UNIFORM CONVERGENCE OF MARTINGALES IN THE ONE-DIMENSIONAL BRANCHING RANDOM WALK

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**Abstract**

In the supercritical branching random walk an initial person has children whose positions are given by a point process $Z^{(1)}$. Each of these then has children in the same way, with the positions of children in each family, relative to their parent's, being given by independent copies of $Z^{(1)}$, and so on. For any value of its argument, $\lambda$, the Laplace transform of the point process of $n^{th}$ generation people, normalized by its expected value, is a martingale, the usual branching process martingale being a special case. Here it is shown that under certain conditions these martingales converge uniformly in $\lambda$, almost surely and in mean. A consequence of this result is that the limit is, in an appropriate region, analytic in $\lambda$.

**1. Introduction.** This paper considers the one dimensional supercritical branching random walk. The process starts with a single initial ancestor at the origin. She has children, forming the first generation, with their positions on the real line, $R$, being given by a point process $Z^{(1)}$. Each of these children then has offspring in a similar way, with the positions of each new family relative to their parent being given by independent copies of $Z^{(1)}$. This gives the point process of second generation individuals, denoted by $Z^{(2)}$. Subsequent generations are formed similarly, yielding $Z^{(n)}$ as the $n^{th}$ generation point process. Let $\{ z_r^{(n)} : r\}$ be an enumeration of the positions of the $n^{th}$ generation people.

Let $\mu$ by the intensity measure of $Z^{(1)}$ then, as is well known, $\mu^{n*}$ (the n-fold convolution of $\mu$) is the intensity measure of $Z^{(n)}$. As the process is supercritical we have $\mu(R) > 1$. Let $m(\lambda)$ be the Laplace transform of $\mu$. Then

$$\begin{aligned} m(\lambda) &= \int e^{-\lambda x}\mu(dx) \\ &= E\int e^{-\lambda x} Z^{(1)}(dx) \\ &= E\sum_r e^{-\lambda z_r^{(1)}} \end{aligned}$$

and hence

$$\begin{aligned} m(\lambda)^n &= \int e^{-\lambda x}\mu^{n*} \\ &= E\int e^{-\lambda x} Z^{(n)}(dx) \\ &= E\sum_r e^{-\lambda z_r^{(n)}}. \end{aligned}$$

We will adopt the convention that the real and imaginary parts of $\lambda$ are $\theta$ and $\eta$ respectively, so that $\lambda = \theta + i\eta \in C$, where $C$ is the complex numbers.

The description of the process given above implies that, for any set $A$,

$$Z^{(n+1)}(A) = \sum_r Z_{n,r}^{(1)}(A - z_r^{(n)}), \tag{1.1}$$

where $\{Z_{n,r}^{(1)} : r\}$ are independent copies of $Z^{(1)}$ with $Z_{n,r}^{(1)}$ giving the relative positions of the family of $z_r^{(n)}$. Let $F^{(n)}$ be the $\sigma$-field containing all information about the first $n$ generations. Then (1.1) implies that

$$\begin{aligned} E(\int e^{-\lambda x} Z^{(n+1)}(dx) \mid F^{(n)}) &= \sum_r E(\int e^{-\lambda x} Z_{n,r}^{(1)}(dx - z_r^{(n)}) \mid F^{(n)}) \\ &= \sum_r E\left(\int e^{-\lambda(x + z_r^{(n)})} Z_{n,r}^{(1)}(dx) \mid F^{(n)}\right) \\ &= \sum_r E(\int e^{-\lambda x} Z_{n,r}^{(1)}(dx) \mid F^{(n)}) e^{-\lambda z_r^{(n)}} \\ &= m(\lambda)\int e^{-\lambda x} Z^{(n)}(dx). \end{aligned}$$

Therefore, if $m(\lambda)$ is finite and non-zero,

$$W^{(n)}(\lambda) = m(\lambda)^{-n}\int e^{-\lambda x} Z^{(n)}(dx)$$

is a martingale with respect to $F^{(n)}$. These martingales and variants of them for similar processes have been considered often, for example, by Watanabe (1967), Joffe et al. (1973), Kingman (1975), Biggins (1977), Wang (1980), Uchiyama (1982) and Neveu (1988).

Notice that $W^{(n)}(\theta)$ is a non-negative martingale and so converges almost surely for all $\theta$. However only when it converges also in mean can we be confident that the limit is not trivial. In particular $W^{(n)}(0)$ is the classical branching process martingale, for which the condition

$$EW^{(1)}(0)\log^{+}(W^{(1)}(0))<\infty$$

is necessary and sufficient for convergence in mean. Let $\Omega^0$ be the interior of $\{\lambda : m(\theta) < \infty\}$; we will assume throughout that $\Omega^0$ is non-empty. Then, if $\theta \in \Omega^0$, it is shown in Biggins (1977) that the conditions

$$EW^{(1)}(\theta)\log^{+}(W^{(1)}(\theta)) < \infty \tag{1.2}$$

and

$$\theta \in \{\theta : -\log(m(\theta)) < -\theta m'(\theta)/m(\theta)\} \tag{1.3}$$

are necessary and sufficient for $EW(\theta) = 1$, and hence for $W^{(n)}(\theta)$ to converge in mean.

The set described by (1.3 ) is actually an interval and, to avoid complications with end points, we will let $T$ be its intersection with $\Omega^0$, so that $T$ is an open interval. Hence

$$T = \{\theta : \theta \in \Omega^0, -\log(m(\theta)) < -\theta m'(\theta)/m(\theta)\}. \tag{1.4}$$

The main results here concern the convergence of $W^{(n)}(\lambda)$ or $W^{(n)}(\theta)$ as a sequence of functions. Under suitable conditions there is an open set $\Lambda^*$ in $C$, containing $T$, with $W^{(n)}(\lambda)$ converging uniformly on any compact subset of $\Lambda^*$, almost surely and in mean. A consequence of this is that the limit, $W(\lambda)$, is actually analytic on $\Lambda^*$. Under a rather weaker moment condition, a slightly different approach yields the uniform convergence of $W^{(n)}(\theta)$ to $W(\theta)$ on compact subsets of $T$, implying that $W(\theta)$ is continuous on $T$.

Joffe et al. (1973) give a result on the uniform convergence of $W^{(n)}(i\eta)$ for a particular case of the process considered here. They adopt an elegant approach through results on convergence of martingales taking values in a Banach space, with a vital step in the proof being the verification that the limit $W(i\eta)$ is continuous. In contrast here the convergence of $W^{(n)}(\lambda)$ to $W(\lambda)$ will be tackled directly, yielding the convergence of the appropriate Banach space valued martingale and the properties of the limit as consequences.

A more detailed study of the convergence of $W^{(n)}(\lambda)$ to $W(\lambda)$ is contained in Biggins (1989). There a uniform convergence result is obtained for the branching random walk on $R^p$ both in discrete and continuous time. An application of these results is also given, large deviation results for $Z^{(n)}$ being obtained using them.

**2. The Main Results.** We consider first the convergence of $W^{(n)}(\lambda)$ in a suitable region of $C$.

**Theorem 1.** *If for some* $\gamma > 1$

$$EW^{(1)}(\theta)^{\gamma} < \infty \textit{ for all } \theta \in T \tag{2.1}$$

*and* $\Lambda^*$ *is defined by*

$$\Lambda^* = \{\lambda \in \Omega^0 : m(\alpha\theta)/|m(\lambda)|^{\alpha} < 1 \textit{ for some } \alpha \in (1,\gamma]\}$$

*then* $W^{(n)}(\lambda)$ *converges uniformly, to* $W(\lambda)$, *on any compact subset of* $\Lambda^*$, *almost surely and in mean.*

It is not too hard to show that

$$\Lambda^* = \bigcup_{1<\alpha\leq\gamma} \operatorname{int}\{\lambda \in \Omega^0 : m(\alpha\theta)/|m(\lambda)|^{\alpha} < 1\} \tag{2.2}$$

so $\Lambda^*$ is open. Furthermore computing the derivative of $m(\alpha\theta)/m(\theta)^{\alpha}$ with respect to $\alpha$ and setting $\alpha = 1$ confirms that the set $T$, defined at (1.4 ), is the intersection of $\Lambda^*$ with the real axis.

If $F$ is a compact subset of $\Lambda^*$ the assertion of the theorem is that

$$\sup\{\ |W^{(N)}(\lambda) - W^{(n)}(\lambda)|\ : \lambda \in F, N \geq n\ \}$$

converges to zero almost surely and in mean as $n \to \infty$. It will then follow that $W(\lambda)$ exists on $F$ and that

$$\sup\{|W^{(n)}(\lambda) - W(\lambda)|\ : \lambda \in F\ \}$$

also converges to zero in both senses.

The approach we will take to this result relies heavily on Cauchy's integral formula and so does not lend itself to considering convergence for $\theta \in T$ alone.

Furthermore, with this approach, I can see no way to escape from the rather strong moment condition (2.1). A weaker moment condition suffices in considering the convergence of $W^{(n)}(\theta)$ on $T$ to which we now turn.

**Theorem 2.** *If*

$$EW^{(1)}(\theta)\,(\log^+(W^{(1)}(\theta)))^{3/2} < \infty \textit{ for all } \theta \in T \tag{2.3}$$

*then* $W^{(n)}(\theta)$ *converges uniformly to* $W(\theta)$ *on compact subsets of* $T$, *almost surely and in mean.*

The two moment conditions (2.1) and (2.3) are easily seen to be equivalent to the alternatives resulting if $W^{(1)}(\theta)$ is replaced by $\int e^{-\theta x} Z^{(1)}(dx)$. Now, if $g$ is any convex function, $Eg\left(\int e^{-\theta x} Z^{(1)}(dx)\right)$ is convex in $\theta$ and so its domain of finiteness must be an interval. Furthermore it must be uniformly bounded on compact subsets of the interior of this interval. The conditions (2.1) and (2.3) simply insist that, for particular $g$, this interval of finiteness should include all of $T$.

Differentiation of $E(W^{(n+1)}(\theta) \mid F^{(n)}) = W^{(n)}(\theta)$ shows that $W^{(n)}(\theta)'$ is also a martingale, as are all higher derivatives. The proof of Theorem 2 yields, in the course of its proof, the following result about this martingale.

**Theorem 3.** *If* $\theta \in T$ *with*

$$EW^{(1)}(\theta)\,(\log^+(W^{(1)}(\theta)))^{3/2} < \infty \tag{2.4}$$

*and*

$$E|W^{(1)}(\theta)'|\,\log^+(|W^{(1)}(\theta)'|) < \infty \tag{2.5}$$

*then* $W^{(n)}(\theta)'$ *converges almost surely and in mean.*

At the end of section 4 it is indicated how the method of proof extends to higher derivatives, at some notational expense, but the details are not considered here. It is perhaps worth noting that in proving Theorem 2 we will show that the moment condition (2.3) implies (2.4) and (2.5). Of course under the stronger moment condition (2.1) Theorem 1 holds, and then all derivatives converge.

The proof of the main results relies heavily on the following lemma which is proved in Biggins (1989).

**Lemma 1.** *If* $\{X_r\}$ *are independent complex random variables with* $E(X_r) = 0$ *or, more generally, martingale differences, then*

$$E|\Sigma X_r|^\alpha \le 2^\alpha \Sigma E|X_r|^\alpha$$

*for* $1 \le \alpha \le 2$.

This, together with Jensen's inequality, immediately yields the following result for the branching random walk.

**Lemma 2.** *If, given* $F^{(n)}$, $\{X_r\}$ *are independent identically distributed copies of* $X$ *with* $EX = 0$ *then*

$$E\left|\sum_r \frac{e^{-\lambda z_r^{(n)}}}{m(\lambda)^n} X_r\right| \le 2\left(\frac{m(\alpha\theta)^n}{|m(\lambda)|^{\alpha n}} E|X|^\alpha\right)^{1/\alpha}$$

*for* $1 \le \alpha \le 2$.

**3. Proof of Theorem 1.** It will suffice to show that for each $\lambda_0 \in \Lambda^*$ uniform convergence holds in a disc centred at $\lambda_0$ for then a standard covering argument completes the proof. Denote the disc of radius $\rho$ centred at $\lambda_0$ by $D_{\lambda_0}(\rho)$. Given $\lambda_0 \in \Lambda^*$, we can, because of the representation (2.2), find $\alpha \in (1, \gamma]$ and $\rho$ such that

$$D_{\lambda_0}(3\rho) \subset \{\lambda \in \Omega^0 : m(\alpha\theta) / |m(\lambda)|^\alpha < 1\}. \tag{3.1}$$

We will demonstrate convergence on $D_{\lambda_0}(\rho)$.

Let $\Gamma$ be the boundary of $D_{\lambda_0}(2\rho)$ and suppose $f$ is analytic on $D_{\lambda_0}(3\rho)$, then, writing

$$\Gamma = \{z(t) : z(t) = \lambda_0 + 2\rho e^{2\pi i t},\ t \in [0,1)\}, \tag{3.2}$$

Cauchy's integral formula gives

$$f(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\zeta}dz$$

$$= \int_0^1 \frac{f(z(t))}{z(t)-\zeta} 2\rho e^{2\pi i t} dt.$$

Therefore

$$\sup\{|f(\zeta)| : \zeta \in D_{\lambda_0}(\rho)\} \le 2\int_0^1 |f(z(t))| dt, \tag{3.3}$$

where $z(t)$ parameterises the boundary of $\Gamma$ and is defined at (3.2 ).

As $W^{(N)}(\zeta) - W^{(n)}(\zeta)$ is an analytic function on $\Lambda^*$, (3.3 ) yields immediately that

$$\sup\{|W^{(N)}(\zeta) - W^{(n)}(\zeta)| \ : \zeta \in D_{\lambda_0}(\rho)\}$$

$$\leq 2\int_0^1 |W^{(N)} - W^{(n)}|\, dt$$

$$\leq 2\int_0^1 \sum_{r=n}^{N-1} |W^{(r+1)} - W^{(r)}|\, dt$$

$$\leq 2\int_0^1 \sum_{r=n}^{\infty} |W^{(r+1)} - W^{(r)}|\, dt, \tag{3.4}$$

where we have suppressed $z(t)$ in the integrand. The proof will be complete if we show that (3.4 ) converges to zero almost surely and in mean as $n \to \infty$, and for this it suffices to verify that it has finite expectation. Now note that

$$E\int_0^1 \sum_{n=0}^{\infty} |W^{(n+1)} - W^{(n)}|\, dt$$

$$= \int_0^1 \sum_{n=0}^{\infty} E|W^{(n+1)} - W^{(n)}|\, dt$$

$$\leq \sup\{\sum_n E|W^{(n+1)}(\lambda) - W^{(n)}(\lambda)| : \lambda \in \Gamma\ \}. \tag{3.5}$$

Furthermore, with the obvious extension of the notation of (1.1 ),

$$W^{(n+1)}(\lambda) - W^{(n)}(\lambda) = \sum_r \frac{e^{-\lambda z_r^{(n)}}}{m(\lambda)^n}\left\{W_{n,r}^{(1)}(\lambda) - 1\right\} \tag{3.6}$$

and so Lemma 2 can be applied to estimate (3.5 ).

Observe first that

$$E|W^{(1)}(\lambda) - 1|^{\alpha} \leq 2^{\alpha}(E|W^{(1)}(\lambda)|^{\alpha} + 1)$$

$$\leq 2^{\alpha}\left(\frac{m(\theta)^{\alpha}}{|m(\lambda)|^{\alpha}} EW^{(1)}(\theta)^{\alpha} + 1\right)$$

and so, by virtue of the moment condition (2.1 ) and the remarks immediately after Theorem 2, is uniformly bounded for $\lambda \in \Gamma$. Hence, applying Lemma 2,

$$\sum_n E\left|W^{(n+1)}(\lambda) - W^{(n)}(\lambda)\right| \le K \sum_n \left(\frac{m(\alpha\theta)}{|m(\lambda)|^{\alpha}}\right)^{n/\alpha}$$

for $\lambda \in \Gamma$ but, because of (3.1 ) and (3.2 ),

$$\sup\left\{\frac{m(\alpha\theta)}{|m(\lambda)|^{\alpha}} \;:\; \lambda \in \Gamma\right\} < 1$$

and so (3.5 ) is indeed finite.

**4. Proof of Theorems 2 and 3.** Just as in the proof of Theorem 1 it will suffice to show that for each $\theta_0 \in T$ uniform convergence holds in an interval around $\theta_0$. As $\theta_0 \in T$, there is a $\beta > 1$ with

$$\frac{m(\beta\theta_0)}{m(\theta_0)^{\beta}} < 1$$

so we may choose $\rho$ sufficiently small that

$$(\theta_0 - 2\rho, \theta_0 + 2\rho) \subset \left\{\theta \;:\; \frac{m(\beta\theta)}{m(\theta)^{\beta}} < 1\right\} \tag{4.1}$$

(which implies that $\theta_0 \pm 2\rho \in T$). Now let $D = (\theta_0 - \rho, \theta_0 + \rho)$. Then

$$\delta = \sup\left\{\frac{m(\beta\theta)^{1/\beta}}{m(\theta)} \;:\; \theta \in D\right\} < 1. \tag{4.2}$$

In place of the bound (3.3), based on Cauchy's integral formula, we employ the following simple estimate

$$\sup\{|f(\theta)| \;:\; \theta \in D\} \le f(\theta_0) + \int_D |f'(\zeta)|\, d\zeta.$$

Hence

$$\sup\left\{\left|W^{(N)}(\theta) - W^{(n)}(\theta)\right| \;:\; \theta \in D\right\}$$

$$\le \left|W^{(N)}(\theta_0) - W^{(n)}(\theta_0)\right| + \int_D \left|\left(W^{(N)}(\theta) - W^{(n)}(\theta)\right)'\right| d\theta$$

$$\leq \sum_{r=n}^{\infty} \left| W^{(r+1)}(\theta_0) - W^{(r)}(\theta_0) \right|$$

$$+ \sum_{r=n}^{\infty} \int_D \left| (W^{(r+1)}(\theta) - W^{(r)}(\theta'))' \right| d\theta \tag{4.3}$$

and the result will be proved by showing that the final bound here has finite expectation. The second term on the right is the harder one to deal with so our discussion focuses on it. In considering it Theorem 3 will be proved as a by-product. Much as in (3.5), it will be enough to show that

$$\sup\{ \sum_n E\left| (W^{(n+1)}(\theta) - W^{(n)}(\theta))' \right| : \theta \in D \} < \infty.$$

Differentiation of (3.6) gives

$$(W^{(n+1)}(\theta) - W^{(n)}(\theta))'$$

$$= \sum_r \frac{e^{-\theta z_r^{(n)}}}{m(\theta)^n} \left\{ \left( -z_r^{(n)} - \frac{nm'(\theta)}{m(\theta)} \right) (W_{n,r}^{(1)}(\theta) - 1) + W_{n,r}^{(1)}(\theta)' \right\} \tag{4.4}$$

Notice that these are the differences of the martingale $\{W^{(n)}(\theta)'\}$ and, under suitable moment conditions, Lemma 1 would apply to (4.4 ) to give a bound on the expectation of its absolute value. When the resulting bounds have a finite sum (over $n$) this yields the convergence almost surely and in mean of the martingale. However to allow weaker moment conditions we will use a truncation technique and so need variants of Lemmas 1 and 2, which we now discuss. In these lemmas $\{I_r\}$ will be indicator functions with $I_r^c = 1 - I_r$. These will be used to isolate cases where $|X_r|$ is big.

**Lemma 3.** *Let $\{X_r, I_r\}$ be a sequence of random vectors which are, given G, independent with $E(X_r \mid G) = 0$ and let N be a (possibly infinite) G measurable random integer then*

$$E\left|\sum^N X_r\right| \leq 2E\sum^N E(|X_r I_r| \mid G) + 2^3 \left( E\sum^N E\left( |X_r I_r^c|^\alpha \mid G \right) \right)^{1/\alpha}$$

*for* $1 \leq \alpha \leq 2$.

**Proof.** As $E(X_r \mid G) = 0$

$$E(X_rI_r^c|\,G) = -E(X_rI_r|\,G)\,,$$

consequently

$$\Sigma X_r = \Sigma X_rI_r + \Sigma E(X_rI_r|\,G) + \Sigma(X_rI_r^c - E(X_rI_r^c|\,G))\,, \tag{4.5}$$

where, of course, the $N$ has been suppressed in the notation. Now

$$E|\Sigma X_rI_r| \le E\Sigma E(|X_rI_r|\,|G) \tag{4.6}$$

and

$$E|\Sigma E(X_rI_r|\,G)| \le E\Sigma E(|X_rI_r|\,|\,G)\,. \tag{4.7}$$

The final term on the right of (4.5) requires more work. Note first that for any random variable $X$ and $\alpha \geq 1$

$$\begin{aligned} E|X - E(X)|^{\alpha} &\le 2^{\alpha}(E|X|^{\alpha} + |E(X)|^{\alpha}) \\ &\le 2^{\alpha+1}E|X|^{\alpha}. \end{aligned}$$

Now applying Jensen's inequality, Lemma 1 and the inequality just derived we see that

$$\begin{aligned} E\left|\sum (X_rI_r^c - E(X_rI_r^c|\,G))\right| &\le \left(E\left|\sum (X_rI_r^c - E(X_rI_r^c|\,G))\right|^{\alpha}\right)^{1/\alpha} \\ &= \left(EE\left|\sum (X_rI_r^c - E(X_rI_r^c|\,G))\right|^{\alpha}\middle|\,G\right)^{1/\alpha} \\ &\le \left(E2^{\alpha}\sum 2^{\alpha+1}E\left(\left|X_rI_r^c\right|^{\alpha}\middle|\,G\right)\right)^{1/\alpha} \\ &\le 2^3\left(E\sum E\left(\left|X_rI_r^c\right|^{\alpha}\middle|\,G\right)\right)^{1/\alpha} \end{aligned} \tag{4.8}$$

for $1 \le \alpha \le 2$. Combining the bounds (4.6 ), (4.7 ) and (4.8 ) with (4.5 ) completes the proof.

It is worth stating the following special case of Lemma 3 which has conditions appropriate to our context.

**Lemma 4.** *If, given $G$, $\{X_r, I_r\}$ are independent identically distributed copies of $\{X, I\}$ with $EX = 0$, and $\{C_r\}$ are $G$ measurable, then*

$$E\left|\sum C_rX_r\right| \le 2E|XI|\,E\sum|C_r| + 2^3(E|XI^c|^{\alpha})^{1/\alpha}(E\sum|C_r|^{\alpha})^{1/\alpha}. \tag{4.9}$$

We now return to consideration of (4.4 ), applying the lemma just obtained to its two parts separately. Consider first

$$E\left|\sum_r \frac{e^{-\theta z_r^{(n)}}}{m(\theta)^n}\left(-z_r^{(n)}-\frac{nm'(\theta)}{m(\theta)}\right)(W_{n,r}^{(1)}(\theta)-1)\right|, \tag{4.10}$$

which is the more complicated of the two. Obviously we take $F^{(n)}$ as $G$. Let

$$C_r = \frac{e^{-\theta z_r^{(n)}}}{m(\theta)^n}\left(-z_r^{(n)}-\frac{nm'(\theta)}{m(\theta)}\right),$$

$$X_r = W_{n,r}^{(1)}(\theta)-1$$

and for the indicator variables let

$$I_r = I\left\{1+W_{n,r}^{(1)}(\theta)>c^n\right\} \tag{4.11}$$

where $c>1$ will be fixed later. Let $F_\theta$ be the probability measure of the random variable in this indicator; we will use this in bounding the terms obtained in applying Lemma 4. Specifically, observe that

$$E|XI| \le \int_{c^n}^{\infty} xF_\theta(dx) \tag{4.12}$$

and

$$E|XI^c|^\alpha \le \int_1^{c^n} x^\alpha F_\theta(dx) \tag{4.13}$$

To simplify the expressions resulting from the calculation of the other components of (4.9 ) let the probability measure $\mu_\theta$, which has mean zero, be given by

$$\mu_\theta(dx) = \frac{e^{-\theta x}}{m(\theta)}\mu\left(dx+\frac{m'(\theta)}{m(\theta)}\right).$$

Then it is straightforward to check that

$$E\sum_r \frac{e^{-\theta z_r^{(n)}}}{m(\theta)^n} f\left(z_r^{(n)}+\frac{nm'(\theta)}{m(\theta)}\right) = \int f(x)\,\mu_\theta^{n*}(dx),$$

and hence

$$E\sum_r \frac{e^{-\alpha\theta z_r^{(n)}}}{m(\alpha\theta)^n} f\left(z_r^{(n)}+\frac{nm'(\theta)}{m(\theta)}\right) = \int f(x+nk(\alpha,\theta))\,\mu_{\alpha\theta}^{n*}(dx),$$

where

$$k(\alpha,\theta) = \frac{m'(\alpha\theta)}{m(\alpha\theta)} - \frac{m'(\theta)}{m(\theta)}.$$

Therefore

$$E\sum |C_r| \le \int |x|\,\mu_\theta^{n*}(dx) \tag{4.14}$$

and

$$E\sum |C_r|^\alpha \le \left(\frac{m(\theta\alpha)}{m(\theta)^\alpha}\right)^n \int |x+nk(\alpha,\theta)|^\alpha \mu_{\alpha\theta}^{n*}\ (dx). \tag{4.15}$$

Applying Lemma 4 to (4.10) using the above calculations will still result in rather unwieldy expressions, so we now restrict attention to $\theta \in D$. Note first that, as $\int x^2 \mu_\theta(dx)$ is continuous on $\Omega^0$ and hence uniformly bounded on $D$,

$$\begin{aligned}\left(\int |x|^\alpha \mu_\theta^{n*}(dx)\right)^{1/\alpha} &\le \left(\int x^2 \mu_\theta^{n*}(dx)\right)^{1/2}\\ &= n^{1/2}\left(\int x^2\mu_\theta(dx)\right)^{1/2}\\ &\le Kn^{1/2},\end{aligned}$$

for $1 \le \alpha \le 2$. (We will use $K$ for a generic constant independent of $\theta \in D$.) Now we take $\alpha = \beta$, so that (4.1) holds. Hence

$$\sup\{k(\beta,\theta):\ \theta \in D\} < \infty$$

so that

$$\int |x+nk(\beta,\theta)|^\beta \mu_{\beta\theta}^{n*}\ (dx) \le Kn^\beta,$$

and furthermore (4.2) holds. With these estimates and the bounds (4.12), (4.13), (4.14), and (4.15) we can now apply Lemma 4 to see that (4.10) is bounded by

$$K\left(n^{1/2}\int_{c^n}^{\infty} xF_\theta(dx) + n\delta^n\left(\int_1^{c^n} x^\beta F_\theta(dx)\right)^{1/\beta}\right)$$

which is in turn bounded by

$$K\left(n^{1/2}\int_{cn}^{\infty} xF_\theta(dx) + n\delta^n c^{n(\beta-1)/\beta}\left(\int_1^\infty xF_\theta(dx)\right)^{1/\beta}\right) \tag{4.16}$$

Now we choose $c > 1$ but small enough that $\delta c^{(\beta-1)/\beta} < 1$. We now want to show that (4.16 ) is summable over $n$, and for Theorem 2 this must be so uniformly in $\theta \in D$. The sum of the second term here will be (uniformly) bounded provided that $\int_1^\infty xF_\theta(dx)$ is. Turning to the first term

$$\sum_n n^{1/2}\int_{c^n}^\infty xF_\theta(dx) = \int_1^\infty x\sum_n n^{1/2} I\left(n \le \frac{\log x}{\log c}\right)F_\theta(dx)$$

$$\le K\int_1^\infty x(\log x)^{3/2}F_\theta(dx) \quad . \tag{4.17}$$

Hence the moment condition (2.4) (or (2.3)) ensures that both sums are finite.

Consideration of the second part of (4.4),

$$E\left|\sum_r \frac{e^{-\theta z_r^{(n)}}}{m(\theta)^n} W_{n,r}^{(1)}(\theta)'\right|,$$

is similar, the indicator variables now being

$$I_r = I\left\{1 + \left|W_{n,r}^{(1)}(\theta)'\right| > c^n\right\}$$

with $F_\theta^1$ as the associated probability measure. The calculation analogous to (4.17 ) no longer involves $n^{1/2}$ and so leads to the moment condition (2.5 ). This completes the proof of Theorem 3.

For the proof of Theorem 2 we must show in addition that

$$\sup\left\{\int x(\log x)^{3/2}F_\theta(dx) \ : \ \theta \in D\right\} < \infty \tag{4.18}$$

and

$$\sup\left\{\int x(\log x)F_\theta^1(dx) \ : \ \theta \in D\right\} < \infty \tag{4.19}$$

To do this let $\theta_1 = \theta_0 - 2\rho$ and $\theta_2 = \theta_0 + 2\rho$; recall that $\rho$ was chosen so that $\theta_1$ and $\theta_2$ are in $T$. Note that

$$\sup\left\{\frac{(1+|x|)e^{-\theta x}}{e^{-\theta_1 x} + e^{-\theta_2 x}} : \theta \in D\right\} < \infty \tag{4.20}$$

and so, for $\theta \in D$,

$$W^{(1)}(\theta)+\left|W^{(1)}(\theta)\right| \le K\sum_r (1+\left|z_r^{(1)}\right|)e^{-\theta z_r^{(1)}}$$

$$\le K\sum_r \left(e^{-\theta_1 z_r^{(1)}}+e^{-\theta_2 z_r^{(1)}}\right)$$

$$\le K(W^{(1)}(\theta_1)+W^{(1)}(\theta_2))$$

$$\le K \max\left\{W^{(1)}(\theta_1), W^{(1)}(\theta_2)\right\}; \qquad (4.21)$$

hence (2.3 ) does indeed guarantee (4.18) and (4.19 ).

This shows that the second term on the right of (4.3 ) has finite expectation. A similar, but much more straightforward, analysis shows the finiteness of the expectation of the first term. (The analysis also establishes that, when $\theta \in T$, (1.2 ) is indeed sufficient for $EW(\theta) = 1$.) This completes the proof of Theorem 2.

If we considered higher derivatives of $W^{(n)}(\theta)$ the analogue of (4.4 ) would now involve each of the derivatives of $W_{n,r}^{(1)}(\theta)$ up to the degree in question. These can be analysed as the components of (4.4 ) were. However the powers of $n$ multiplying the two parts of (4.16 ) will depend on which term in the analogue of (4.4 ) we are considering. The argument dealing with the second term in (4.16 ) is unaffected by any higher powers of $n$ but they do change the power of $\log x$ appearing in (4.17 ). It is worth noting too that an estimate like (4.20 ) will still work if higher powers of $|x|$ are included. Hence a bound like (4.21 ) also holds for higher derivatives. Combining these considerations we see that analogues of Theorems 2 and 3 can be obtained for higher derivatives by suitably strengthening the moment conditions.

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