate (resp. repair rate) if T is a lifetime (resp. repair time).

It is well known that a distribution F with bounded hazard rate $h(t) \leq \lambda$ can be generated by a shock model, where $(S_n, n \in \mathbb{N})$ is a Poisson process with constant intensity λ and the damages Y_n are conditionally independent $\{0,1\}$-valued random variables with $\Pr(Y_n = 1 \mid S_n, n \in \mathbb{N}) = \lambda^{-1}h(S_n)$. The unit survives as long as there has been no shock damage. This amounts to taking $X(t) = \sum_{n \leq N(t)} Y_n$ and $c = 0$ in the shock model. Given that a shock with damage 1 has occurred at $\widehat{S}$ one can start a new lifetime $\widehat{S}$ instead of $h(t)$, etc. This shows a possibility to construct renewal processes $(\widehat{S}, n \in \mathbb{N})$ with interarrival time distribution F by "thinning" the Poisson process $(S_n, n \in \mathbb{N})$. In an analogue fashion, one can generate Markov processes and Markov renewal processes, see e.g. Miller (1979), Sonderman (1980), Brown (1984), Shaked and Shantikumar (1988), and the book of Stoyan (1983), p. 68.

1.2. Usefulness of Stochastic Ordering in Reliability. The first goal of reliability theory is to deduce the probabilistic properties of the system from the given information regarding its structure and its units. This is in most cases a hard problem. Therefore simplified and easier to handle systems are used to get approximations for the original system. This amounts to comparing stochastic performance processes. By establishing a stochastic order of these processes bounds for the characteristics of the original system are obtained.

Another problem in reliability theory is to examine which modifications of a system result in an improvement. Again, the usefulness of the stochastic comparison of the respective performance processes is obvious. But stochastic ordering is also useful for the comparison of different input data, as can be seen in our standard example. One would hope that the system is improved if one uses shorter repair times. Since the repair times are stochastic variables, stochastic ordering is needed to define the meaning of "shorter."

Another modification which hopefully results in an improvement of a system is the so called "preventive replacement." This means that a working unit is not only replaced upon failure, but also when its current working period has a given age T ("age replacement policy"). another well known policy is to replace a working unit upon failure and at times $T, 2T, 3T, \cdots$. This is called "block replacement policy" because it is usual in practice to replace not only a single unit but a block or set of units at times $T, 2T, 3T, \cdots$. Of course preventive replacement can only be useful it the residual lifetime of a working unit is shorter than the lifetime of a new unit. Again stochastic ordering is needed to compare the residual lifetimes of units with different age. This is done by introducing several formal notions of "aging".

1.3. Stochastic Order. We refer to Mosler (1982) and to Stoyan (1983) for the background and the properties of stochastic orders.

Let $(\mathcal{X}, \mathcal{B}, P^i)$, $i = 1, 2$, be two probability spaces and let $\mathcal{X}$ be endowed

with a partial order. We write $P^1 \leq_d P^2$ iff $\int f dP_1 \leq \int f dP_2$ for all increasing P^1 and P^2 integrable real functions f, $P^1 \leq_c P^2$ iff $\mathcal{X}$ is convex and $\int f dP^1 \leq \int f dP^2$ for all increasing convex P^1 and P^2 integrable real functions f, $P^1 \leq_{cv} P^2$ iff $\mathcal{X}$ is convex and $\int f dP^1 \leq \int f dP^2$ or all increasing, concave P^1 and P^2 integrable functions f. We write increasing instead of nondecreasing, better instead of not worse, etc.

If X^1, X^2 are $\mathcal{X}$-valued random variables with distributions P^1, P^2, we write $X^1 \leq_d X^2$ instead of $P^1 \leq_d P^2$, etc. If P^i has a distribution function F^i then $F^1 \leq_d F^2$ and $P^1 \leq_d P^2$ are defined to be equivalent.

Some other types of increasing functions are of interest in reliability. In the two-state case, e.g. the lifetime L of a system without spares and repair and with component lifetimes, $L_1, \cdots, L_n$ is given by

$$L = \max_j \min_{i \in \mathcal{P}_j} L_i =: \zeta(L_1, \cdots, L_n)$$

where $\mathcal{P}_1, \cdots, \mathcal{P}_k$ are the path sets of system, see Barlow and Proschan (1975). (A path set $\mathcal{P}$ is a subset of the set of units such that the system is up if all units in $\mathcal{P}$ are up.) Except when $k = 1$ (series system) or $|\mathcal{P}_j| = 1$ for all j (parallel system) the function ζ is increasing but neither convex nor concave. So one can use the class R of functions $f = g \circ \zeta$, where $\zeta : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is of the type given above and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, to define a stochastic order $\leq_R$ on the space of n-dimensional positive random vectors $L = (L_1, \cdots, L_n)$: $L^1 \leq_R L^2$ iff $Ef(L^1) \leq Ef(L^2)$ for all $f \in R$ such that these expectations exist (see Arjas (1981)).

2. Notions of Aging. In this section we give an outline of the main ideas used to get a mathematical description of aging of a reliability system. First we introduce the well known and widely used classical notions of aging for a single lifetime (see e.g. Barlow and Proschan (1975), p. 54, 84, 159). Then we give a short survey of some generalizations. We stress the importance of making clear which information about the system available up to time ϑ is used to describe the distribution of the residual lifetime (resp. lifetimes) past ϑ.

Let L denote a positive random variable, called lifetime, with distribution function F and $F(0) = 0$. In reliability theory it is often convenient to describe F by its cumulative hazard $H(t) := -\log(\bar{F}(t))$ where $\bar{F}(t) := 1 - F(t)$ denotes the survival function. If F is absolutely continuous with density f then the hazard rate (or failure rate) h is defined to be $h(t) := \lim_{\delta \to 0} \Pr(L \leq t + \delta \mid$

$L > t) = f(t)/\bar{F}(t)$. So we have

$$\bar{F}(t) = \exp(-H(t)) = \exp\left(-\int_0^t h(u)du\right).$$

Let $\rho_\vartheta L = \max\{0, L - \vartheta\}$ denote the residual lifetime of L past time ϑ. Then the conditional distribution $F(t \mid \vartheta) := \Pr(\rho_\vartheta L \leq t \mid L > \vartheta) = F(t+\vartheta)/\bar{F}(\vartheta)$ of $\rho_\vartheta L$ is used to get formal notions of aging mentioned above. It is clear that $F(t \mid \vartheta)$ has the cumulative hazard $H(t \mid \vartheta) = H(t+\vartheta) - H(\vartheta)$ and the hazard rate $h((t \mid \vartheta) = h(t + \vartheta)$.

The classical notions of aging which describe the case that the performance of the system declines with increasing age ϑ are defined as follows:

F has the property:

- IFR (Increasing Failure Rate). If $F(\cdot \mid \vartheta_2) \leq_d F(\cdot \mid \vartheta_1)$ for all $0 \leq \vartheta_1 < \vartheta_2$. (If F has a density, this is equivalent to: the failure rate h is an increasing function.)
- IFRA (Increasing Failure Rate Average). If $t \to H(t)/t$ is an increasing function. (Equivalent to: $\overline{F}(\alpha t) \geq F^\alpha(t)$ for all $0 < \alpha < 1$ and $t > 0$.)
- NBU (New Better than Used). If $F(\cdot \mid \vartheta) \leq_d F = F(\cdot \mid 0)$ for all $\vartheta > 0$.
- DMRL (Decreasing Mean Residual Lifetime). If $\vartheta \to E(\rho_\vartheta L \mid L > \vartheta) = \int_0^\infty \overline{F}(t \mid \vartheta)dt$ is a real decreasing function.
- NBUE (New Better than Used in Expectation). If EL exists and $E(\rho_\vartheta L \mid L > \vartheta) \leq EL$ for all $\vartheta > 0$.
- If $\mu = EL$ is finite and $\frac{1}{\mu}\int_t^\infty \overline{F}(u)du \leq \exp(-t/\mu)$ for all $t \geq 0$. (The expression on the left hand side is the survival function of the time to next failure in a stationary renewal process with failure lifetime distribution F.)

The intuitive meaning of these notions is clear. In an analogue way notions of aging are defined to describe the case that the performance of the system improves as the age ϑ increases.

The following implications are well known:

$$\text{IFR} \Rightarrow \left\{\begin{array}{c}\text{IFRA} \Rightarrow \text{NBU} \\ \\ \text{DMRL}\end{array}\right\} \Rightarrow \text{NBUE} \Rightarrow \text{HNBUE}.$$

Applications of reliability theory show that it is worthwhile to generalize these notions in the following way: Consider an n-dimensional vector $(L_1, \cdots, L_n)$ of lifetimes and/or use information gathered about the system up to time ϑ other than the fact that the system is up at time ϑ. So in the shock model introduced

in Section 1.1 one could e.g. use the information $L > \vartheta$ (as in the notions above) or $L > \vartheta$, $S_1, \cdots, S_{N(t)}$ or $L > \vartheta$, $S_1, \cdots, S_{N(t)}$, $Y_1, \cdots, Y_{N(t)}$. In the general case let the sub σ-field $\mathcal{F}_\vartheta$ of $\mathcal{F}$ describe the known information about the reliability system up to age ϑ. Suppose $\{L_i > \vartheta\} \in \mathcal{F}_\vartheta$ for $i = 1, \cdots, n$ $\mathcal{F}_\vartheta \subset \mathcal{F}_{\vartheta_1}$ for all $0 \leq \vartheta < \vartheta_1$. (That means: At time ϑ it is known, whether the unit i is up and no information known at ϑ is lost later on.) Using the joint conditional distribution $F(t_1, \cdots, t_n \mid \mathcal{U}_\vartheta) := \Pr(\rho_\vartheta L_1 \leq t_1, \cdots, \rho_\vartheta L_n \leq \mathcal{F}_\vartheta)$ instead of $F(t \mid \vartheta) : \Pr(\rho_\vartheta L_1 \leq t \mid L > \vartheta)$ one can define and use in straightforward manner the properties IFR, NBU, DMRL and NBUE. As to IFR and NBU one could get further generalizations by taking $\leq_c$ or $\leq_{cv}$ or $\leq_R$ instead $\leq_d$.

In many cases the lifetime L of a system is given by its performance process $X = (X(t), t \geq 0)$, with values in the state space $\mathcal{X}$. Let $\mathcal{U}$ (resp. $\mathcal{D}$) denote the set of up-states (resp. down-states) of the system and suppose $\mathcal{D}$ to be a nonempty decreasing set and $\mathcal{D} \neq \mathcal{X}$. Moreover, we assume that $X(0) \in \mathcal{U}$. Then the lifetime of the system (defined as the time to first failure of the system) has the survival function $\overline{F}_\mathcal{D}(t) = \Pr(X(u) \in \mathcal{U}$ for all $u \in [0, t])$ and hence the distribution function $F_\mathcal{D}(t) = 1 - \overline{F}_\mathcal{D}(t)$.

In a natural way the notions of aging can be extended to the performance process X by saying that the process X has the IFR (resp. IFRA, NBU) property if $F_\mathcal{D}$ has the classical IFR (resp. IFRA, NBU) property for all nonempty decreasing subsets $\mathcal{D}$ of $\mathcal{X}$ with $\mathcal{D} \neq \mathcal{X}$. Many authors have dealt with the important problem of establishing aging properties for classes of performance processes. See e.g. Esary, Marshall, and Proschan (1973), Abdel-Hameed and Proschan (1975), Marshall and Shaked (1979), Ross (1979, 1981), Gottlieb (1980), Block and Savits (1981), Klefsjö (1981), Savits and Shaked (1981), Griffith (1982), Ohi and Nishida (1983), Marshall and Shaked (1986), and Shaked and Shantikumar (1987).

A further point of investigation is to establish generalizations of classical closure properties of aging, e.g.: If in the two-state case the structure function of a system is increasing and if the lifetimes of the components are independent and IFRA then the lifetime of the system is IFRA too. Many authors have dealt with this topic, e.g. Ahmed (1990), Arjas (1981), Basu and Kirmani (1986), Block and Savits (1979), Deshpande, Kocher and Singh (1986), Klefsjö (1982), Launer (1984), Marshall and Shaked (1986), ad Shaked (1983).

Weaker orders have also been considered. For instance (Ahmed (1990)) F has the generalized HNBUE property if $\int_0^\infty \int_0^\infty \overline{F}(t+u)dudt \leq \int_0^\infty \overline{F}(t)dt \cdot \int_0^\infty \overline{F}(u)du$. (This can be considered as a generalization of: F is NBU if $\overline{F}(t+u) \leq \overline{F}(t) \cdot \overline{F}(u)$; F is NBUE if $\int_0^\infty \overline{F}(t+u)du \leq \overline{F}(t) \cdot \int_0^\infty \overline{F}(u)du$.)

A typical result: If in the shock model of Section 1.1 the shock epochs

form a Poisson process and the distribution given by $\overline{P}_1, \overline{P}_2, \cdots$ is GHNBUE, then the lifetime L is GHNBUE.

3. Comparison of Performance Process. As we have seen in Section 1.2, it is a natural and important problem of reliability theory to compare performance processes of reliability systems.

3.1. Performance Process for the Standard Reliability System. We consider here only a class of processes $X = (X(t), t \geq 0)$ which arise in a natural way from our standard reliability system in Section 1.1, if one supposes that the lifetime distributions have densities $f_i(t)$ and hazard rates (failure rates) $\mu_i(t)$ and the repair times have densities $g_i(T)$ and hazard rates (Repair rates) $\lambda_i(t)$.

For simplicity we consider first the case $n = 3$ (units) $m = 1$ (working station) and $\sigma = 1$ (repair facility). Consider e.g. $X(s) = (X_1(s), X_2(s), X_3(s)) = (r, w, s) =: x$ (at time s unit 1 is in repair, unit 2 is working, and unit 3 is a spare, see Section 1.1). So at time s units 1 and 2 are active and unit 3 is not active. Let $\Theta_i(s)$ denote the age of the current active (resp. inactive) period of unit i at epoch s. Let $\Theta(s) = (\Theta_1(s), \Theta_2(s), \Theta_3(s))$. So in the situation $X(s) = (r, w, s) = x$ we have $\Theta(s) = (\vartheta_1, \vartheta_2, \vartheta_3) =: \vartheta$ where ϑ_1 is the age of the current repair period of unit 1 and ϑ_2 the age of the current working period of unit 2. Let T_s denote the epoch of the first transition of the performance process X after time s. The transition is caused by the active unit, which has the active period ending first past time s. Let $\rho_{\vartheta_1} R_1$ (resp. $\rho_{\vartheta_2} L_2$) denote the residual repair time of unit 1 (resp. the esidual working time of unit 2). If $\rho_{\vartheta_1} R_1 < \rho_{\vartheta_2} L_2$ then a transition $x = (r, w, s) \to (s, w, s) = y$ occurs, and moreover if $\rho_{\vartheta_1} R_1 \leq t$, then $T_s \leq s + t$. If $\rho_{\vartheta_2} L_2 < \rho_{\vartheta_1 R_1}$, then a transition $x = (r, w, s) \to (r, q, w) =: z$ occurs, and moreover if $\rho_{\vartheta_2} L_2 \leq t$, then $T_s \leq s + t$.

Now we use the assumptions of independence in the standard system and the conditional survival functions $\overline{F}_i(t \mid \vartheta_i)$, $\overline{G}(T \mid \vartheta_i)$, the conditional densities $f_i(t \mid \vartheta_i)$, $g_i(t \mid \vartheta_i)$ and the corresponding hazard rates $\mu_i(t + \vartheta_i)$, $\lambda_i(t + \vartheta_i)$ (see Section 2) of the working and repair times:

$$\begin{aligned}
&\Pr(X(T_s) = y, T_s \leq s + t \mid X(s) = x,\ \Theta(s) = \vartheta,\ (X(u),\ \Theta(u),\quad 0 \leq u < s)) \\
&= \int_{u=1}^{t} g_1(u \mid \vartheta_1) \cdot \overline{F}_2(u \mid \vartheta_2) du \\
&= \int_{u=1}^{t} \lambda_1(u + \vartheta_1) \exp\left(-\int_0^u \lambda_1(v + \vartheta_1) dv\right) \cdot \exp\left(-\int_0^u \mu_2(v + \vartheta_2) dv\right) du \\
&= \int_{u=0}^{t} \lambda_1(u + \vartheta_1) \exp\left(-\int_0^u (\lambda_1(v + \vartheta_1) + \mu_2(v + \vartheta_2)) dv\right) du.
\end{aligned}$$

In the same way we have

$$\Pr(X(T_s) = z,\ T_s \le s+t \mid X(s) = x,\ \Theta(s) = \vartheta,\ (X(u),\ \Theta(u),\quad 0 \le u \le s)) = \int_{u=0}^{t} \mu_2(u+\vartheta_2) \exp\left(-\int_0^u \lambda_1(v+\vartheta_1) + \mu_2(v+\vartheta_2)dv\right) du.$$

To get a uniform notation, we put

$$q_{xy}(\vartheta) := \lambda_1(\vartheta_1),\ q_{xz}(\vartheta) := \mu_2(\vartheta_1),\ q_{x\tilde{y}} := 0 \text{ for } \tilde{y} \in \mathcal{X}^3 \backslash \{y, x\},$$
$$q_x(\vartheta) := \sum_{\tilde{y} \in \mathcal{X}^3} q_{x\tilde{y}}(\vartheta) \text{ and } e = (1,1,1).$$

Then we have

$$P(X(T_x) = \tilde{y},\ T_s \le s+t \mid X(s) = x,\ \Theta(s) = \vartheta,\ (X(u),\ \Theta(u),\quad 0 \le u < s)) = \int_0^t q_{x\tilde{y}}(\vartheta + ue) \exp\left(-\int_0^u (\vartheta + ve)dv\right) du.$$

Taking into account that for $s \le t < s + T_s$ no transitions occur, we have $X(t) + X(s)$ and $\Theta(t) = \vartheta + (t-s)e$ for $s \le t < s + T_s$. At epoch T_s the X-process enters the new state $X(T_s)$ and $\Theta(T_s)$. If we have e.g. $X(s) = x = (r, w, s)$ and $X(T_s) = z = (r, q, w)$ then

the repair of unit 1 continues: $\Theta_1(T_s) = \vartheta_1 + (T_s - s)$,

unit 2 begins to queue (is not active): $\Theta_2(T_s) = 0$, and

unit 3 starts fresh for a working period: $\Theta_3(T_s) = 0$.

Generally speaking, $\Theta(T_s)$ is a function of $X(s)$, $X(T_s)$, $\Theta(s)$, and $T_s - s$: $\Theta(T_s) = \psi(X(s), X(T_s), \Theta(s), T_s - s)$, where the function ψ results from the description of the reliability system. Now we can construct the process $(X(t), \Theta(t), t \ge 0)$ for $T_s \le t \le T_s'$ where T_s' denotes the next transition epoch of x past T_s, etc. We say that $(X(t), \Theta(t), t \ge 0)$ is a reliability process since it describes a typical reliability system.

Along the lines of this simple example, we give a formal description of what we call a reliability process. Let $\mathcal{X}^n$ be the state space of n units of a reliability system where $\mathcal{X}$ is a partially ordered finite set containing a nonempty subset $\mathcal{A}$ of "active states." Let $X_i(t)$ denote the state of unit i at time t and put $X(t) = (X_1(t), \cdots, X_n(t))$. If $X_i(t) \in \mathcal{A}$ (resp. $\notin \mathcal{A}$), let $\Theta_i(t)$ denote the age of the current active (resp. inactive) period of unit i at time t, and let $\Theta(t) = (\Theta_1(t), \cdots, \Theta_n(t))$. We assume that the process $(X, \Theta) = (X(t), \Theta(t), t \ge 0)$ with state space $\mathcal{X}^n \times \mathbb{R}_+^n$ has right continuous sample paths and is constructed in the following way:

(1) $(X(0), \Theta(0))$ has a given distribution π_0.

(2) For all $s \geq 0$ let T_s denote the epoch of the first transition of the process $(X(t), t \geq 0)$ after time s.

At time s the units j such that $X_j(s) \in \mathcal{A}$ have active periods. The active period ending first causes the next transition at time T_s. Let $\nu(x,y)$ denote the number of the unit which causes the transition from $X(s) = x$ to $X(T_s) = y$.

For $x \neq y \in \mathcal{X}^n$ let $\vartheta \to q_{xy}(\vartheta) \in \mathbb{R}_+^n$, $\vartheta \in \mathbb{R}_+^n$, be locally bounded Borel functions (transition rates) such that

$$\Pr(T_s \leq s+t, X(T_s) = y \mid X(s) = x, \Theta(s) = \vartheta, (X(u), \Theta(u);\ 0 \leq u < s))$$
$$\int_0^t q_{xy}(\vartheta + ue) \exp\left(-\int_0^t q_x(\vartheta + ve)dv\right) du \text{ where } q_x(\vartheta) = \sum_{y \in \mathcal{X}^n} q_{xy}(\vartheta)$$

and $e = (1, \cdots, 1) \in \{0,1\}^n$.

(3) $X(t) = X(s)$, $\Theta(t) = \Theta(s) + (t-s)e$ for $s \leq t < T_s$.

(4) $\Theta(T_s) = \psi(X(s), X(T_s), \Theta(s), T_s - s)$, where $(x, y, \vartheta, t) \mapsto \psi(x, y, \vartheta, t) \in \mathbb{R}_+^n$, $x, y \in \mathcal{X}^n$, $\vartheta \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+^n$ is a given function.

(5) If $q_x(\vartheta) \equiv 0$ the state x is absorbing, i.e. if the X-process enters state x at epoch s, we have $X(t) = x$ for all $t \geq s$.

REMARK. We see that $(X(t), \Theta(t); t \geq 0)$ is a Markov process and the law of this process is given by π, ψ, and $q_{xy}(\vartheta)$, $x \neq y \in \mathcal{X}^n$, where $q_{xy}(\vartheta)$ can be considered as the conditional transition rate of the X-process given $\Theta(t) = \vartheta$.

Of course $(X(t), \Theta(t), t \geq 0)$ could also be considered as a semi-Markov process if one takes $(X_n, S_n, n \in \mathbb{N})$ as an embedded Markov renewal process, where S_n denotes the nth transition epoch of the X-process and $X_n = X(S_n)$.

3.2. Stochastic Comparison of Two Processes. Now we wish to compare stochastically two reliability processes (X, Θ) and $(\widehat{X}, \widehat{\Theta})$. An inspection of the proofs of the comparison theorem of Sonderman (1980) for semi-Markov processes X or the well known comparison theorem for Markov processes X (see e.g. Stoyan (1983), p. 62–) without nuisance process Θ shows that the proofs can be extended to the reliability process (X, Θ).

THEOREM 1. *Let (X, Θ) and $(\widehat{X}, \widehat{\Theta})$ be two reliability processes according to the description given above. If*

$$X(0) \leq_d \widehat{X}(0), \tag{1}$$

$$\sum_{y \in \mathcal{U}} q_{xy}(\vartheta) \leq \sum_{y \in \mathcal{U}} \hat{q}_{\hat{x}y}(\hat{\vartheta}) \quad \text{for all } x \leq \hat{x}, \text{ all } \vartheta, \hat{\vartheta}$$
$$\text{and all increasing } \mathcal{U} \subset \mathcal{X}^n \text{ such that } x \notin \mathcal{U},\ \hat{x} \notin \mathcal{U}, \tag{2}$$

$$\sum_{y\in\mathcal{D}} q_{xy}(\vartheta) \geq \sum_{y\in\mathcal{D}} \hat{q}_{\hat{x}y}(\hat{\vartheta}) \text{ for all } x \leq \hat{x}, \text{ all } \vartheta, \hat{\vartheta} \tag{3}$$
$$\text{and all decreasing } \mathcal{D} \subset \mathcal{X}^n \text{ such that } x \notin \mathcal{D},\ \hat{x} \notin \mathcal{D},$$

then there exist processes Y, $\hat{Y}$ defined on a common probability space such that Y (resp. $\hat{Y}$) has the same distribution as X (resp. $\hat{X}$) and

$$\Pr(Y(t) \leq \hat{Y}(t) \text{ for all } t \geq 0) = 1 \text{ and therefore } X \leq_d \hat{X}.$$

REMARK. This theorem remains valid for a countable state space if one adds some regularity conditions.

EXAMPLE. If we take in the standard reliability system $F_1 = F_2 = \cdots = F_n$ with repair rate $\lambda(t)$, the conditions (2) and (3) reduce to

$$\lambda(t) \leq \hat{\lambda}(\hat{t}) \text{ for all } t, \hat{t} \geq 0$$

$$\mu(t) \geq \hat{\mu}(\hat{t}) \text{ for all } t, \hat{t} \geq 0.$$

This is equivalent to the following condition. There exist constants c, d such that

$$\lambda(t) \leq c \leq \hat{\lambda}(\hat{t}) \text{ for all } t, \hat{t} \geq 0$$
$$\mu(t) \geq d \geq \hat{\mu}(\hat{t}) \text{ for all } t, \hat{t} \geq 0.$$

This means that there exists a system $\widetilde{X}$ with exponential repair times and lifetimes which is stochastically "between" X and $\hat{X}$: $X \leq_d \widetilde{X} \leq_d \hat{X}$.

As Sonderman (1980) points out, the conditions (2), (3) of Theorem 1 are very strong. Of course these assumptions can be weakened by making stronger assumptions in some other respect. For instance, Shaked and Shantikumar (1988) consider a reliability process with state space $\mathcal{X}^n = \{r, w\}^n$ where both states r and w are considered active. This means e.g. that it is not possible to have spares or to queue for repair ($n = m = \sigma$ in the standard reliability system). As a trade-off for this restriction of the state space Shaked and Shantikumar get a theorem with weaker assumptions for the transition rates. Define $\vartheta(w, x) = (\vartheta_1(w, x), \cdots, \vartheta_n(w, x)) \in \mathbb{R}_+^n$, where $\vartheta_i(w, x) = \vartheta_i$ if $x_1 = w$ and $= 0$ otherwise; $\vartheta(r, x)$ is defined in an analogue way. Moreover let

$$\alpha_i(x, \vartheta) = q_{xy}(\vartheta) \text{ if } \nu(x, y) = i,$$

that is, i is the unit responsible for the transition from x to y. (See number (2) of the description of a reliability process.)

THEOREM 2 (Shaked and Shantikumar, 1988). *Consider two reliability processes (X, Θ) and $(\hat{X}, \hat{\Theta})$ as in Theorem 1. Assume that in these two processes each unit of the system can be only in one of the states w (= working)*

or r *(= repair). Assume that for all* $x, \hat{x} \in \mathcal{X}^n$ *and* $\vartheta, \hat{\vartheta} \in \mathbb{R}_+^n$ *such that* $\vartheta(w,x) \leq \hat{\vartheta}(w,x)$ *and* $\vartheta(r,\hat{x}) \geq \hat{\vartheta}(r,\hat{x})$ *we have*

$$\alpha_i(x,\vartheta) \left\{ \begin{matrix} \geq \\ \leq \end{matrix} \right\} \hat{\alpha}_i(\hat{x},\hat{\vartheta}) \text{ if } \begin{cases} x_i = w \\ \hat{x}_i = r. \end{cases}$$

Assume the initial conditions $(X(0), \Theta(0)) = (x(0), \vartheta(0))$ *and* $(\widehat{X}(0), \widehat{\Theta}(0)) = (\hat{x}(0), \hat{\vartheta}(0))$ *such that* $x(0) \leq y(0)$, $\vartheta(w, x(0)) \leq \hat{\vartheta}(w, x(0))$, $\vartheta(r, \hat{x}(0)) \geq \hat{\vartheta}(r, \hat{x}(0))$. *Then* $X \leq_d \widehat{X}$.

REMARK. Indeed Shaked and Shantikumar (1988) show more. Define on $\mathcal{X}^n \times \mathbb{R}_+^n$ the following partial order: $(x,\vartheta) \leq (\hat{x},\hat{\vartheta})$ if $x \leq \hat{x}$ and $\vartheta(w,x) \leq \hat{\vartheta}(w,x)$ and $\vartheta(r,\hat{x}) \geq \hat{\vartheta}(r,\hat{x})$. Then $(X,\Theta) \leq_d (\widehat{X},\widehat{\Theta})$.

EXAMPLE. (Same assumptions as in the last example and additional $n = m = \sigma$: no spares and no queueing.) Here the conditions concerning α_i, $\hat{\alpha}_i$ in Theorem 2 are equivalent to

$$\mu(t) \geq \hat{\mu}(\hat{t}) \text{ for all } \hat{t} \geq t \geq 0$$
$$\lambda(t) \leq \hat{\lambda}(\hat{t}) \text{ for all } t \geq \hat{t} \geq 0.$$

Indeed these conditions are weaker than those in the last example. (But e.g. in the important case of IFR lifetimes and repair times the conditions are equivalent.)

REMARK. If one considers the standard reliability system (spares and queueing allowed) with the assumption $F_1 = \cdots = F_n$ and $G_1 = \cdots = G_n$, it is not necessary to distinguish between the units. So it is convenient to use the performance process $X(t) :=$ number of units in up state at time t. Again (X,Θ) is a Markov process. Using the ideas of the proof of Theorem 2 one can see that if the same conditions hold as in the last example and if both systems start with fresh units, then $X \leq_d \widehat{X}$.

REMARK. Apparently (see e.g. the proof of Theorem 2 and the papers mentioned after the shock model in Section 1.1) a very frequently used tool to get comparisons of performance process X and $\widehat{X}$ with continuous time, is to find a point process $0 = S_0 < S_1 < S_2 \cdots$ such that $X(S_n) \leq_d \widehat{X}(S_n)$ implies $(X(t), S_n \leq t < S_{n+1}) \leq_d (\widehat{X}(t), S_n \leq t < S_{n+1})$.

Franken, Kirstein and Streller (1984) call $(X(t), \Theta(t), S_n, t \geq 0, n \in \mathbb{N})$ a process with an embedded point process (PEP). Then, according to the results of Strassen (1965) and Kamae, Krengel and O'Brien (1977), to get the result $X \leq_d \widehat{X}$ one only has to establish the implication if $X(S_n) \leq_d \widehat{X}(S_n)$, then $X(S_{n+1}) \leq_d \widehat{X}(S_{n+1})$. Assumptions that guarantee the implication are well known if $(X(S_n))$ and $(\widehat{X}(S_n))$ are (conditional) Markov chains.

In many cases it is convenient for mathematical reasons to take a Poission process $(S_n, n \in \mathbb{N})$. For a reliability process (X, Θ) such that $q_x(\vartheta) \leq \lambda = \text{const.}$ for all $x \in \mathcal{X}^n$, $\vartheta \in \mathbb{R}_+^n$ the following expansion holds

$$\begin{aligned}
&\Pr(X(T_s) = y,\ T_s \leq t + s \mid X(s) = x,\ \Theta(s) = \vartheta) \\
&= \int_0^t q_{xy}(\vartheta + ue) \exp\left(-\int_0^u q + x(\vartheta + ve)dv\right) du \\
&= \int_0^t q_{xy}(\vartheta + ue) \exp\left(\lambda u \int_0^u (1 - \lambda^{-1} q_x(\vartheta + ve))dv\right) e^{-\lambda u} du \\
&= \sum_{n=1}^{\infty} \int_0^t \left[\lambda^{-1} q_{xy}(\vartheta + ue) \prod_{k=1}^{n-1} \int_0^u (1 - \lambda^{-1} q_x(\vartheta + v_k e)) \frac{1}{u} du_k\right] \lambda \frac{(\lambda u)^{n-1}}{(n-1)!} du.
\end{aligned}$$

As we will see, the last formula shows that (X, Θ) can be constructed as follows. Transitions of X can occur only at the epochs of a Poisson process $(S_n, n \in \mathbb{N})$ with parameter λ. If $X(s) = x$ and the next epoch of (S_n) after s is at $s + v$, then let a transition $X(s) = x \to X(s + v) = y$ take place with conditional probability $\lambda^{-1} q_{xy}(\vartheta + ve)$ and let no transition take place with conditional probability $1 - \lambda^{-1} q_{xy}(\vartheta + ve)$. If there is a transition, take $\Theta(s + v) = \psi(x, y, \vartheta, v)$, otherwise take $\Theta(s + v) = \vartheta + ve$.

Then the term in the square brackets represents the conditional probability of the event that at the first $n - 1$ epochs of (S_n) after s there is no transition and at the nth epoch (at the time $s + u$) a transition $x \to y$ takes place, given that the nth epoch of (S_n) after s takes place at $s + u$. (If in a Poisson process the nth epoch after s takes place at $s + u$, the first $n - 1$ epochs are conditionally independent and uniformly distributed on $(s, s + u)$.) The term after the brackets is the probability (density) that the nth epoch of (S_n) after s takes place at $s + u$. So the last representation of $\Pr(X(T_s) = y,\ T_s \leq s + t \mid X(s) = x,\ \Theta(s) = \vartheta)$ shows that conditionally upon (S_n) and Θ the process $(X(S_n), n \in \mathbb{N}_0)$ is a Markov chain with transition probabilities

$$\begin{aligned}
&\Pr(X(S_{n+1}) = y \mid X(S_n) = x,\ \Theta(S_n) = \vartheta,\ S_{n+1} - S_n = v) \\
&\qquad = \begin{cases} \lambda^{-1} q_{xy}(\vartheta + ve) & \text{and } \Theta(S_{n+1}) + \psi(x, y, \vartheta, v) \text{ if } y \neq x \\ (1 - \lambda^{-1} q_x(\vartheta + ve)) & \text{and } \Theta(S_{n+1}) = \vartheta + ve \text{ if } y = x. \end{cases}
\end{aligned}$$

Now the well known conditions which establish the implication $X(S_n) \leq_d \widehat{X}(\widehat{S}_n) \Rightarrow X(S_{n+1}) \leq_d \widehat{X}(\widehat{S}_{n+1})$ are used to get conditions (2), (3) in Theorem 1 (where one has taken into account the special form of the transition matrix and the fact that the partial order is used: $(x, \vartheta) \leq (\hat{x}, \hat{\vartheta})$ if $x \leq \hat{x}$).

Of course it is somewhat artificial to take a Poisson process (S_n) to get the embedded Markov chains $(X(S_n))$, $(\widehat{X}(S_n))$ because the Poisson process

has only a mathematical and not a "physical" meaning with respect to the processes X, $\widehat{X}$. To get another embedded point process, one could couple the two independent processes (X, Θ) and $(\widehat{X}, \widehat{\Theta})$ and take (S_n) as the joint process of the transition epochs $T_1 T_2, \cdots$ and $\widehat{T}_1 \widehat{T}_2, \cdots$ of (X, Θ), $(\widehat{X}, \widehat{\Theta})$ due to the construction given before Theorem 1. We have tried this embedded point process for two alternating renewal processes X, $\widehat{X}$ and indeed obtained weaker (but more complicated) conditions.

4. Optimal Maintenance. The standard models can be found e.g. in Barlow and Proschan (1975). Many problems in this field have the structure of a stochastic dynamic programming problem (see e.g. Derman (1970)). As an example we consider a Markov decision process with discrete time.

EXAMPLE. Optimal Replacement. Let $\mathcal{S}_n$ be a system consisting of n two state components. Here the state space is $\mathcal{X} : \{0,1\}^n$ and let $e = (1, \cdots, 1)$ denote the best state. ($x \leq y$ is interpreted as y is better than x.) The state of the system is known at times $t = 0, 1, 2, \cdots$. If the system is in state x at time t, then one has the option to replace some of the components. Let z be the state after replacement. Usually only $z \in A_x$ is allowed, where A_x is a given subset of $\{z \in \mathcal{X}^n : z \geq x\}$. A replacement gives rise to replacement costs $r(x, z) \geq 0$, where $r(x, x) = 0$.

Moreover the system induces a cost $c(z)$ per period if it starts at t in state z (after replacement). Given that the system is in state z at time t (after replacement), then let $P(z, y)$ be the probability that the system is in state y at time $t+1$ (before replacement), where $P(z, y))$ is a stochastic matrix. Then the minimal expected discounted total costs $v(x)$ given that $\mathcal{S}$ starts in state x, satisfy the optimality equation

$$v(x) = \min_{z \in A_x} \left[r(x, z) + c(z) + \alpha \sum_y P(z, y) v(y) \right],$$

where α is the discount factor.

Under reasonable conditions on $r(x, z)$ and $c(z)$ one gets that $v(x)$ is decreasing in x if

$$P(x, \cdot) \leq_d P(y, \cdot) \text{ for all } x \leq y,$$

where $P(x, \cdot)$ denotes the distribution $B \to \sum_{z \in B} P(x, z)$, $B \subset \mathcal{X}^n$.

In this case one has a chance to get good structures for the optimal replacement rules (see e.g. Ross (1979), Gaede (1983), (1984), Ohnishi, Kawai and Mine (1986), Tjims and van der Schouten (1984), and Hinderer (1982)).

If one has not full information about the state $X(t)$ at time t, then it is rather complicated to get v and the optimal replacement rule (see e.g. Schneeberger (1988)). Sometimes in this case a martingale representation of the two

state process $Z(t) = \varphi(X_1(t), \cdots, X_n(t)) = \varphi(X(t))$ is useful, where φ is the structure function of a system with n components described by the processes $X_1(t), \cdots, X_n(t)$. A typical assumption is that only the state $Z(t)$ and some of the components can be observed. In this case let $X(t)$ be partitioned as $X(t) = (X_0(t), X_N(t))$ where $X_0(t)$ represents the observed components and let us assume that we have a time t the information

$$\mathcal{F}_t = \sigma - \text{algebra generated by } (X_0(s), Z(s), \quad 0 \leq s \leq t).$$

Let μ_t be a process s.t. $P(Z(t+dt) - Z(t) = -1 \mid \mathcal{F}_t) = \mu_t dt + o(dt)$. Then we have

$$Z(t) = Z(0) = \int_0^t \mu_s ds + M(t),$$

where $M(t)$ is a $\mathcal{F}_t$-martingale. This is used e.g. by Jensen (1986) to obtain optimal replacement rules.

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