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# SECOND ORDER BONFERRONI-TYPE, PRODUCT-TYPE AND SETWISE PROBABILITY INEQUALITIES

BY HENRY W. BLOCK*, TIMOTHY COSTIGAN, AND ALLAN R. SAMPSON*

*University of Pittsburgh, The Ohio State University, and University of Pittsburgh*

This review paper considers a variety of lower bounds to $\text{Prob}(X_1 \le c, \cdots, X_n \le c)$ involving one and two-dimensional marginal probabilities. Some of these bounds, e.g., Bonferroni-type, do not require conditions on the random variables. Others of these inequalities, e.g., product-type, require positive dependence conditions on $X_1, \cdots, X_n$ for the inequalities to hold. Because all of the two-dimensional bounds depend on the labeling order of the random variables, various permutation-optimized versions of the bounds are described. Relationships among the various bounds are also considered.

**1. Introduction and Overview.** In many statistical applications such as moving window detection and the calculation of the expected stopping time and its variance in sequential analysis, one wishes to find a constant $c$ such that

$$P(c) = \text{Prob}(X_1 \le c, \cdots, X_n \le c) = 1 - \alpha, \tag{1.1}$$

where $\boldsymbol{X} = (X_1, \cdots, X_n)$ is a vector of dependent random variables and $0 \le \alpha < 1$ is given. The constant $c$ which satisfies (1.1) is called the *multivariate $\alpha$-level critical value.* Exact calculation of $c$ can entail extensive computation. Sometimes the calculation of $c$ is intractable (even with the use of a computer) because it requires iteration and high dimensional multivariate numerical integration. This is particularly true in the types of applications noted above. Due to this dilemma, which is known as the "curse of dimensionality," there has been considerable interest in finding easily computable approximations to $P(c)$ which can be used in an iterative search procedure to find an approximation to the multivariate $\alpha$-level critical value.

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While for some applications any good approximation to $P(c)$ is satisfactory, in order to control the experimentwise error rate in multiple hypothesis testing applications, the approximation used in the iterative search scheme should be a lower bound to $P(c)$. With a lower bound approximation, the approximate multivariate $\alpha$-level critical value is guaranteed to be conservative, i.e., the experimentwise error rate will be less than or equal to $\alpha$. For this reason we concern ourselves in this review paper on finding lower bounds to $P(c)$. Moreover to reduce computational complexity, we want these lower bounds to depend on "low" dimension marginal distributions of $\boldsymbol{X}$. In general, we say that an approximation is of order $i$ if it depends on marginal probabilities of $\boldsymbol{X}$ of order $i$ or less.

Two popular first-order approximations are based on Bonferroni's inequality and a first-order product-type inequality (Slepian (1962), Sidak (1967)). The Bonferroni inequality always gives a conservative estimate to the multivariate $\alpha$-level critical value, while we can only be sure that the first-order product bound is conservative if $\boldsymbol{X}$ satisfies positive dependence conditions (usually, positive lower orthant dependent, (PLOD), see Dykstra *et al.*, (1973) and the Appendix). However, if these PLOD conditions are satisfied, then the product-type bound is sharper than the Bonferroni bound (e.g., see Fuchs and Sampson (1987)). In particular, if $\boldsymbol{X}$ is distributed according $N(\mathbf{0}, \boldsymbol{\Sigma})$, then $|X_1|, \cdots |X_n|$ are PLOD (Sidak (1967)). Consequently, the product bound should always be used rather than the Bonferroni bound in two-sided significance testing between multivariate normal distributions. Applications of this result include Games (1977), Rao, Marsh and Winwood (1985), and Rao and Marsh (1987). An appealing feature of the Bonferroni and first-order product-type bounds is that other than variances, one does not need to know the covariance structure of $\boldsymbol{X}$.

Regardless of the covariance structure, first order bounds are known to be overly conservative. Holm (1979) developed a sequentially rejective procedure to be used with the Bonferroni bound which identifies individual hypotheses to be rejected. This procedure reduces the degree of conservatism of the Bonferroni method yet controls the experimentwise error rate under the complete and partial null hypotheses when the test statistics are independent. Tabulations aid in the implementation of the Holm sequentially rejective procedure. Recent modifications of the procedure are due to Shaffer (1986), Simes (1986) and Holland and Copenhaver (1987). A review of these sequentially rejective procedures is given by Hochberg and Tamhane (1987).

Several authors have shown that first-order approximations are unacceptably conservative under known strong positive dependence (Glaz and Johnson (1984), Schwager (1984) and Stoline (1983)). Recent approaches utilize second-order (and higher-order) bounds which can exploit the strong positive dependence structure, if it is present, and are consequently much sharper lower bounds than the first-order bounds. One of the main focuses of this paper is the review and comparison of these bounds.

We note that a simple method of improving Bonferroni bounds is based on the well known principle of inclusion and exclusion (see Feller, 1968, Chapter IV.5). This principle states that by subtracting off all $\binom{n}{2}$-bivariate marginal probabilities from the usual Bonferroni bound, an upper bound to $P(c)$ is obtained, then by adding all $\binom{n}{3}$ trivariate marginal probabilities, a lower bound is obtained, etc. This sequence of alternating lower and upper bounds are applicable regardless of the underlying distributions. See Barlow and Proschan (1981) for applications of these higher-order Bonferroni bounds to reliability analysis in engineering. Tong (1980) offers a good review of Bonferonni and other "distribution free" inequalities such as Chebychev-type inequalities and Kolmogorov-type inequalities. For recent developments of bounds based on the inclusion-exclusion criteria, also see Hoppe (1985), Rescei and Seneta (1987) and Seneta (1988).

There are two reasons for further restricting our attention to second-order bounds. First of all, the computationally feasible second-order bounds are often surprisingly accurate, representing a substantial improvement over first-order bounds. (See Glaz and Johnson (1984), Worsley (1982) and Bauer and Hackl (1985)). Secondly, the relationships among various bounds are more easily understood by restricting our attention to second-order bounds, rather than higher-order bounds. (However, some of the results presented are extendable to third-order bounds. See Glaz and Johnson (1984), Hoover (1990 a,b) and Glaz (1990).)

We employ the notation $B_i = \{X_i \leq c\}$ and $A_i = \{X_i > c\}$, for $i = 1, \cdots, n$, so that

$$P(c) = P\left(\bigcap_{i=1}^{n} B_i\right) = P\left(\left(\bigcup_{i=1}^{n} A_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right),$$

where $P(A)$ denotes the probability of a set $A$. Consequently, a lower bound to $P(c)$ is attained by obtaining an upper bound to $P\left(\bigcup_{i=1}^{n} A_i\right)$. Second-order Bonferroni-type lower bounds to $P(c)$ are obtained in this manner. However, we often express these bounds as lower bounds to $P\left(\bigcap_{i=1}^{n} B_i\right)$ in this paper in order to compare them to second-order product-type bounds which are obtained in this fashion. For convenience throughout we assume $0 < P(A_i) < 1$.

While

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j),$$

Hunter (1976) used techniques of graph theory to show that an upper bound to $P\left(\bigcup_{i=1}^{n} A_i\right)$ is obtained if $n-1$ appropriately chosen bivariate probabilities are subtracted from $\sum_{i=1}^{n} P(A_i)$. We refer to these bounds as second-order Bonferroni-type bounds.

These second-order Bonferroni-type bounds can be readily compared to second-order product-type bounds which are generalizations of the Slepian-Sidak bound. The same relationships hold for these second-order bounds as the known relationship between the first-order product and Bonferroni bounds. First, positive dependence conditions must be verified to insure that product-type approximations are bounds, unlike the Bonferroni-type bounds. Second, the apparent greater generality of the Bonferroni-type bounds is often vacuous because Bonferroni-type bounds often degenerate, i.e., yield bounds of the form $P(c) \geq k$, where $k < 0$. In contrast product-type bounds never degenerate. (This is particularly relevant when obtaining bounds of high dimensional probabilities in which marginal probabilities are not small.) Third, if the bound conditions of the product-type approximations are satisfied then they are sharper bounds than the corresponding Bonferroni-type bounds. Fourth, product-type bounds have an advantage in that they are exact under independence, unlike Bonferroni-type bounds. The reader is referred to Glaz (1990), Hoover (1990b), and Kenyon (1986, 1987, and 1988) for a numerical comparison of higher-order product-type and Bonferroni-type bounds.

Further development of the Bonferroni-type bounds has been done by Worsley (1982) and Hoover (1990a). Bauer and Hackl (1985) and Worsley (1982) consider the application of these bounds to a wide range of multiple testing problems. Specific applications of these bounds in the applied literature include flexible sequential monitoring schemes (Bauer 1986), multiple comparisons (Stoline 1983), stepwise regression (Bjornstadt and Butler 1988) and multiple forecasts in ARIMA models (Ravishanker *et al.*, 1987).

Glaz and Johnson (1984) and Block, Costigan, and Sampson (1988 a,b) (alternately BCS), develop higher order product-type bounds. Specific applications of these bounds in the applied literature include approximating the operating characteristics of sequential monitoring schemes (Glaz and Johnson (1986) and Kenyon (1988)), multiple comparisons (Kenyon (1986)), moving sums (Glaz and Johnson (1988)), group sequential analysis of litter matched data (Milhako (1987)), and ordered multivariate exponential distributions (Sarkar and Smith (1986)).

Chhetry, Kimeldorf and Sampson (1989), recently developed second and higher-order setwise bounds. These bounds are a different type of extension of the first-order product bounds than the product-type bounds considered by Glaz and Johnson. In fact, setwise bounds can be viewed as a compromise between Bonferroni-type bounds and product-type bounds in the following sense. Setwise approximations require positive dependence conditions (unlike Bonferroni-type bounds), but these conditions are weaker than the required product-type positive dependence bound conditions. Consequently, setwise bounds are applicable in more general situations than product-type bounds. However, Glaz and Johnson product-type bounds yield less conservative estimates than setwise bounds under weak positive dependence conditions (e.g., pairwise PQD (Lehmann 1966) and the Appendix) of certain bivariate subvectors. In particular, product-type bounds are sharper than setwise bounds whenever product-type bound conditions are satisfied. Advantages which both setwise bounds and product-type bounds share over the Bonferroni-type bounds are their nondegeneracy and exactness under independence.

This paper reviews and compares second-order product-type bounds, Bonferroni-type bounds and setwise bounds. Bound conditions are developed, relative efficiences of the bounds are defined and situations where the various bounds are most appropriate are listed.

A discussion of other uses of probability bounds concludes this paper.

**2. Standard Second-Order Bonferroni-Type Bounds.** Writing $\bigcup_{i=1}^{n} A_i$ as the disjoint union

$$A_1 \cup (A_2 \cap A_1^c) \cup \cdots \cup (A_n \cap A_1^c \cap \cdots \cap A_{n-1}^c),$$

we have

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= P(A_1) + P(A_2 \cap A_1^c) + \cdots + P(A_n \cap A_1^c \cap \cdots \cap A_{n-1}^c) \\ &\leq P(A_1) + \sum_{i=2}^{n} P(A_i \cap A_{i-1}^c) \end{aligned}$$

which yields

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i=2}^{n} P(A_i \cap A_{i-1}). \tag{2.1}$$

An equivalent way of writing this bound is

$$P(c) \geq \sum_{i=2}^{n} P(X_{i-1} \leq c, X_i \leq c) - \sum_{i=2}^{n-1} P(X_i \leq c) \equiv b_2(c). \tag{2.2}$$

We refer to expressions (2.1) and (2.2) as the standard second-order Bonferroni-type bound. As will be shown in the following sections, this bound is optimal among other types of second-order Bonferroni-type bounds in most applications involving repeated significance testing.

**3. Standard Second-Order Product-Type Bounds.** It is easy to see that

$$P(c) = P(X_1 \le c) \prod_{i=2}^{n} P(X_i \le c \mid X_1 \le c, \cdots, X_{i-1} \le c).$$

By conditioning on the immediate predecessor rather than all predecessors one obtains the conditional approximation

$$\begin{aligned} P(c) &\approx P(X_1 \le c) \prod_{i=2}^{n} P(X_i \le c \mid X_{i-1} \le c) \\ &= \frac{\prod_{i=2}^{n} P(X_{i-1} \le c, X_i \le c)}{\prod_{i=2}^{n-1} P(X_i \le c)} \equiv \beta_2(c). \end{aligned} \tag{3.1}$$

We refer to (3.1) as the standard second-order product-type approximation and view this approximation as a second-order generalization of Slepian-Sidak bounds, which we denote by

$$\beta_1(c) = \prod_{i=1}^{n} P(X_i \le c).$$

Glaz and Johnson (1984) introduce these second (and higher order) product-type bounds $\beta_2(c)$ (and $\beta_3, \cdots, \beta_{n-1}$) (see Glaz and Johnson for the definitions of $\beta_3, \cdots, \beta_{n-1}$). They prove the following theorem.

THEOREM 1. *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be an arbitrary random vector. If $\boldsymbol{X}$ has a density which is multivariate totally positive of order 2 ($MTP_2$) (see Karlin and Rinott (1980) and the Appendix), then*

$$P(c) \ge \beta_{n-1}(c) \ge \cdots \ge \beta_2(c) \ge \beta_1(c).$$

For $\boldsymbol{X}$ distributed according to $N(\mathbf{0}, \Sigma)$, $MTP_2$ is equivalent to either of the following equivalent conditions

(i) $\sigma^{ij} \le 0$ for $1 \le i < j \le n$, where $\Sigma^{-1} = (\sigma^{ij})$ (Barlow and Proschan (1981))

(ii) $\sigma_{ij\cdot} \ge 0$ for $1 \le i < j \le n$, where $\sigma_{ij\cdot}$ is the partial covariance of $X_i$ and $X_j$ given the other $n-2$ components (Bolviken (1982) and Karlin and Rinott (1983)).

BCS (1988a) obtain substantially weaker positive dependence conditions based on conditional setwise dependence which guarantee that $P(c) \geq \beta_2 \geq \beta_1$. These conditions do not require that the distribution of $\boldsymbol{X}$ be $MTP_2$. BCS (1988a) show, for example, that the multivariate exponential distribution of Marshall and Olkin (1967) satisfies their conditions for upper orthant probabilities but is not $MTP_2$.

For the multivariate normal case the BCS conditions reduce to the following.

THEOREM 2. *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be distributed according to $N(0, \Sigma)$ with $\sigma_{ii} = 1$ for $i = 1, \cdots, n$. Then $P(c) \geq \beta_2(c) \geq \beta_1(c)$ if the following conditions are satisfied:*

$$\text{(i) } \Sigma \geq 0 \tag{3.2a}$$

$$\text{(ii) } \sigma_{ij \cdot j-1} \geq 0, \text{ for } i < j-1, \ j = 3, \cdots, n, \tag{3.2b}$$

*where $A = \{a_{ij}\} \geq 0$ denotes $a_{ij} \geq 0, 1 \leq i, j \leq n$ and $\sigma_{ij \cdot j-1}$ is the conditional covariance of $X_i$ and $X_j$ given $X_{j-1}$.*

We note that condition (3.2a) alone insures that $P(c) \geq \beta_1(c)$. Also the conditions of Theorem 2 are weaker than the $MTP_2$ condition of Theorem 1 (Karlin and Rinott (1983)). For instance, (3.2a) and (3.2b) are satisfied when group sequential analysis is applied for bivariate normal response, although the test statistic is not $MTP_2$ (see BCS (1988a)).

The conditions of Theorem 2 are sufficient but not necessary for $P(c) \geq \beta_2(c) \geq \beta_1(c)$. BCS (1988b) employ techniques of graph theory to develop other conditions. Also Glaz (1990) computationally verifies that $P(c) \geq \beta_2(c) \geq \beta_1(c)$, for moving averages of order 10, yet none of the sufficient bound condition of BCS (1988a, 1988b) are satisfied.

**4. Second-Order Setwise Bounds.** Chhetry, Kimeldorf and Sampson (1989) develop second-order setwise approximations of the following form:

$$\tau_2(c) \equiv \begin{cases} \prod_{i=1}^{m} P(X_{2i-1} \leq c, X_{2i} \leq c), & \text{if } n = 2m \qquad (4.1a) \\ \\ P(X_1 \leq c) \prod_{i=1}^{m} P(X_{2i} \leq c, X_{2i+1} \leq c), & \text{if } n = 2m+1. \ (4.1b) \end{cases}$$

For simplicity we restrict our attention to the case $n = 2m$.

Chhetry, Kimeldorf and Sampson introduce a concept called setwise PLOD and show that if $(X_1, X_2), (X_3, X_4), \cdots, (X_{2m-1}, X_{2m})$ are setwise PLOD then

$$P(c) \geq \tau_2(c).$$

For the multivariate normal case this reduces to the following theorem.

THEOREM 3. *Let $X$ be distributed according to $N(\mathbf{0}, \Sigma)$. If $\sigma_{ij} \geq 0$ for $|j-i| \geq 2$, and $\sigma_{2i,2i+1} = 0$ for $i = 1, \cdots, m-1$ then $P(c) \geq \tau_2(c)$.*

Note that one may have $\sigma_{2i-1,2i} < 0$ for $i = 1, \cdots, m$ and still have $P(c) \geq \tau_2(c)$.

**5. Relationships Among Standard Second-Order Bonferroni, Product and Setwise Bounds.** We note the following important properties:

(1) one does not have to verify bound conditions for $b_2$ to hold, unlike $\beta_2$ and $\tau_2$;

(2) the bounds $\beta_2$ and $\tau_2$ are never degenerate, unlike $b_2$;

(3) the bounds $\beta_2$ and $\tau_2$ are exact under independence, unlike $b_2$.

The following theorem is proved by BCS (1988b), Hoover (1990b), and Glaz (1990).

THEOREM 4. $\beta_2(c) \geq b_2(c)$.

It follows from Theorem 4, that when second-order product-type bounds are applicable, (i.e., when Theorem 1 and 2 are satisfied so that $P(c) \geq \beta_2(c) \geq \beta_1(c)$), they are superior to second-order Bonferroni-type bounds.

Hoover (1990b) defines $\frac{\beta_2(c)}{b_2(c)}$ as a measure of efficiency of $\beta_2$ relative to $b_2$. The ratio is only slightly larger than 1, for moderate $n$, large $c$ and strong positive dependence. However, if $n$ is large, $\beta_2$ can be considerably sharper than $b_2$, even for large $c$. This is true in a moving window detection application considered by Glaz (1988) in which $b_2$ often degenerates.

We now consider the relationship between setwise and product-type bounds. We first note that both of these approximations can be viewed as alternative generalizations of Slepian-Sidak bounds, and both behave similarly in terms of the preceding properties (1), (2), and (3). BCS (1988b) show the following.

THEOREM 5. *Let $X = (X_1, \cdots, X_{2m})$ be an arbitrary random vector. If $(X_{2i}, X_{2i+1}))$ is PQD, for $i = 1, \cdots, m-1$, then $\beta_2(c) \geq \tau_2(c)$. If $(X_{2i}, X_{2i+1})$, $i = 1, \cdots, m$, are NQD (see Lehmann (1966) and the Appendix), then $\tau_2(c) \geq \beta_2(c)$.*

In the multivariate normal case, Theorem 5 reduces to the following:

THEOREM 6. *Let $X = (X_1, \cdots, X_{2m})$ be distributed according to $N(0, \Sigma)$. Then $\beta_2(c) \geq (\leq) \tau_2(c)$ provided $\sigma_{2i,2i+1} \geq (\leq) 0$, $i = 1, \cdots, m-1$.*

We note that the positive dependence condition of Theorems 5 and 6 are relatively weak. Clearly, they are satisfied whenever $X$ is $MTP_2$ or the bound conditions for $P(c) \geq \beta_2(c) \geq \beta_1(c)$ given in Theorem 2 are satisfied. When these strong positive dependence conditions are satisfied, then $\beta_2(c)$

is sometimes much sharper than $\tau_2(c)$. This can be seen by examining $\frac{\beta_2(c)}{\tau_2(c)}$ which measures the efficiency of $\beta_2$ relative to $\tau_2$. Specifically,

$$\frac{\beta_2(c)}{\tau_2(c)} = \prod_{i=1}^{m-1} \left[ \frac{P(X_{2i} \le c, X_{2i+1} \le c)}{P(X_{2i} \le c)P(X_{2i+1} \le c)} \right]. \tag{5.1}$$

When $P(X_{2i} \le c, X_{2i+1} \le c) >> P(X_{2i} \le c)P(X_{2i+1} \le c)$, the ratio in (5.1) will be much larger than 1, indicating that the product-type bound is much sharper than the setwise bound. Of course, under the corresponding negative dependence conditions the ratio in (5.1) is less than 1.

BCS (1988a) demonstrated that setwise positive dependence concepts play a key role in obtaining weak conditions for $P(c) \ge \beta_2(c) \ge \beta_1(c)$. In fact, we believe that higher order setwise and product-type bounds can be combined to achieve the most accurate bounds in applications involving the multivariate normal with empirical covariance matrices.

Neither the setwise nor the Bonferroni-type bounds dominate the other. It is apparent from the preceding properties (2) and (3) that $\tau_2 \ge b_2$ for large $n$ and that $\tau_2 \ge b_2$ when the components of $\boldsymbol{X}$ are independent. On the other hand, under certain notions of strong positive dependence $b_2 \ge \tau_2$, because $\tau_2$ does not fully exploit the dependence structure.

**6. General Second-Order Bonferroni-Type and Product-Type Bounds.** Hunter (1976) and Worsley (1982) further developed second-order Bonferroni-type bounds, by defining a bound for each spanning tree $T$ corresponding to the "bivariate probability structure." We denote these bounds by $b_2(T)$ and describe them below. See Worsley (1982) for some interesting applications of these bounds. See both of the preceding papers for suitable background on the use of graph and spanning trees for this type of problem.

Corresponding to the sets $A_1, \cdots, A_n$, we identify the complete $n$-vertices graph, i.e., all the vertices are connected by edges, and with vertex $i$ corresponding to set $A_i$. A spanning tree, $T$, is a subgraph of this complete graph satisfying: (a) no cycles are present in the graph, and (b) there is a path from any vertex to any other. To develop the bounds $b_2(T)$, Worsley (1982) uses the notion of an increasing representation of a spanning tree. For every spanning tree, $T$, there exists at least one permutation $\boldsymbol{P} = (P_1, \cdots, P_n)$ of $(1, \cdots, n)$ such that

$$T = \{(P_i^*, P_i), i = 2, \cdots, n\} \tag{6.1}$$

where $P_i^* \in \{P_1, \cdots, P_{i-1}\}$, and $T$ is described by the set of connected vertices, i.e., edges. In this case call the permutation $\boldsymbol{P}$ an increasing representation and (6.1) an *increasing representation* for $T$.

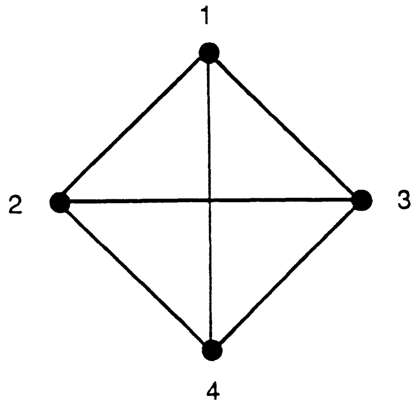


**Figure 6.1**
Complete Graph for $A_1, A_2, A_3$, and $A_4$

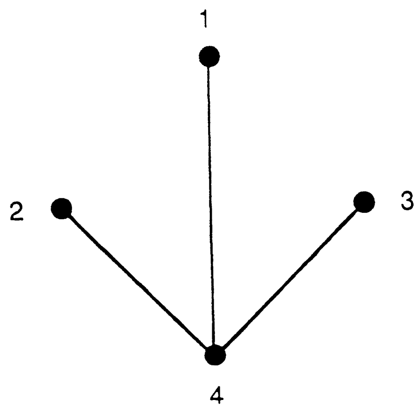


**Figure 6.2**
A Spanning Tree for Graph in Figure 6.1

For example, for sets $A_1$, $A_2$, $A_3$, and $A_4$, Figure 6.1 gives the corresponding complete graph and Figure 6.2 gives a spanning tree for this graph.

Two possible increasing representations for the spanning tree $T = \{(2,4), (3,4), (1,4)\}$ are the permutations $(4,2,3,1)$ and $(4,1,2,3)$. However, $(1,3,2,4)$ is not an increasing representation because no predecessors of 3 are connected to 3 by edges in this particular spanning tree.

Using the increasing representation and arguments similar to those employed in Section 2, it follows that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i=2}^{n} P(A_{P_i} \cap A_{P_i^*}).$$

This bound is independent of the specific increasing representation and can be expressed as

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{(i,j)\in T} P(A_i \cap A_j). \tag{6.2}$$

Equation (6.2) is known as Hunter's inequality (1976). An equivalent representation of (6.2) is

$$P(c) \geq \sum_{(i,j)\in T} P(X_i \leq c, X_j \leq c) - \sum_{i=1}^{n}(d_i - 1)P(X_i \leq c) \equiv b_2(T), \tag{6.3}$$

where $d_i$ is the degree of vertex $i$ within the tree (i.e., the number of vertices connected to $A_i$ by edges in $T$).

For the string-like tree $T_1 = \{(i-1, i)i = 2, \cdots, n\}$, (6.2) and (6.3) reduce to (2.1) and (2.2), the standard second-order Bonferroni-type bound. If one substitutes the star-like tree $T^i = \{(i,j), j = 1, \cdots, n; j \neq i\}$ in (6.2), one obtains

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{j=1}^{n} P(A_j) - \sum_{\substack{j=1 \\ j\neq i}}^{n} p(A_i \cap A_j),$$

which is an inequality due to Kounias (1968). Averaging over $i$ one obtains a bound due to Kwerel (1975).

BCS (1988b) use graph theory techniques similar to those used by Hunter (1976) and Worsley (1982) to obtain general product-type bounds corresponding to each spanning tree, $T$, of the bivariate probability structure, denoted by $\beta_2(T)$. Modifying the Glaz-Johnson technique, BCS estimate $P(X_{P_i} \leq c \mid X_{P_1} \leq c, \cdots, X_{P_{i-1}} \leq c)$ by $P(X_{P_i} \leq c \mid X_{P_i^*} \leq c)$. One obtains the bound

$$\frac{\prod_{i=2}^{n} P(X_{P_i^*} \leq c, X_{P_i} \leq c)}{\prod_{i=3}^{n} P(X_{P_i^*} \leq c)}.$$

BCS (1988b) rewrite this expression as

$$\frac{\prod_{(i,j)\in T} P(X_i \leq c, X_j \leq c)}{\prod_{j=1}^{n} P(X_i \leq c)^{d_i - 1}} = \beta_2(T),$$

and call $\beta_2(T)$ the second order product-type bound corresponding to $T$.

For the string like tree $\beta_2(T)$ reduces to $\beta_2(c)$ defined in (3.1).

For the star-like tree $T^i = \{(i,j), j = 1, \cdots, n; j \neq i\}$, we have

$$\beta_2(T^i) = \frac{\prod_{\substack{j=1 \\ j\neq i}}^{n} P(X_i \leq c, X_j \leq c)}{P(X_i \leq c)^{n-2}},$$

which is the product-type analogue of Kounias' (1968) bound.

General Bonferroni-type bounds $b_2(T)$ are always lower bounds to $P(c)$, while positive dependence conditions must be satisfied for $P(c) \geq \beta_2(T)$.

THEOREM 7. *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random vector. If $\boldsymbol{X}$ has a density which is $MTP_2$ then*

$$P(c) \geq \beta_2(T) \geq \beta_1(c).$$

Positive dependence conditions of BCS (1988a) can be modified to obtain weaker positive dependence conditions which guarantee that $P(c) \geq \beta_2(T) \geq \beta_1(c)$. We now state these conditions for the multivariate normal case.

THEOREM 8. *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be distributed according to $N(0, \Sigma)$. Let $T$ be a spanning tree for the probability graph with an increasing representation $T = \{(P_i^*, P_i), i = 2, \cdots, n\}$. Then*

$$P(c) \geq \beta_2(T) \geq \beta_1(c),$$

*provided the following two conditions are satisfied*

(i) $\Sigma \geq 0$ (6.4a)

(ii) $\mathrm{Cov}(X_{P_h}, X_{P_i} \mid X_{P_i^*}) \geq 0$, $i = 3, \cdots, n$, $h < i$, $h \neq i^*$. (6.4b)

Again, these conditions are substantially weaker than the $MTP_2$ condition. For the string-like tree with the usual increasing representation $T_1 = \{(i-1, i), i = 2, \cdots, n\}$, the conditions in (6.4) reduce to (3.2).

Interestingly, different increasing representations for the same tree $T$ can yield different conditions of the form (6.4). BCS (1988b) exhibit all increasing representations of the string like tree $T_1$, to obtain a variety of bound conditions for $P(c) \geq \beta_2(c) \geq \beta_1(c)$ other than those presented in (3.2).

For the star-like tree $T^i = \{(i, j)\ , j = 1, \cdots, n, j \neq i\}$, bound conditions corresponding to all increasing representations reduce to the following two conditions:

(i) $\Sigma \geq 0$

(ii) $\sigma_{hj \cdot i} \geq 0$, for $1 \leq h < j \leq n$, $h, j \neq i$.

If the bound conditions for $P(c) \geq \beta_2(T) \geq \beta_1(c)$, are satisfied then $\beta_2(T)$ is superior to $b_2(T)$ by the following theorem which is proven in BCS (1988b).

THEOREM 9. *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be an arbitrary random vector. For any spanning tree $T$ of the probability graph, $\beta_2(T) \geq b_2(T)$.*

We consider finding optimal $\beta_2(T)$ and $b_2(T)$ in the next section.

**7. Optimized Bounds.** Seneta (1988) uses optimization arguments similar to those used in Section 2 and concludes

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq P(A_1) + \sum_{j=2}^{n} \max_{i<j} P(A_i^c \cap A_j)$$

which reduces to

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{j=2}^{n} \max_{i<j} P(A_i \cap A_j). \tag{7.1}$$

Maximizing (7.1) over all permutations $\boldsymbol{P} = (P_1, \cdots, P_n)$ of the orderings of the components yields

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \max_{P} \left[\sum_{j=2}^{n} \max_{i<j} P(A_{P_i} \cap A_{P_j})\right], \tag{7.2}$$

where $\max_P$ denotes the maximum over all permutations.

This will clearly yield an improvement of the bound given in (2.1), but (7.2) apparently requires extensive computation. We observe that Seneta (1988) apparently fails to notice that (7.2) coincides with the maximal Hunter bound of the form (6.2) which can be obtained with much less computation. Hunter (1976) shows that $b_2(T_0) \geq b_2(T)$ for all spanning trees $T$, where $T_0$ is obtained by applying Kruskal's (1956) maximal spanning tree to the network function $f((i,j)) = P(X_i > c, X_j > c) = P(A_i \cap A_j)$. The Kruskal algorithm is easy to compute. For instance, in the multivariate normal case with equal marginals, $T_0$ is calculated from examination of the covariance matrix, without evaluating marginal probabilities. Alternatively $b_2(T_0) \geq b_2(T)$ where $T_0$ is obtained by applying Kruskal's algorithm to $f((i,j)) = P(X_i \leq c, X_j \leq c) - P(X_i \leq c) - P(X_j \leq c)$.

Although the maximization approach of Seneta is unnecessary for second-order bounds, it can be applied to obtain an optimal third-order $\beta_i$ product-type bound. However, the maximization is often computationally infeasible. BCS (1988b) show that $\beta_2(T_0^*) \geq \beta_2(T)$ for any spanning tree T where $T_0^*$ is attained by applying Kruskal's maximal spanning tree to the network function

$$f((i,j)) = P(X_i \leq c, X_j \leq c)/(P(X_i \leq c)P(X_j \leq c)).$$

In the equal marginal case, $T_0^* = T_0$ so that the optimal product-type bound and optimal Bonferroni-type bound occur at the same spanning tree. Suppose for a moment that we have equal marginals. In light of Theorem 9, the optimal product-type bound $\beta_2(T_0)$ will be superior to $b_2(T_0)$ if $\boldsymbol{X}$ is

$MTP_2$ or if the conditions of (6.4) are satisfied in the multivariate normal case.

For many applications involving equal marginals the string-like tree or the star-like tree yield the optimal Bonferroni-type and optimal product-type bounds. The following theorem gives conditions under which string-like and star-like trees yield optimal bounds (Worsley (1982), BCS (1988b)).

THEOREM 10. *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random vector satisfying $P(X_1 \leq c) = \cdots = P(X_n \leq c)$.*

(a) $\beta_2(T_1) \geq \beta_2(T)$ *and* $b_2(T_1) \geq b_2(T)$, *for all spanning trees if*

$$P(X_{i-1} \leq c, X_i \leq c) \geq P(X_h \leq c, X_i \leq c)$$

*for all* $h < i-1, i = 3, 4, \cdots, n$.

(b) $\beta_2(T^i) \geq \beta_2(T)$ and $b_2(T^i) \geq b_2(T)$, *for all spanning trees* $T$ *if*

$$P(X_i \leq c, X_j \leq c) \geq P(X_h \leq c, X_j \leq c)$$

for $j = 1, \cdots, n, j \neq i$, *for all* $h \neq i, j$.

In the multivariate normal case the conditions of the previous theorem can be interpreted in terms of correlations.

THEOREM 11. *Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ have a multivariate normal distribution with $\sigma_{ii} = 1$ for $i = 1, \cdots, n$.*

(a) $\beta_2(T_1) \geq \beta_2(T)$ *for all trees* $T$ *if* $\sigma_{i,i-1} \geq \sigma_{hi}$ *for* $h < i-1, i = 3, \cdots, n$.

(b) $\beta_2(T^i) \geq \beta_2(T)$ *for all trees* $T$, *if* $\sigma_{ij} \geq \sigma_{jh}$ *for all* $j, h \neq i, 1 \leq j < h \leq n$.

In the unequal marginal case the maximal spanning tree will not necessarily be a string-like or a star-like tree. When applications based on empirical data such as when $n$ measurements are collected on a number of individuals, the optimal bound will often not correspond to a string-like tree due to the lack of covariance structure. In such applications lacking covariance structure, such as when some correlations are small but others involving non-adjacent components are large, the optimized bounds $\beta_2(T_0)$ and $b_2(T_0)$ will be considerably sharper than the standard second-order bounds, $\beta_2(T_1)$ and $b_2(T_1)$.

There has been virtually no development of optimized setwise bounds, where we define an optimized setwise bound as

$$\max_{\boldsymbol{P}} \prod_{i=1}^{m} P(X_{P_{2i-1}} \leq, X_{P_{2i}} \leq c),$$

where $n = 2m$, $\boldsymbol{P} = (P_1, \cdots, P_n)$ is a permutation of $(1, \cdots, n)$, and the maximization is over all permutations.

Use of this bound is not always apparently computationally feasible and further development is necessary for its use. However, for moderate n, it is sometimes possible to calculate this bound by inspection of the correlation structure.

**8. Discussion.** The results of the preceding Sections also hold for finding lower bounds to upper orthant probabilities. These results also apply to bounding $P(X_1 \leq c_1, \cdots, X_n \leq c_n)$ where the $c_i$'s are unequal.

Second-order product-type and Bonferroni-type probability bounds can be applied to $|X_1|, \cdots, |X_n|$ to obtain estimates of multivariate $\alpha$-level critical values in two-sided testing applications. In our view, a useful extension of the current theory would be to obtain results similar to Theorems 2 and 8 for two-sided testing.

Under strong negative dependence conditions, it follows that $\beta_1(c) \geq \beta_2(c) \geq P(c) \geq b_2(c) \geq b_1(c)$ so that product-type bounds complement Bonferroni-type bounds (Glaz and Johnson (1984)). One such application of this result is to multinomial probabilities (Glaz (1990)).

Another useful extension of the current theory would be to obtain weak bound conditions for $P(c) \geq \beta_2(c) \geq \beta_1(c)$ in applications involving the multivariate $t$-distribution, such as multiple comparisons. Simulation results on the feasibility of the use of product-type bounds for multivariate $t$-distributions would also be helpful. Such research may lead to the use of product-type bounds in applications in which Bonferroni-type bounds are currently employed (Bjornstadt and Butler (1988) and Stoline (1983)).

**9. Other Probability Inequalities.** In this review paper we concentrate on certain probability approximations which are expressed in terms of lower dimensional marginal probabilities and for which sufficient conditions for the approximations to be bounds involve positive dependence concepts.

For the reader who wishes to become more familiar with the more general subject of probability inequalities, see the book by Tong (1980) and the survey articles of Eaton (1982) and Block and Sampson (1982).

With regard to specific topics, there is an extensive literature on inequalities on a symmetric convex set (Tong (1980; Chapter 4) and Eaton (1982)). The better known results in this area are due to Anderson (1955), Sherman (1955) and Pitt (1977). There is also an extensive literature on inequalities by mixture (see Shaked (1977, 1979)). Chhetry, Kimeldorf and Sampson (1989) utilize notions of positive dependence by mixture in their study of setwise positive dependence.

Many probability inequalities are based on majorization (see Tong (1980; Chapter 6) and Eaton (1982)). Primary sources for majorization are Marshall and Olkin (1974, 1979). Among other areas, majorization is useful in obtaining monotonicity results. It is also applicable in situations in which multiple hypotheses are tested, as are the probability bounds presented in this paper. For example, Gail and Simon (1985) recently develop tests for prespecified treatment covariate interactions in clinical trials by using the basic theory of majorization.

Other probability inequalities based on positive dependence are reviewed in Block and Sampson (1982) and Tong (1980). Many of these inequalities are based on the parametric form of the density, e.g., multivariate normal, $t$, chi-squared, and F, as well as multinomial.

Das Gupta *et al.* (1972) provide inequality results for spherically symmetric distributions. These authors also derive some bounds for spherically symmetric distributions with non-zero means. They also derive upper as well as lower bounds. The most important results of Das Gupta et al (1972) are contained in Tong (1980) for the special case of multivariate normal distributions.

## Appendix: Definitions Pertaining to Positive Dependence Concepts

Definition A.1. $(X_1, X_2, \cdots, X_n)$ is *positive (negative) lower orthant dependent, PLOD* if for all real numbers $c_1, c_2, \cdots, c_n$

$$P(X_1 \leq c_1, X_2 \leq c_2, \cdots, X_n \leq c_n) \geq (\leq) \prod_{i=1}^{n} P(X_i \leq c_i). \tag{A.1}$$

Definition A.2. $(X_1, X_2, \cdots, X_n)$ is *positive (negative) upper orthant dependent* if for all real numbers $c_1, c_2, \cdots, c_n$

$$P(X_1 > c_1, X_2 > c_2, \cdots, X_n > c_n) \geq (\leq) \prod_{i=1}^{n} P(X_i > c_i). \tag{A.2}$$

For bivariate distributions Definitions A.1 and A.2 are equivalent and the term *positive (negative) quadrant dependence, PQD, (NQD)* is used.

Definition A.3. A function $f : R^2 \to R^+$ is *totally positive of order 2*, $TP_2$, if whenever $x' \geq x$, $y' \geq y$

$$f(x', y')f(x, y) \geq f(x', y)f(x, y'). \tag{A.3}$$

Definition A.4. A function $f : R^n \to R^+$ is *totally positive of order 2 in pairs, $TP_2$ in pairs*, if $f(X_1, \cdots, X_n)$ is a $TP_2$ function of $X_i$ and $X_j$ in the sense of (A.3) while the other variables are held fixed for $1 \le i < j \le n$. A random vector $(X_1, X_2, \cdots, X_n)$ is $TP_2$ in pairs if it has a joint density function which is $TP_2$ in pairs.

DEFINITION A.5. A function $f : R^n \to R$ is *multivariate totally positive of order 2*, $MTP_2$, if for all $\boldsymbol{x}, \boldsymbol{y} \in R^n$

$$f(\boldsymbol{x} \wedge \boldsymbol{y}) f(\boldsymbol{x} \vee \boldsymbol{y}) \ge f(\boldsymbol{x}) f(\boldsymbol{y}), \tag{A.5}$$

where $\boldsymbol{x} \wedge \boldsymbol{y} = (\min(x_1, y_1), \min(x_2, y_2), \cdots, \min(x_n, y_n))$ and $\boldsymbol{x} \vee \boldsymbol{y} = (\max(x_1, y_1), \max(x_2, y_2), \cdots, \max(x_n, y_n))$. A random vector is $MTP_2$ if its density is $MTP_2$.

A density which is $MTP_2$ is also $TP_2$ in pairs. In fact, $MTP_2$ and $TP_2$ in pairs are equivalent when the support of $f$ is a product space.

## REFERENCES

ANDERSON, T. W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities, *Proc. Amer. Math. Soc.* **6**, 170–176.

BARLOW, R. E. and PROSCHAN, F. (1981). *Statistical Theory of Reliability and Life Testing: Probability Models.* To Begin With, Silver Springs, MD.

BAUER, P. (1986). Approximation of discrete sequential boundaries, *Biometrika* **73**, 759–760.

BAUER, P. and HACKL, P. (1985). The application of Hunter's inequality in simultaneous testing, *Biometrical J.* **27**, 25–36.

BJORNSTADT, J. F. and BUTLER, R. W. (1988). The equivalence of backward elimination and multiple comparisons, *J. Amer. Statist. Assoc.* **83**, 136–144.

BLOCK, H. W., COSTIGAN, T. and SAMPSON, A. R. (1988a). Product-type probability bounds of higher order, University of Pittsburgh Technical Report.

BLOCK, H. W., COSTIGAN, T. and SAMPSON, A. R. (1988b). Optimal product-type probability bounds, University of Pittsburgh Technical Report.

BLOCK, H. W. and SAMPSON, A. R. (1982). Inequalities on distributions: bivariate and multivariate, in *The Encyclopedia of Statistical Sciences*, (Eds., S. Kotz and N. Johnson) Vol. 4, pp. 76-82, John Wiley and Sons, New York.

BOLVIKEN, E. (1982). Probability inequalities for the multivariate normal with non-negative partial correlations, *Scand. J. Statist.* **9**, 49–58.

CHHETRY, D., KIMELDORF, G. and SAMPSON, A. R. (1989). Concepts of setwise dependence, *Probab. Engrg. Inform. Sci.* **3**, 367–380.

Das Gupta, S., Eaton, M. L., Olkin, I., Perlman, M. D., Savage, L. J. and Sobel, M. (1972). Inequalities on the probability content of convex regions for elliptically contoured distributions. *Proc. Sixth Berkeley Symp. Probab. Statist.* Vol. II, 241–265.

Dykstra, R. L., Hewett, J. E. and Thompson, W. A. (1973). Events which are almost independent, *Ann. Statist.* **1**, 674–681.

Eaton, M. L. (1982). A review of selected topics in multivariate probability inequalities, *Ann. Statist.* **10**, 11–43.

Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, Vol. 1, 3rd Ed., John Wiley & Sons, New York.

Fuchs, C. and Sampson, A. R. (1987). Simultaneous confidence intervals for the general linear model, *Biometrics* **43**, 457–469.

Gail, M. and Simon, R. (1985). Testing for qualitative interactions between treatment effects and patient subsets, *Biometrics* **41**, 361–372.

Games, P. (1977). An improved $t$-table for simultaneous control of $g$ contrasts, *J. Amer. Statist. Assoc.* **72**, 531–534.

Glaz, J. (1990). A comparison of Bonferroni-type and product type inequalities in presence of dependence, in *Topics in Statistical Dependence*, (Eds. H. W. Block, A. R. Sampson, T. H. Savits), IMS Lecture Notes – Monograph Series, IMS, Hayward, CA.

Glaz, J. and Johnson, B. McK. (1984). Probability inequalities for multivariate distributions with dependence structures, *J. Amer. Statist. Assoc.* **79**, 436–441.

Glaz, J. and Johnson, B. McK. (1986). Approximating boundary crossing probabilities with application to sequential tests, *Sequential Anal.* **1**, 37–72.

Glaz, J. and Johnson, B. McK. (1988). Boundary crossing for moving sums, *J. Appl. Probab.* **25**, 81–88.

Hochberg, Y. and Tamhane, A. C. (1987). *Multiple Comparison Procedures.* John Wiley & Sons, New York.

Holland, B. S. and Copenhaver, M. D. (1987). An improved sequentially rejective Bonferroni test procedure, *Biometrics* **43**, 417–423.

Holm, S. (1979). A simple sequentially rejective multiple test procedure, *Scand. J. Statist.* **6**, 65–70.

Hoover, D. R. (1990a). Subset complement addition upper bounds, an improved inclusion-exclusion method, *J. Statist. Plann. Inference* **29**, 195–202.

Hoover, D. R. (1990b). Improved Bonferroni and Sidak/Slepian bounds: comparisons and applications, *Comm. Statist. Theory Methods* **19**, 1623–1637.

HOPPE, F. M. (1985). Iterating Bonferroni bounds, *Statist. Probab. Lett.* **3**, 121–125.

HUNTER, D. (1976). An upper bound for the probability of a union, *J. Appl. Probab.* **13**, 597–603.

KARLIN, S. and RINOTT, Y. (1980). Classes of orderings of measures and related correlation inequalities-I: Multivariate totally positive distributions, *J. Multivariate Anal.* **10**, 476–498.

KARLIN, S. and RINOTT, Y. (1983). $M$-Matrices as covariance matrices of multinormal distributions, *Linear Algebra Appl.* **52/53**, 419–438.

KENYON, J. R. (1986). Calculating improved bounds and approximations for multiple comparisons, *Computer Science and Statistics: Proceedings 18th Symposium Interface*, 367–371.

KENYON, J. R. (1987). Bounds and approximations for multivariate probabilities: Their calculation and application, University of Connecticut Doctoral Dissertation.

KENYON, J. R. (1988). Calculating improved bounds and approximations for sequential testing procedures, (Submitted for publication).

KOUNIAS, E. G. (1968). Bounds for the probability of a union, with applications, *Ann. Math. Statist.* **39**, 2154–2158.

KWEREL, S. M. (1975). Most stringent bounds on aggregated probabilities of partially specified dependent probability systems, *J. Amer. Statist. Assoc.* **70**, 472–479.

KRUSKAL, J. B. (1956). On the shortest spanning subtree of a graph and the traveling salesman problem, *Proc. Amer. Math. Soc.* **7**, 48–50.

LEHMANN, E. (1966). Some concepts of dependence, *Ann. Math. Statist.* **73**, 1137–1153.

MARSHALL, A. and OLKIN, I. (1967). A multivariate exponential distribution, *J. Amer. Statist. Assoc.* **62**, 30–44.

MARSHALL, A. and OLKIN, I. (1974). Majorization in multivariate distributions, *Ann. Statist.* **73**, 1137–1153.

MARSHALL, A. and OLKIN, I. (1979). *Inequalities. Theory of Majorization and Its Application.* Academic Press, New York.

MILHAKO, D. (1987). Repeated significance testing in survival analysis using a litter-matched design, *Biometrics* **43**, 115–125.

PITT, L. D. (1977). A Gaussian correlation inequality for symmetric convex sets, *Ann. Probab.* **5**, 470–474.

RAO, B. R., MARSH, G. M. and WINWOOD, J. (1985). Sidak-type simultaneous prediction intervals for the mortality measure RSRR, about the corresponding SePMRi for several competing risks of death in an epidemiological study, *J. Statist. Plann. Inference* **12**, 311–329.

RAO, B. R. and MARSH, G. M. (1987). Approximate methodologies for proportional mortality analysis in epidemiologic studies involving competing risks of death regardless of covariance structure, *Biometrical J.* **29**, 525–540.

RAVISHANKER, N., HOCHBERG, Y. and MELNICK, E. (1987). Approximate simultaneous intervals for multiple forcasts, *Technometrics* **29**, 371–376.

RESCEI, E. and SENETA, E. (1987). Bonferroni-type inequalities, *Adv. Appl. Probab.* **19**, 508–511.

SARKAR, S. K. and SMITH, W. (1986). Probability inequalities for ordered $MTP_2$ random variables, *Sankhȳa* **48**, 119–135.

SCHWAGER, S. J. (1984). Bonferroni sometimes loses, *Amer. Statist.* **38**, 192–197.

SENETA, E. (1988). Degree, iteration and permutation in improving Bonferroni-type bounds, *Austral. J. Statist.* **30A**, 27–38.

SHAFFER, J. P. (1986). Modified sequential rejective multiple test procedures, *J. Amer. Statist. Assoc.* **81**, 826–831.

SHAKED, M. (1977). A family of concepts of positive dependence for exchangeable random variables, *Ann. Statist.* **5**, 505–515.

SHAKED, M. (1979). Some concepts of positive dependence for bivariate interchangeable distributions, *Ann. Inst. Statist. Math.* **31**, 67–84.

SHERMAN, S. (1955). A theorem on convex sets with applications, *Ann. Math. Statist.* **26**, 763–766.

SIDAK, Z. (1967). Rectangular confidence regions for means of multivariate normal distributions, *J. Amer. Statist. Assoc.* **62**, 626–633.

SIMES, R. J. (1986). An improved Bonferroni procedure for multiple tests of significance, *Biometrika* **73**, 741–744.

SLEPIAN, D. (1962). On one-sided barrier problem for Gaussian noise, *Bell System Tech. J.* **41**, 463–501.

STOLINE, M. R. (1983). The Hunter method of simultaneous inference and its recommended use for applications having large known correlation structured, *J. Amer. Statist. Assoc.* **78**, 366–370.

TONG, Y. L. (1980). *Probability Inequalities in Multivariate Distributions*, Academic Press, Inc., New York.

WORSLEY, K. J. (1982). An improved Bonferroni inequality and applications, *Biometrika* **69**, 297–302.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260

DEPARTMENT OF STATISTICS
THE UNIVERSITY OF OHIO
COLUMBUS, OHIO 43120-1247