A CENTRAL LIMIT THEOREM FOR EVOLVING RANDOM FIELDS

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Abstract

A functional central limit theorem is proved for certain random fields whose domain has both a temporal and a spacial component. These processes are made up of dependent summands which are measurable with respect to an increasing filtration. Temporal limit theory for semi-martingales is utilized to provide spacial finite-dimensional convergence. Consequently the limiting random fields have independent increments in time, and can be thought of as evolving random fields. In deriving a tightness result the notion of majorizing measures is employed to allow local spacial variability. Thus a functional central limit theorem for evolving random fields with a rather general dependence structure is given here for the first time. Comparison with results available for empirical processes suggest that this result is close to optimal.

Introduction. In this paper we consider conditions for weak convergence of a sequence of random fields which are individually evolving over time. For motivation though, let's first examine a given sequence of random fields at some fixed time point. Let **X** be an arbitrary index set and for each $n \geq 1$, let $\{Y_{n,i}(x) : 1 \leq i \leq n,\ x \in X\}$ be a sequence of random functions $Y_{n,i} : X \to \mathbb{R}$. Let

$$S_n(x) = \sum_{i \leq n} Y_{n,i}(x) \qquad \text{for} \qquad x \in X.$$

Thus the sequence $\{S_n(x) : n \geq 1,\ x \in X\}$ is a sequence of random fields. We are interested in general conditions under which S_n converges weakly. We wish to allow dependence among the individual summands, the $Y_{n,i}$'s. We do this by considering conditions of the martingale type, involving only conditional first and second moments. (These are more desirable than mixing type conditions which involve the entire distribution.) In addition, we must constrain our indexing family X to satisfy a complexity condition involving the notion of majorizing measures (or, alternatively, metric entropy). There will be a natural topology on **X** associated with our problem.

In two papers Goldie and Greenwood (1986a, b) found conditions for weak convergence of sequences of set-indexed random fields. An unexpectedly difficult aspect of the problem was the characterization of the limiting distribution in a way which allowed the identification of the limiting finite-dimensional distributions. The first of the two papers was devoted to this problem, and the solution there was rather unsatisfactory in that mixing conditions were used rather than martingale type conditions.

An entirely different way of characterizing finite-dimensional laws is available if one considers, instead of a sequence of random fields $\{S_n(x) : x \in X\}$, a sequence of *evolving* random fields

$$S_n(x,t) = \sum_{i \le \tau(n,t)} Y_{n,i}(x) \quad \text{for} \quad x \in X,\ t \in [0,1].$$

(Here $\tau(n,t)$ depends on both n and t, perhaps randomly.) The idea of an evolving random field is appealing for modeling because most naturally occurring random fields are, in fact, either evolving over time or have arrived at the state in which we observe them by development through time. There are many examples where an evolving random field model is appropriate and weak limit theory is needed. Picture empirical data being gathered on a d-dimensional screen through real time. This example has been studied extensively as a random field, with time fixed at the point when sampling ends, but is also of interest as an evolving random field. A limit result can now be stated which allows dependence in both space and time of the empirical data. Another kind of example is a measure-valued process obtained from a branching diffusion. Here a functional limit theorem for the evolving random field has not yet been obtained (using our topology) but the conditions to be checked can be seen using our Theorem 2.1. Further applications arise related to e.g. U-statistics, chi-squared tests for spatial processes, and robust estimation for autoregressive process (see Koul (1989)).

An evolving random field has a natural partial ordering along the time axis. Each finite collection of points $\underset{\sim}{x} = (x_1,\dots,x_k) \subset X$ corresponds to a sequence of k-dimensional stochastic processes in the time parameter:

$$S_n(\underset{\sim}{x},t) = (S_n(x_1,t),\dots,S_n(x_k,t)),\ t \in [0,1].$$

The proof of weak convergence of evolving random fields will reduce to proving *finite-dimensional convergence* (i.e. weak convergence of $\{S_n(\underset{\sim}{x},t) : t \in [0,1]\}$) and *tightness* of $\{S_n(x,t) : x \in X,\ t \in [0,1]\}$. (See Theorem 2.1.)

If each $S_n(\underset{\sim}{x},t)$ is a Markov process in t we can prove weak convergence using the theory found, e.g. in the book of Ethier and Kurtz (1986). For a variety of models, results about weak convergence of interacting particle systems as in the book of Liggett (1985) might be used. If $S_n(\underset{\sim}{x}, t)$ is a sequence of semimartingales with respect, e.g., to the filtration $F_{n,t} = \sigma\{S_n(x,s) \ : x \in X,\ s \le t\ \}$ then conditions for convergence of the finite-dimensional processes $S_n(\underset{\sim}{x}, t)$ appear in the book of Jacod and Shiryaev (1987).

In this paper we find conditions for weak convergence of a sequence of evolving random fields where the finite-dimensional processes $\{S_n(\underset{\sim}{x}, t) \ : t \in [0,1]\}$, are semimartingales. The sense of weak convergence here is different from and, in fact, much stronger than the sense usually employed in the subject called "measure-valued processes" or "super-processes." There, the processes are considered as *indexed* by t, only, and *evaluated* on x. The topology relative to which convergence is proved is then much weaker than the one we consider and in fact, by a theorem of Mitoma (c.f. Walsh (1984)), weak convergence of $S_n(x,t)$, for fixed $x \in X$ is sufficient for tightness of the "super-processes." Here our main challenge will be to prove tightness. The weak convergence obtained is, of course, correspondingly more useful.

The requisite tightness, or asymptotic equicontinuity, component of the weak convergence of $\{S_n(x,t) : x \in X, t \in I\}$ is obtained by restricting the complexity of X with a *majorizing measure condition* in conjunction with a *bracketing condition.* The concept of majorizing measures dates back to the early 1970's, see Preston (1972) and Fernique (1974). Fernique (1974) gave a characterization of X, together with its canonical metric, in terms of majorizing measures which was sufficient for the a.s. continuity of a Gaussian process $\{Z(x) : x \in X\}$. Quite recently Talagrand (1986) gave a necessary condition for the a.s. continuity of Z in terms of majorizing measures. (See Theorem 1.1.) The use of a majorizing measure condition in this context allows more local variability in the richness of X than a metric entropy condition would allow.

Now let $\Delta_{n,i}^{(\delta)}(\ \cdot\ ) = \left\{ \sup_{y:\sigma(\ \cdot\ , y) < \delta} |Y_{n,i}(y) - Y_{n,i}(\ \cdot\ )| \right\}^*$ where, for $G \in B(X)$, G^* denotes the P-outer envelope of G. Conditions which restrict the magnitude of the $\Delta_{n,i}^{(\delta)}$'s are termed *bracketing conditions.* The use of bracketing conditions first arose in the study of the central limit theorem for the set-indexed empirical measure, under the name *inclusion conditions* for families of sets. Dudley (1978) was one of the first to use them in developing tightness results for empirical processes; tying the L_P variability of the brackets (for $p > 2$) to constraints on metric entropy. The development of a new method of proof of asymptotic equi-continuity allowed Ossiander (1985) and (1986) to impose only L_2 constraints on the variability of the brackets through the use of a simple metric entropy condition. Using a refinement of the same method, while relying upon the recent work of Talagrand (1987), this last result was further improved by Andersen, Giné, Ossiander and Zinn (AZOG) (1988) by linking constraints on the weak-L_2 variability of the brackets to (natural) majorizing measure conditions. Ledoux and Talagrand (1989) have a comprehensive study of recent results in this area. A key ingredient of all of the above work is the assumption of independence of the $Y_{n,i}$'s which is not made here. Instead the $Y_{n,i}$'s are taken to be measurable with respect to an increasing filtration and the following key assumptions on the $Y_{n,i}$'s are made:

(i) uniform *conditional* asymptotic negligibility,

(ii) uniform *conditional* asymptotic centering,

(iii) a *conditional* bound on the weak-L_2 norm, uniform over a collection of brackets.

Thus in this paper functional central limit theorems for evolving random fields with a rather general dependence structure are obtained for the first time.

In section 1 we introduce notation and discuss the continuity of Gaussian processes. Some relevant results involving majorizing measures are presented. A definition of weak convergence and conditions implying the weak convergence of evolving random fields are given in section 3. This section contains a statement of our main central limit theorem, as well as several variants. Finite dimensional convergence and some characteristics of the limiting Gaussian process are discussed in section 3. Section 4 contains some exponential probability bounds for martingales. The crucial tightness result is proved in section 5. It depends on an exponential probability bound which may be of independent interest.

1. Notation, Definitions, and Preliminaries. Let(Ω, M, P) be a complete probability space and let **X** be some (arbitrary) topological space. Let B(**X**) denote the set of all bounded real-valued functions on **X** and **B**(**X**) denote the smallest σ-field containing all sets of the form $\{f \in B(X) : f(x_j) \in B_j, j = 1, \ldots, m\}$ where m is an arbitrary integer, the x_j are points in **X**, and the B_j are half-open intervals in $\mathbb{R}$.

Definition 1.1. A *random field*, Y, on X is a measurable mapping Y from (Ω, M) into (B(X), **B**(X)). Thus Y(x,w) denotes the value of the function in B(X) picked out by w at the point x. Generally, we suppress w through-out and simply write Y(x).

We assume throughout that the random fields we work with are separable in the following sense.

Definition 1.2. A *separable* random field is a random field, Y, for which there exists a countable set $D \subset X$ and a fixed event N for which P(N) = 0, such that for any closed interval $C \subset R$ and open set $U \subset X$ the two sets $\{w: Y(x,w) \in C,\ x \in U\}$ and $\{w : Y(x,w) \in C,\ x \in U \cap D\}$ differ (at most) by a subset of N.

For each $n \geq 1$, let $\{F_{n,i} : i \geq 1\}$ be an increasing filtration on the probability space (Ω, F, P). Let $F_{n,0} = \{\emptyset, \Omega\}$ be the trivial σ-algebra. For each $n \geq 1$ and $t \in I = \{t : 0 \leq t \leq 1\}$ let $\tau(n,t)$ be a stopping time with respect to $\{F_{n,i} : i \geq 0\}$. Also let $\{Y_{n,i} : i \geq 1\}$ be an array of separable random fields on **X** with each $Y_{n,i}$ being measurable with respect to $F_{n,i}$. Let

$$S_n(x,t) = \sum_{1 \leq i \leq \tau(n,t)} Y_{n,i}(x) \text{ for } x \in X \text{ and } t \in I.$$

Note that S_n is a random field on $X \times I$.

The next definition is standard.

Definition 1.3. A (centered) *Gaussian* random field on **X** is a random field on **X** for which each finite linear combination $\sum_{1 \leq j \leq m} a_j Y(x_j)$ is a centered Gaussian random variable. That is, each linear combination has a density of the form $(2\pi\sigma^2)^{-1/2} \exp\{-y^2/2\sigma^2\}$ for some $\sigma > 0$.

We utilize the following two definitions in discussing our limiting process. Let $I = \{t : 0 \leq t \leq 1\}$ denote the unit inteval, and let $\{F_t : 0 \leq t \leq 1\}$ be an increasing filtration contained in F. That is, $F_s \subset F_t \subset F$ for $0 \leq s < t \leq 1$.

Definition 1.4. A F_t-*evolving* random field, Z, on $X \times I$ is a random field on $X \times I$ for which $Z(\cdot,t)$ is measurable with respect to F_t for each $t \in I$ and has independent increments in **I** uniformly over **X**; that is for any x, $y \in X$, Z(x,t) is independent of Z(y,s) - Z(y,t) for $0 \leq t < s \leq 1$.

Definition 1.5. A *standard* F_t-evolving random field, **Z**, on $X \times I$ is a F_t-evolving random field on $X \times I$ for which

$$EZ(x,t) = 0,$$

$$EZ^2(x,0) = 0.$$

and

$$E(Z(x,s) - Z(x,t))^2 = |s - t|\, EZ^2(x,1)$$

for all $x \in X$ and $s, t \in I$. We say that Z is a *(standard) evolving random field* if Z is a (standard) F_t-evolving random field for some increasing filtration $\{F_t : 0 \leq t \leq 1\}$ contained in F.

Clearly a Gaussian random field Z on $X \times I$ is a standard evolving random field if and only if $Z(x, \cdot)/(EZ^2(x,1))^{1/2}$ is a standard Brownian motion for each fixed x.

The following two definitions are due to Fernique (1974).

Definition 1.6. Let d be a (pseudo-) metric on **X**. A Borel (sub-) probability measure ν on **X** is a *majorizing measure* for **X** with respect to d if

$$\sup_{x \in X} \int_0^\infty ln^{1/2}\ (1/\nu(B_d(x,u)))\, du < \infty. \tag{1.1}$$

Here, $B_d(x,u) = \{y \in X : d(x,y) < u\}$ is the u-ball in (**X**,d) centered at x.

Definition 1.7. A sub-probability measure μ is a *discrete majorizing measure* if it is a majorizing measure with support being a countably dense subset of **X**.

Let $\{Z(x) : x \in X\}$ be a mean zero Gaussian process on the space **X**. Let d denote the canonical L_2 pseudo-metric generated on **X** by **Z**, so

$$d(x,y) = (E(Z(x) - Z(y))^2)^{1/2}. \tag{1.2}$$

The following theorem is due to Fernique (1974) and Talagrand (1987).

Theorem 1.1. Suppose that the Gaussian process $\{Z(x) : x \in X\}$ is separable in (**X**,d). Then: (i) (Fernique) If (**X** ,d) admits a majorizing measure, ν, then Z has bounded sample paths a.s.. If, in addition,

$$\lim_{\delta \to 0} \sup_{x \in X} \int_0^\delta ln^{1/2}(1/\nu\ (B_d(x,u)))\, du = 0 \tag{1.3}$$

then the sample paths of Z are uniformly continuous with respect to d a.s.

(ii) (Talagrand) If Z has bounded sample paths a.s., then (X,d) admits a majorizing measure. If the sample paths of Z are uniformly continuous with respect to d a.s. then (X,d) admits a majorizing measure ν, satisfying (1.3.).

It is easy to see that if (X,d) admits a majorizing measure ν, it also must admit a discrete majorizing measure μ. If (1.3) holds for ν, the following lemma, due to Andersen (1986), allows us to assume that μ has a particular structure.

Lemma 1.1 If (X,d) admits a majorizing measure for which (1.3) holds, then for any $\beta \in (0,1)$ there exists a discrete majorizing measure μ satisfying the following. For each $x \in X$ and $k \geq 0$,

$$\text{there exists } x_k \text{ in the support of } \mu \text{ with } d(x,x_k) < \beta^k, \tag{1.4}$$

and

$$\lim_{K \to \infty} \sup_{x \in X} \sum_{k \geq K} \beta^k ln^{1/2}(1/\mu(x_k)) = 0. \tag{1.5}$$

We may also assume that

$$\text{if } x_k = y_k \text{ for a pair } x, y \in X, \text{ then } x_j = y_j \text{ for } j < k. \tag{1.6}$$

Furthermore

$$\lim_{K \to \infty} \sup_{x \in X} \sum_{k \geq K} \beta^k ln^{1/2}(1/\beta^k \prod_{j \leq k} \mu(x_j)) = 0. \tag{1.7}$$

Notice that (1.5) is the discrete analogue of (1.3), and that (1.7) follows easily from (1.5) using Kronecker's Lemma and interchange of summation. The following relationship between majorizing measures and metric entropy is useful.

Definition 1.8. The metric entropy of X with respect to d is given by

$H(\delta,X,d) = \ln \min\{N :$ there exists $x_1, ..., x_n \in X$ such that $d(x_i,x) < \delta$ for every $x \in X$ for some $i = 1, ..., n\}$.

It is easy to verify the following lemma.

Lemma 1.2. If

$$\int_0^1 H^{1/2}(u, X, d)\, du < \infty \tag{1.8}$$

then there exists a majorizing measure μ such that (1.3) holds.

2. Weak Convergence and the Central Limit Theorem. In this section we see how weak convergence of $\{S_n(x,t) : x \in X,\ t \in I\}$ follows from finite-dimensional convergence and asymptotic equi-continuity (tightness). We use the definitions and viewpoint of Gaenssler and Schneemeier (1986). An alternate approach using the Hoffmann-Jorgensen definition of weak convergence (which would yield a slightly more general result) is that of Andersen and Dobrić (1987). Results of later sections are called in as needed in order to state our main results here.

Let $l^\infty(X,I) = \{f : X\times I \to \mathbf{R} : \sup_{(x,t)\in X\times I} |f(x,t)| < \infty\}$ denote the space of real-valued bounded functions on $X\times I$ and equip it with the supremum norm, so that $\|f\|_{X\times I} = \sup_{(x,t)\in X\times I} |f(x,t)|$. For any metric r on $X\times I$, let $g_r = \{f \in l^\infty(X,I) :$ f is uniformly r-continuous on $X\times I\}$. For a mean zero standard evolving Gaussian process $\{Z(x,t) : x\in X, t\in I\}$, let σ be the canonical L_2 metric on $X\times I$ given by

$$\sigma((x,s),(y,t)) = (E(Z(x,s)-Z(y,t))^2)^{1/2}. \tag{2.1}$$

Let d be the metric on **X** given by

$$d(x,y) = \sigma((x,1)(y,1)) \tag{2.2}$$

and ρ be the metric on **I** given by

$$\rho(s,t) = |s-t|^{1/2} \sup_{x\in X} (\mathrm{Var}\ Z(x,1))^{1/2}. \tag{2.3}$$

Letting τ denote the metric on $X\times I$ given by

$$\tau((x,s),(y,t)) = d(x,y) + \rho(s,t), \tag{2.4}$$

it is easy to see that τ dominates σ. If Z is separable and uniformly $\sigma(\tau)$-continuous, then from Talagrand's result (Theorem 1.1 (ii)) we know that $(X\times I,\sigma)$ $((X\times I,\tau))$ is totally bounded. Then $g_\sigma(g_\tau)$ is separable and closed in $(l^\infty(X\times I), \|\cdot\|)$; c.f. Corollary 2 of Gaenssler and Schneemeier (1986). Let **B** be the σ-algebra of all Borel sets in $(l^\infty(X\times I), \|\cdot\|)$ and let $\mathbf{B}_b$ denote the sub-σ-algebra of **B** generated by the open $\|\cdot\|$-balls in $l^\infty(X\times I)$.

Let $\{S_n : n \geq 1\}$ and Z all be defined on a common probability space (Ω, F, P), with each S_n being an element of $(X\times I, \mathbf{B})$ and Z being an element of $(l^\infty(X\times I), \mathbf{B}_b)$.

Definition 2.1. The sequence S_n *converges in distribution* to Z ($S_n \xrightarrow{D} Z$) if and only if

$$P(Z \in g_\sigma) = 1 \tag{2.5}$$

and

$$\lim_{n \to \infty} Ef(S_n) = Ef(Z) \tag{2.6}$$

for all $f \in \{g : \mathbf{X} \times \mathbf{I} \to R$: g is bounded, uniformly $\|\cdot\|$-continuous and $\mathbf{B}_b$-measurable$\}$.

Theorem 2.1. If $\{Z(x,t) : x \in \mathbf{X}, t \in \mathbf{I}\}$ is a standard Gaussian evolving random field with

(X,d) admitting a majorizing measure which satisfies (1.3), (2.7)

for any finite collection $x_1, ..., x_k \in \mathbf{X}$

$$(S_n(x_1, \cdot), ..., S_n(x_k, \cdot)) \xrightarrow{D} (Z(x_1, \cdot), ..., Z(x_k, \cdot)), \tag{2.8}$$

and, for all $\eta > 0$,

$$\lim_{\delta \to 0} \overline{\lim_{n}} P\left[\sup_{t \in I, d(x,y) < \delta} |S_n(x,t) - S_n(y,t)| > \eta\right] = 0 \tag{2.9}$$

then $S_n \xrightarrow{D} Z$.

Proof: Condition (2.8) of the theorem identifies the limiting distribution and using Lemma 3.2, (2.7) guarantees its continuity. In view of the domination of σ by τ, Theorem 2.12 of Andersen and Dobrić (1987), and Theorem 1 of Gaenssler and Schneemeier (1986), it remains for us to show that (2.9) (together with (2.6), (2.7), and (2.8)) imply that for any $\eta > 0$,

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} P^* \left[\sup_{1 \le i \le I_\delta, \tau((x_i,t_i),(x,t)) < \delta} |S_n(x_i,t_i) - S_n(x,t)| > \eta\right] = 0,$$

where $\{(x_i,t_i) : 1 \le i \le I_\delta\}$ is a (finite) δ-net in $(\mathbf{X},\tau)$. Note that

$$B_\tau((x,t),\delta) \subset B_d(x,\delta) \times B_\rho(t,\delta)$$

and

$$|S_n(x_i,t_i) - S_n(x,t)| \le |S_n(x_i,t_i) - S_n(x_i,t)| + |S_n(x_i,t) - S_n(x,t)|.$$

Thus

$$P^*[\sup_{1 \le i \le I_\delta, \tau((x_i,t_i),(x,t)) < \delta} |S_n(x_i,t_i) - S_n(x,t)| > \eta]$$

$$\le P^*[\sup_{1 \le i \le I_\delta, d(x_i,x) < \delta, t \in \mathbf{I}} |S\ (x_i,t) - S_n(x,t)| > \eta/2]$$

$$+ P^*[\sup_{1 \le i \le I_\delta, \rho(t_i,t) < \delta} |S_n(x_i,t_i) - S_n(x_i,t)| > \eta/2]$$

$$\le P^*[\sup_{t \in \mathbf{I}, d(x,y) < \delta} |S_n(x,t) - S_n(y,t)| > \eta/2]$$

$$+ P^*[\sup_{1 \le i \le I_\delta, \rho(s,t) < \delta} |S_n(x_i,s) - S_n(x_i,t)| > \eta/2].$$

From (2.9), if we let n go to infinity, and then δ go to zero, the first term goes to zero. From (2.8) (together with (2.6)),

$$\lim_{n \to \infty} P^* \left[\sup_{1 \le i \le \mathbf{I}_\delta, \rho(s,t) < \delta} |S_n(x_i,s) - S_n(x_i,t)| > \eta/2 \right]$$

$$= P^* \left[\sup_{1 \le i \le \mathbf{I}_\delta, \rho(s,t) < \delta} |Z(x_i,s) - Z(x_i,t)| > \eta/2 \right].$$

Now from (2.5) we see that this term decreases to zero as δ decreases. □

Notice that a result parallel to Theorem 2.1 is easily available for many cases when the limiting distribution is that of a general evolving Gaussian random field. The needed ingredient is a metric τ, dominating the canonical metric σ, with τ, being of the form given in (2.4), where d is as defined in (2.2) and $\rho(s,t)$ $= \sup_{x \in \mathbf{X}} (E(Z(x,s) - Z(x,t))^2)^{1/2}$ with Z(x,t) being uniformly ρ-continuous in t over **X**.

The following corollaries are easy consequences of Theorem 2.1 and Theorem 5.1, 5.2, and 5.3 respectively.

Corollary 2.1. If (2.7) holds,

(3.1), (3.2), and (3.3) of Lemma 3.1 (finite-dimensional convergence) are satisfied, (2.10)

and

conditions (5.7), (5.8), (5.9) and (5.10) of Theorem 5.1 giving the uniform modulus of continuity are satisfied, (2.11)

then $S_n \xrightarrow{D} Z$.

Note that the definition of convergence in distribution implies that the Gaussian process **X** is a.s. σ-continuous on $\mathbf{X} \times \mathbf{I}$. Other variants are:

Corollary 2.2. If (2.7) and (2.10) hold, as well as

(5.12) and (5.13) of Theorem 5.2, (2.12)

then $S_n \xrightarrow{D} Z$.

Corollary 2.3. If (2.7) and (2.10) hold, as well as

(5.12) and (5.14) of Theorem 5.3, (2.13)

then $S_n \xrightarrow{D} Z$.

The following three corollaries are easy consequences of the three corollaries above and Lemma 1.2. They replace the majorizing measure condition (1.3) with the metric entropy condition (1.8).

Corollary 2.4. If (1.8), (2.10), and (2.11) hold, then $S_n \xrightarrow{D} Z$.

Corollary 2.5. If (1.8), (2.10), and (2.12) hold, then $S_n \xrightarrow{D} Z$.

Corollary 2.6. If (1.8), (2.10), and (2.13) hold, then $S_n \xrightarrow{D} Z$.

Notice that if a weak convergence result for random fields whose domain lacks a temporal component is of interest, a simpler version of Theorem 2.1 (and consequently Corollaries 2.1 through 2.6 that follow) is readily available. In particular, consider the random field

$$S_n(x) = \sum_{i \le \tau(n)} Y_{n,i}(x) \text{ for } x \in \mathbf{X},$$

where the $Y_{n,i}$'s are measurable with respect to an increasing filtration and $\tau(n)$ is a stopping time. Think of this as reducing the temporal component of the domain to a single value; i.e. $\mathbf{I} = \{1\}$. Conditions (2.8) and (2.9) of Theorem 2.1

would thus be simplified, and Corollaries 2.1 through 2.6 would hold as stated for $\mathbf{I} = \{1\}$ throughout.

3. Finite Dimensional Convergence and the Limiting Process. In the case in which the prelimiting random fields have independent "increments" in t, e.g. for $0<s<t, S_n(s, \cdot), S_n(t, \cdot)-(S_n \cdot s,)$ are independent processes, the usual central limit theorem can be used to identify the limit law which will also, of course, have independent increments. However when a more complex 'temporal' dependence structure exists in S_n the identification of the distribution of the limiting becomes a serious problem. Here we state sufficient conditions for identification of a more general Gaussian limit by calling in general results for weak convergence of $\mathbf{R}^k$ -valued semimartingales. The following lemma is a reformulation of e.g. Thm 2.27 of Jacod and Shiryaev (1987).

Lemma 3.1. (Convergence of finite-dimensional distributions in **X**.) If for all $t \in \mathbf{I}$, $\varepsilon \in (0,1]$, $\underset{\sim}{a} = (a_1,\ldots,a_k) \in \mathbf{R}^k$, and $\underset{\sim}{x} = (x_1,\ldots,x_k) \in \mathbf{X}^k$ for $k \geq 1$,

$$\sum_{1 \le i \le \tau(n,t)} P\left(\left|\sum_{1 \le j \le k} a_j Y_{n,i}(x_j)\right| > \varepsilon \,\middle|\, F_{n,i-1}\right) \xrightarrow{P} 0, \tag{3.1}$$

$$\sup_{t \in \mathbf{I}} \sum_{1 \le i \le \tau(n,t)} E\left(\sum_{1 \le j \le k} a_j Y_{n,i}(x_j)\, 1_{\left[\left|\sum_{1 \le j \le k} a_j Y_{n,i}(x_j)\right| \le \varepsilon\right]} \,\middle|\, F_{n,i-1}\right) \xrightarrow{P} 0, \tag{3.2}$$

and

$$\begin{aligned}\sum_{1 \le i \le \tau(n,t)} &\left(E\left(\left(\sum_{1 \le j \le k} a_j Y_{n,i}(x_j)\right)^2 1_{\left[\left|\sum_{1 \le j \le k} a_j Y_{n,i}(x_j)\right| \le \varepsilon\right]} \,\middle|\, F_{n,i-1}\right)\right. \\ &\left. -\left(E\left(\sum_{1 \le j \le k} a_j Y_{n,i}(x_j)\, 1_{\left[\left|\sum_{1 \le j \le k} a_j Y_{n,i}(x_j)\right| \le \varepsilon\right]} \,\middle|\, F_{n,i-1}\right)\right)^2\right) \\ &\xrightarrow{P} t \operatorname{Var}\left(\sum_{1 \le j \le k} a_j Z(x_j, 1)\right),\end{aligned} \tag{3.3}$$

where $\{Z(x,1) : x \in \mathbf{X}\}$ is a Gaussian process, then the finite-dimensional distributions in **X** of $\{S_n(\cdot,t) : t \in \mathbf{I}\}$ converge to those of $\{Z(\cdot, t) : t \in \mathbf{I}\}$ where $Z(\cdot,t)$ is a standard evolving Gaussian process.

The finite-dimensional convergence in **X** above gives us a candidate for our limiting process, as well as the canonical metric σ (see 2.1) on the product space

$\mathbf{X}\times\mathbf{I}$. Recall also the definitions of the metrics d, ρ, and τ given in (2.2), (2.3) and (2.4) respectively of section 2. The following lemma gives a proof of the continuity of the limiting distribution under the natural majorizing measure condition.

Lemma 3.2. If the mean-zero Gaussian random field $\{Z(x,1) : x \in \mathbf{X}\}$ is separable and d-continuous a.s., then the standard evolving Gaussian random field $\{Z(x,t) : x \in \mathbf{X}, t \in \mathbf{I}\}$ is σ-continuous and τ-continuous a.s..

Proof. First notice that for all $\delta > 0$, $x \in \mathbf{X}$ and $s \in \mathbf{I}$,

$$B_d(x, \delta/2) \times B_\rho(s, \delta/2) \subset B_\tau((x,s), \delta). \tag{3.4}$$

From Theorem 1.1, (**X** ,d) admits a majorizing measure ν satisfying

$$\lim_{\delta\to 0} \sup_{x\in\mathbf{X}} \int_0^\delta ln^{1/2}(1/\nu(B_d(x,u)))\,du = 0.$$

Let m be the product measure $\nu\times\lambda$ on $\mathbf{X}\times\mathbf{I}$ where λ is Lebesgue measure. Then, setting $c = \sup_{x\in\mathbf{X}} (\mathrm{Var}Z(x,1))^{1/2}$, it follows from (3.4) that

$$\int_0^\delta ln^{1/2}(1/m\ (B_\tau((x,t),u))\,du \le \int_0^\delta ln^{1/2}(1/\lambda\ \ (B_\rho(t,u/2))\ \ \nu(B_d(x,u/2)))\,du$$

$$\le \int_0^\delta ln^{1/2}(1/cu^{1/2})\,du + \int_0^\delta ln^{1/2}(1/\nu(B_d(x,u/2)))\,du,$$

and thus ,

$$\lim_{\delta\to 0} \sup_{(x,t)\in\mathbf{X}\times\mathbf{I}} \int_0^\delta ln^{1/2}(1/m(B_\tau((x,t),u)))\,du = 0. \tag{3.5}$$

Since $B_\tau((x,t),u) \subset B_\sigma((x,t),u)$, clearly (3.5) holds also with the metric τ replaced by σ. The result follows from Theorem 1.1. □

Again a parallel result is easily available for many general evolving Gaussian processes. (See comment following Theorem 2.1.)

4. Exponential Probability Bounds for Martingale Differences. This section contains some inequalities which play an essential part in this derivation of our tightness results. Throughout we assume that $\{Y_i : i \ge 1\}$ is a sequence of real-valued random variables on the probability space (Ω, F, P) which are measurable with respect to an increasing filtration $\{F_i : i \geq 1\}$ with $F_i \subset F$ for each i. We let $F_0 = \{\phi, \Omega\}$ denote the trivial σ-algebra.

The following proposition, a Bernstein-type inequality for martingales, due to Freedman (1973), is a basic ingredient of most of our probability bounds.

Proposition 4.1. . Suppose that $P(\sup_{i \geq 1} |Y_i| > a) = 0$ and $E(Y_i | F_{i-1}) = 0$. For $n \geq 1$, let $V_n = \sum_{1 \leq i \leq n} E(Y_i^2 | F_{i-1})$. Then, for all $\eta, \delta > 0$,

$$P\left[\sum_{1 \leq i \leq n} Y_i > \eta, V_n \leq \delta \text{ for some } n \geq 1\right] \leq [\delta / (a\eta + \delta)]^{(a\eta+\delta)/a^2} e^{n/a}$$

$$\leq \exp\{-\eta^2/2\,(a\eta + \delta)\}.$$

The following simple lemma proves useful. A variant of it may be found in AGOZ (1988).

Lemma 4.1. Let $\{c_i : i \geq 1\}$ be a decreasing sequence of non-negative real numbers. For any $b > 0$ and $n \geq 1$,

$$\sup_{a > 0} a \sum_{1 \leq i \leq n} c_i 1_{[a < c_i \leq b]} \leq 2 \sup_{1 \leq i \leq n} i c_i^2 1_{[c_i \leq b]}.$$

Proof: Notice first that for any $j \geq 1$,

$$\sum_{1 \leq i \leq j} c_i 1_{[c_i \leq b]} = \sum_{1 \leq i \leq j} (i^{1/2} c_i 1_{[c_i \leq b]})\, i^{-1/2}$$

$$\leq \sup_{1 \leq i \leq j} i^{1/2} c_i 1_{[c_i \leq b]} \left(\sum_{1 \leq i \leq j} i^{-1/2}\right)$$

$$\leq 2j^{1/2} \sup_{1 \leq i \leq n} i^{1/2} c_i 1_{[c_i \leq b]}.$$

Fix $a > 0$, and let $j_a = \{j : c_j \geq a > c_{j+1}\}$. Then

$$a \sum_{1 \leq i \leq n} c_i 1_{[a < c_i \leq b]} = a \sum_{1 \leq i \leq j_a} c_i 1_{[a < c_i \leq b]}$$

$$\leq c_{j_a} 1_{[c_{j_a} \leq b]} \sum_{1 \leq i \leq j_a} c_i 1_{[c_i \leq b]}$$

$$\leq 2 j_a^{1/2} c_{j_a} 1_{[c_{j_a} \leq b]} \sup_{1 \leq i \leq n} i^{1/2} c_i 1_{[c_i \leq b]}$$

$$\le 2 \sup_{1 \le j \le n} j^{1/2} c_j 1_{[c_j \le b]} \sup_{1 \le i \le n} i^{1/2} c_i 1_{[c_i \le b]}$$

$$\le 2 \sup_{1 \le j \le n} j c_j^2 1_{[c_j \le b]} . \qquad \square$$

The following lemma is an easy consequence of Freedman's Proposition (4.1).

Lemma 4.2. For any $\eta > \delta > 0$ and sequence of sets $\{A_i : i \ge 1\}$ with $A_i \in F_i$,

$$P\Big(\sum_{1 \le i \le n} 1_{A_i}(Y_i) > \eta,\ \sum_{1 \le i \le n} P(A_i | F_{i-1}) \le \delta \text{ for some n} \Big) \le (e\delta|\eta)^{\eta}.$$

Proof: Since $|1_{A_i}(Y_i) - P(A_i| F_{i-1})| \le 1$, $E(1_{A_i}(Y_i) - P(A_i| F_{i-1})) = 0$, and $E((1_{A_i}(Y_i) - P(A_i | F_{i-1}))^2 | F_{i-1}) \le P(A_i| F_{i-1})$, we may apply Proposition 4.1 above to see that

$$P\Big(\sum_{1 \le i \le n} 1_{A_i}(Y_i) > \eta \text{ and } \sum_{1 \le i \le n} P(A_i| F_{i-1}) \le \delta \text{ for some n}\Big)$$

$$\le P\Big(\sum_{1 \le i \le n} (1_{A_i}(Y_i) - P(A_i| F_{i-1})) > \eta - \delta \text{ and } \sum_{1 \le i \le n} P(A_i| F_{i-1}) \le \delta \text{ for some n}\Big)$$

$$\le (\delta/\eta)^{\eta} e^{\eta}. \qquad \square$$

The following lemma is essential in proving our tightness theorem. It is basically an extension of Lemma 2.16 of AGOZ (1988) to sums whose increments are martingale differences.

Lemma 4.3. Suppose that the Y_i 's are non-negative and let the set

$$\Lambda = \Big[\sup_{a > 0} a^2 \sum_{i \ge 1} P(Y_i > a| F_{i-1}) \le \delta \Big].$$

For all $\eta > \delta$,

$$P\Big(\Big[\sup_{a > 0} a^2 \sum_{1 \le i \le n} Y_i 1_{[a < y_i \le b]} > 2\eta \text{ for some n}\Big] \cap \Lambda \Big)$$

$$\le (e\delta/\eta)^{\eta/b^2}/(1 - e\delta/\eta).$$

Proof: Fix $n \ge 1$ and let $Y_{(i)}$ be the *decreasing* order statistics of $\{Y_1, ..., Y_n\}$. Using Lemma 4.1, we see that for $\eta > 0$

$$\Big[\sup_{a > 0} a \sum_{1 \le i \le n} Y_i 1_{[a < Y_i \le b]} > 2\eta \Big] \subset \Big[\sup_{1 \le i \le n} i Y_{(i)}^2 1_{[Y_{(i)} \le b]} > \eta \Big]$$

$$\subset \Big[\sup_{1 \le i \le n} iY_{(i)}^2 1_{[Y_{(i)} \le b]} > \eta\Big]$$

$$= \bigcup_{\eta/b^2 < i \le n} \big[Y_{(i)}^2 1_{[Y_{(i)} \le b]} > \eta/i\big]$$

$$\subset \bigcup_{\eta/b^2 \le i < n} \Big[\sum_{1 \le j \le n} 1_{[Y_j > (\eta/i)^{1/2}]} \ge i\Big].$$

Notice that, for each $i \ge 1$,

$$\Lambda \subset \Big[(\eta/i) \sum_{1 \le j \le n} P(Y_j > (\eta/i)^{1/2} | F_{j-1}) \le \delta \text{ for all } n\Big]$$

$$= \Big[\sum_{1 \le j \le n} P(Y_j > (\eta/i)^{1/2} | F_{j-1}) \le \delta i\ \eta \text{ for all } n\Big].$$

Thus, using Lemma 4.2, we see that

$$P\Big(\Big[\sup_{a > 0} a \sum_{1 \le i \le n} Y_i 1_{[a < Y_i \le b]} > 2\eta \text{ for some } n\Big] \cap \Lambda\Big)$$

$$\le \sum_{i > \eta/b^2} P\Big(\Big[\sum_{1 \le j \le n} 1_{[Y_j > (\eta/i)^{1/2}]} \ge i \text{ for some } n\Big] \cap \Lambda\Big)$$

$$\le \sum_{i \le \eta/b^2} P\Big(\sum_{1 \le j \le n} 1_{[Y_j > (\eta/i)^{1/2}]} \ge i \text{ and}$$

$$\sum_{1 \le j \le n} P(Y_j > (\eta/i)^{1/2} | F_{j-1}) \le \delta i\ \eta \text{ for some } n\Big)$$

$$\le \sum_{i > \eta/b^2} (e\delta/\eta)^i$$

$$< (e\delta/\eta)^{\eta/b^2} / (1 - e\delta/\eta).$$

5. Tightness. We assume throughout this section that the metric space (X,d) admits a majorizing measure m for which

$$\lim_{\delta \to 0} \sup_{x \in X} \int_0^\delta ln^{1/2}(1/m(B_d(x,u)))\,du = 0. \tag{5.1}$$

Thus, in addition, we have, for each $\beta > 0$, a discrete majorizing measure μ on (X,d) for which (1.4) through (1.7) hold. Fix $\beta > 0$ and, in the notation of Lemma 1.1, set $\mathbf{X}_j = \{x_j : x \in \mathbf{X}\}$, so that $\mathbf{X}_j$ is a finite β^j net in X.

Let $\Delta_{n,i}^{(j)}(x) = \sup_{y:y_j = x_j} |Y_{n,i}(x_j) - Y_{n,i}(y)|$.

Thus $\Delta_{n,i}^{(j)}$ is a local modulus of continuity for $Y_{n,i}$ which is constant on the set $\{y : y_j = x\}$ for $x \in \mathbf{X}_j$. Due to the structure of the x_j's specified by (1.6), we may assume that $\Delta_{n,i}^{(j)}(x)$ decreases in j for each fixed x.

Before continuing, we introduce some technical definitions involving the discrete majorizing measure μ. Fix $0 < \beta < 1$. For $x \in \mathbf{X}$ and $j \geq 0$, set $m_j(x) = \beta^j \prod_{k \leq j} \mu(x_k)$. Define $m : X \times (0,1] \to (0,1]$ as follows:

$$m(x,u) = m_{j+1}(x) + (u - \beta^{j+1})(m_j(x) - m_{j+1}(x))/(\beta^j - \beta^{j+1})$$

for $\beta^{j+1} \leq u < \beta^j$. Notice that, for each fixed x, $m(x,\cdot)$ is a continuous and strictly increasing function with $m(x,u) = m(x_j, u)$ for $\beta^j \leq u \leq 1$. Also, since μ is a majorizing measure, we have $\sum_{x \in \mathbf{X}_j} m(x_j) \leq \beta^j$. Let $g : \mathbf{X} \times (0,1] \to \mathbf{R}^+$ be defined by

$$g(x,u) = ln(1/m(x,u)). \tag{5.2}$$

For each fixed x, the function $g(x,\cdot)$ is continuous and strictly decreasing with $g(x,u) = g(x_j, u)$ for $\beta^j \leq u \leq 1$. If the metric entropy integral condition (1.8) holds, we may assume that m(x,u) (and thus g(x,u)) depend only on u and not on x. Thus, under the assumption of (1.8) rather than (5.1) most of the conditions that follow can be stated much more simply.

The following sets $\Lambda_j^{(1)}$ and $\Lambda_j^{(2)}$ are respectively the random function analogues of the sets Λ of Lemma 4.3 and $[V_n \leq \delta]$ of Proposition 4.1. For each fixed n, denote the entire sequence $\{Y_{n,i} : i \geq 1\}$ by Y_n.

For $j \geq 0$, let

$$\Lambda_j^{(1)}(Y_n) = [\sup_{t \in \mathbf{I}} \sup_{a > 0} a^2 \sum_{1 \leq i \leq \tau(n,t)} P(\Delta_{n,i}^{(k)}(x) > a | F_{n,i-1}) > \beta^{2k} \quad \text{for some } x \in \mathbf{X}, k \geq j] \tag{5.3}$$

for $b > 0$

$$\Lambda_j^{(2)}(Y_n, b) = [\sup_{t \in \mathbf{I}} \sum_{1 \leq i \leq \tau(n,t)} E((Y_{n,i}(x_k) - Y_{n,i}(x_{k-1}))^2 \tag{5.4}$$

$$1_{[|Y_{n,i}(x_k) - Y_{n,i}(x_{k-1})| \le \beta^k/g^{1/2}(x,\beta^k)]} \Big| F_{n,i-1}) > b\beta^{2k}$$

for some $x \in \mathbf{X}$ and $k > j$],

and for $c > 0$

$$\Lambda_j^{(3)}(Y_{n,c}) = [\sup_{t \in \mathbf{I}} \sum_{1 \le i \le \tau(n,t)} P(\,|\,Y_{n,i}(x_k)| > \beta^j/2g(x,\beta^j)\,|\,F_{n,i-1})$$

$$> cg(x,\beta^k) \text{ for some } x \in \mathbf{X}, k \ge j\,]. \qquad (5.5)$$

Notice that if the increments, $Y_{n,i}$, are independent, then all three sets, $\Lambda^{(1)}, \Lambda^{(2)}$ and $\Lambda^{(3)}$ are non-random.

For $a > 0$, let

$$\phi(y,a) = -a1_{[y \le -a]} + y1_{[|y| < a]} + a1_{[y \ge a]}$$

and for $a : \mathbf{X} \to \mathbf{R}^+$, let

$$Y_{n,i}^{a(\cdot)}(x) = \phi(Y_{n,i}(x), a(x))$$

denote the truncated $Y_{n,i}$ process. Notice that we have allowed the truncation level to depend on x. Again, denote the entire process $\{Y_{n,i}^{a(\cdot)} : i \ge 1\}$ by $Y_n^{a(\cdot)}$, and let

$$S_n^{a(\cdot)}(x,t) = \sum_{1 \le i \le \tau(n,t)} (Y_{n,i}^{a(\cdot)}(x) - E(Y_{n,i}^{a(\cdot)}(x)\,|\,F_{n,i-1})) \qquad (5.6)$$

denote the truncated and conditionally centered partial sum process.

We are ready to state our first tightness theorem.

Theorem 5.1 If (X,d) admits a majorizing measure satisfying (5.1),

$$\sup_{x \in \mathbf{X}, t \in \mathbf{I}} \sup_{1 \le i \le \tau(n,t)} |Y_{n,i}(x)| \xrightarrow{P} 0 \qquad (5.7)$$

for all $a > 0$

$$\sup_{x \in \mathbf{X}, t \in \mathbf{I}} \sup_{1 \le i \le \tau(n,t)} P(|Y_{n,i}(x)| > a\,|\,F_{n,i-1}) \xrightarrow{P} 0 \qquad (5.8)$$

and

$$\sup_{x \in \mathbf{X}, t \in \mathbf{I}} \Big| \sum_{1 \le i \le \tau(n,t)} E(Y_{n,i}(x) 1_{[|Y_{n,i}(x)| \le a]} \,|\, F_{n,i-1}) \Big| \xrightarrow{P} 0, \tag{5.9}$$

for some $b \le e^2 (e^2 - \beta^3)/\beta^4$,

$$\overline{\lim_j}\ \overline{\lim_n}\ P^* \left(\Lambda_j^{(1)}(Y_n) \cup \Lambda_j^{(2)}(Y_n, b)\right) = 0 \tag{5.10}$$

then, for all $\eta > 0$,

$$\overline{\lim_j}\ \overline{\lim_n}\ P^* \left(\sup_{x \in \mathbf{X}, t \in \mathbf{I}} |S_n(x,t) - S_n(x_j, t)| > \eta \right) = 0. \tag{5.11}$$

Theorem 5.1 is a corollary of the following theorem which has slightly weaker but more cumbersome hypotheses.

Theorem 5.2. If $(\mathbf{X},d)$ admits a majorizing measure satisfying (5.1), for all $a > 0$ and $\varepsilon > 0$,

$$\overline{\lim_j}\ P^* \left(\sup_{x \in \mathbf{X}, t \in \mathbf{I}} \left|S_n(x,t) - S_n^a(x,t)\right| > \varepsilon \right) = 0 \tag{5.12}$$

and for some $b \le e^2(e^2 - \beta^3)/\beta^4$ and $c \le e^2(e^2 - \beta^3)/2\beta^4$,

$$\overline{\lim_j}\ \overline{\lim_n}\ P^* \left(\Lambda_j^{(1)}(Y_n) \cup \Lambda_j^{(2)}(Y_n, b) \cup \Lambda_j^{(3)}(Y_n, c)\right) = 0,$$

then for all $\eta > 0$, (5.11) holds.

Conditions (5.10) and (5.13) above should be viewed as the conditional analogue of the weak - L_2 and L_2 bracketing conditions for process with independent increments. (c.f. Ossiander (1986) and AGOZ (1988)).

The following lemma helps bridge the distance between Theorem 5.2 and 5.1.

Lemma 5.1. If (5.7) holds, and, for all $a > 0$, (5.8) and (5.9) hold, then for all $a > 0$ and $\varepsilon > 0$, (5.12) holds.

Proof: For any $x \in \mathbf{X}$, $t \in \mathbf{I}$, and $a > 0$,

$$\begin{aligned} &\left|S_n(x,t) - S_n^a(x,t)\right| \\ &\quad = \Big| \sum_{1 \le i \le \tau(n,t)} \left(Y_{n,i}(x) - Y_{n,i}^a(x) + E(Y_{n,i}^a(x) \,|\, F_{n,i-1})\right) \\ &\le \sum_{1 \le i \le \tau(n,t)} \left|Y_{n,i}(x) - Y_{n,i}^a(x)\right| \end{aligned}$$

$$+\left|\sum_{1\le i\le\tau(n,t)} E(Y^a_{n,i}(x)\,|\,F_{n,i-1})\right|$$

$$\le \sum_{1\le i\le\tau(n,t)} |Y_{n,i}(x)|\,1_{[|Y_{n,i}(x)|>a]}$$

$$+\left|\sum_{1\le i\le\tau(n,t)} E(Y_{n,i}(x)\,1_{[|Y_{n,i}(x)|\le a]}\;|F_{n,i-1})\right|$$

$$+\,a\sum_{1\le i\le\tau(n,t)} P(|Y_{n,i}(x)|>a|\,F_{n,i-1})\,.$$

Thus, for any a, $\eta > 0$,

$$P^*(\sup_{s\in \mathbf{X}, t\in \mathbf{I}} |S_n(x,t)-S^a_n(x,t)|>\eta\,)$$

$$\le P^*\,(\sup_{x\in\mathbf{X}}\sup_{1\le i\le\tau(n,t)} |Y_{n,i}(x)|>a\,)$$

$$+P^*\,(\sup_{x\in\mathbf{X}}\left|\sum_{1\le i\le\tau(n,t)} E(Y_{n,i}(x)\,1_{[|Y_{n,i}(x)|\le a]}|\,F_{n,i-1})\right|>\eta/3\,)$$

$$+P^*\,(\sup_{t\in I}\sum_{1\le i\le\tau(n,t)} P(|Y_{n,i}(x)|>a|\,F_{n,i-1})>\eta/3\mathrm{a}). \qquad \square$$

We now prove Theorem 5.1 using Lemma 5.1 and Theorem 5.2.

Proof of Theorem 5.1: Lemma 5.1 implies that under the hypothesis of Theorem 5.1, (5.12) holds. Note that $g(x,\beta^j)$ is constant on the set $\{y\in \mathbf{X} : y_j = x\}$ so that $\beta^j/2g(x,\beta^j)$ takes on a finite number of values which are bounded below away from 0. Also $g(x,\beta^k)$ increases in k, so

$$\Lambda_j^{(3)}(Y_n,c)\subset \Big[\sup_{x\in \mathbf{X}, t\in \mathbf{I}}\sum_{1\le i\le\tau(n,t)} P(\;|Y_{n,i}(x)|\;>\min_{x\in\mathbf{X}_j}\beta^j/2g(x,\beta^j)\,|\,F_{n,i-1})$$

$$>c\min_{x\in\mathbf{X}_j} g(x,\beta^j)\,\Big].$$

Condition (5.8) implies that the probability of this latter set goes to 0 for all $j\ge 0$ and $c > 0$. The proof is completed by noting that

$$(P^*\,(\Lambda_j^{(1)}(Y_n)\cup\Lambda_j^{(2)}(Y_n,b)\cup\Lambda_j^{(3)}(Y_n,c))$$

$$\le P\,(\Lambda_j^{(1)}(Y_n)\cup\Lambda_j^{(2)}(Y_n,b))$$

$+P^* (\Lambda_j^{(3)} (Y_n, c))$. □

Theorem 5.2 is, in turn, proved using the following result.

Theorem 5.3. If (X ,d) admits a majorizing measure satisfying (5.1) and for all a > 0 and ε > 0, (5.12) holds, then for all η > 0 and $b \le 2e^2(e^2 - \beta^3)/\beta^4$,

$$\overline{\lim_j}\ \overline{\lim_n}\ P^*([\sup_{x \in X, t \in I} |S_n(x,t) - S_n(x_j, t)| > \eta] \tag{5.14}$$

$$\cap (\Lambda_j^{(1)} (Y_n^{\beta^j/2g(\cdot,\beta^j)}) \cup \Lambda_j^{(2)} \left(Y_n^{\beta^j/2g(\cdot,\beta^j)}, b)\right)^c \Big) = 0.$$

We use the following two lemmas in deriving Theorem 5.2 from Theorem 5.3.

Proof of Lemma 5.2: Fix j ≥ 0 and to simplify notation, set $a_j(x) = \beta^j/2g^{1/2}(x,\beta^j)$. Clearly, since a_j (x) only depends on x through x_j, $Y_{n,i}^{a_j(y)}(y)$ and $Y_{n,i}^{a_j(x_j)}(x_j)$ are truncated at the same level $(a_j(x_j))$ whenever $y_j = x_j$ and we have then

$$\left|Y_{n,i}^{a_j(y)}(y) - Y_{n,i}^{a_j(x_j)}(x_j)\right| \le |Y_{n,i}(y) - Y_{n,i}(x_j)|.$$

Thus, for k ≥ j, using (1.6) of Lemma 1.1,

$$\sup_{y:\ y_k = x_k} \left|Y_{n,i}^{a_j(y)}(y) - Y_{n,i}^{a_j(x_j)}(x_j)\right| \le \Delta_{n,i}^{(k)}(x) .$$ □

Lemma 5.3. For all c > 0 and j ≥ 0,

$$\Lambda_j^{(2)} (Y_n^{\beta^j/2g(\cdot,\beta^j)}, c) \subset \Lambda_j^{(2)} Y_n, c/2) \cup \Lambda_j^{(3)} (Y_n, c/4) .$$

Proof: Again set $a_j(x) = \beta^j/2g(\cdot,\beta^j)$ for ease of notation and note that a_j (x) depends on x only through x_j. For any a, b > 0, we have

$$(Y_{n,i}^a(x_k) - Y_{n,i}^a(x_{k-1}))^2 1_{[|Y_{n,i}^a(x_k) - Y_{n,i}^a(x_{k-1})| \le b]}$$

$$\le (Y_{n,i}(x_k) - Y_{n,i}(x_{k-1}))^2 1_{[|Y_{n,i}(x_k) - Y_{n,i}(x_{k-1})| \le b]}$$

$$+ (2b)^2 (1_{[|Y_{n,i}(x_k)| > a]} + 1_{[|Y_{n,i}(x_{k-1})| > a]})$$

so, taking conditional expectations, summing, and taking suprema, we have, for any j ≥ 0 and c > 0,

$$\Lambda_j^{(2)}(Y_n^{a_j(\cdot)}, c) \subset \Lambda_j^{(2)}(Y_n, c/2)$$

$$\cup \ [4a_k^2(x) \sum_{1 \le i \le \tau(n,t)} P(|Y_{n,i}(x_k)| > a_j(x) | F_{n,i-1})$$

$$> c\beta^{2k}/4 \text{ for some } x \in \mathbf{X} \text{ and } k \ge j]$$

$$\subset \Lambda_j^{(2)}(Y_n, c/2) \cup \Lambda_j^{(3)}(Y_n, c/4). \qquad \square$$

Proof of Theorem 5.2: It follows from Lemmas 5.2 and 5.3 that for any $\eta > 0$, $j \ge 0$, and $b > 0$,

$$P^*(\sup_{x \in \mathbf{X}, t \in \mathbf{I}} |S_n(x,t) - S_n(x_j, t)| > \eta)$$

$$\le P^*\Big([\sup_{x \in \mathbf{X}, t \in \mathbf{I}} |S_n(x,t) - S_n(x_j,t)| > \eta] \cap (\Lambda_j^{(1)}(Y_n^{\beta^j/2g(\cdot,\beta^j)})$$

$$\cup \Lambda_j^{(2)}(Y_n^{\beta^j 2g(\cdot,\beta^j)}, b))^c\Big)$$

$$+ P^*(\Lambda_n^{(1)}(Y_n) \cup \Lambda_n^{(2)}(Y_n, b/2) \cup \Lambda_n^{(3)}(Y_n, b/4)). \qquad \square$$

Theorem 5.3 depends in turn on the following proposition. The statement of the proposition employs the function $\gamma : \mathbf{X} \times (0,1] \to R^+$ defined by

$$\gamma(x, \delta) = \sum_{j \ge j_\delta + 1} \beta^j ln^{1/2}(1/m_j(x)), \tag{5.15}$$

where $j_\delta = \{j : \beta^{j+1} < \delta \le \beta^j\}$.

Proposition 5.1. If $(\mathbf{X}, d)$ admits a majorizing measure satisfying (5.1),

$$E(Y_{n,i}(x) | F_{n,i-1}) = 0 \text{ for } x \in \mathbf{X} \text{ and } i \ge 1$$

and, for some $\beta \in (0,1)$ and $j_0 \ge 0$,

$$P^*(\sup_{i \ge 1} |Y_{n,i}(x)| > \beta^{j_0}/2g^{1/2}(x, \beta^{j_0}) \text{ for any } x \in \mathbf{X}) = 0, \tag{5.16}$$

where g is as defined in (5.2), then for $b \le 2e^2(e^2 - \beta^3)/\beta^4$,

$$P^*([\sup_{t \in \mathbf{I}} |S_n(x,t) - S_n(x_{j_0}, t)| > (1 + 12e^2/\beta^5)\gamma(x, \beta^{j_0}) \text{ for some } x \in \mathbf{X}] \tag{5.17}$$

$$\cap \Big(\Lambda_{j_0}^{(1)}(Y_n) \cup \Lambda_{j_0}^{(2)}(Y_n, b))^c) \le c\beta^{j_0}$$

where $c = c_\beta$ is a universal constant depending only on β.

Proof of Theorem 5.3: For any $x \in \mathbf{X}, t \in \mathbf{I}$, $j \geq 0$, and $a: \mathbf{X} \to R^+$ which depends on x only through x_j,

$$|S_n(x,t) - S_n(x_j,t)| \leq \left|S_n(x,t) - S_n^{a(\cdot)}(x,t)\right| + \left|S_n(x_j,t) - S_n^{a(\cdot)}(x_j,t)\right|$$
$$+ \left|S_n^{a(\cdot)}(x,t) - S_n^{a(\cdot)}(x_j,t)\right|.$$

Thus, for any $j \geq 0$,

$$\sup_{x \in \mathbf{X}, t \in \mathbf{I}} |S_n(x,t) - S_n(x_j,t)|$$
$$\leq \sup_{x \in \mathbf{X}, t \in \mathbf{I}} \left|S_n^{\beta^j/2g^{1/2}(x,\beta^j)}(x,t) - S_n^{\beta^j/2g^{1/2}(x,\beta^j)}(x_j,t)\right|$$
$$+ 2 \sup_{x \in \mathbf{X}, t \in \mathbf{I}} \left|S_n(x,t) - S_n^{\beta^j/2g^{1/2}(x,\beta^j)}(x,t)\right|.$$

Fix $\eta > 0$ and choose j_0 sufficiently large to have

$$(1 + 12e^2/\beta^5) \sup_{x \in \mathbf{X}} \gamma(x, \beta^{j_0}) < \eta/2.$$

(Recall from (1.5) of Lemma 1.1 that $\sup_{x \in \mathbf{X}} \gamma(x,\delta) \to 0$ as $\delta \to 0$.) Then for $j \geq j_0$

$$P^*([\sup_{x \in \mathbf{X}, t \in \mathbf{I}} (|S_n(x,t) - S_n(x_j,t)| > \eta] \cap (\Lambda_j^{(1)}(Y_n^{\beta^j/2g(\cdot,\beta^j)}) \cup$$
$$\Lambda_j^{(2)}\left(Y_n^{\beta^j/2g(x,\beta^j)}, b\right)\Big)^c\Big)$$
$$\leq P^*([\sup_{t \in \mathbf{I}} |S_n(x,t) - S_n(x_j,t)| > (1 + 12e^2/\beta^5)\gamma(x,\beta^j) \text{ for some } x \in \mathbf{X}]$$
$$\cap (\Lambda_j^{(1)}(Y_n^{\beta^j/2g(\cdot,\beta^j)}) \cup \Lambda_j^{(2)}(Y_n^{\beta^j/2g(\cdot,\beta^j)}, b))^c)$$
$$+ P^*(\sup_{y \in \mathbf{X}_j, t \in \mathbf{I}, x \in \mathbf{X}} \sup_{x_j = y} \left|S_n(x,t) - S_n^{\beta^j/2g(y,\beta^j)}(x,t)\right| > \eta/4)$$
$$\leq c\beta^j + o(1). \qquad \square$$

It remains to prove Proposition 5.1. It proceeds as follows. First we stratify $S_n(x,t)$ and $S_n(x_{j_0},t)$ by partitioning the sample space. Then, within each stratum, for a particular $j \geq j_0$ we compare $S_n(x,t)$ to $S_n(x_{j_0},t)$ in two stages. The first comparison gives rise to two remainder terms, one of which is contained in $\Lambda_{j_0}^{(1)}$ and the other of which is bounded in probability on $\Lambda_{j_0}^{(1)}$ using Lemma 4.3. The second comparison gives rise to a third remainder term, which we bound in probability on the set $\Lambda_{j_0}^{(2)}$.

Before we begin the proof, however, we need some technical lemmas which will be used in controlling the rate at which our approximations x_j converge to x. The following property of the function g defined in (5.2) above is of central importance.

Lemma 5.4. The function $g : \mathbf{X} \times (0,1] \to R^+$ satisfies

$$\int_0^\delta g^{1/2}(x,u)\,du < ((1-\beta)/\beta)\gamma(x,\delta) \tag{5.18}$$

as well as

$$\sup_{x\in\mathbf{X}} \int_0^\delta g^{1/2}(x,u)\,du < ((1-\beta)/\beta) \sup_{x\in\mathbf{X}} \gamma(x,\delta). \tag{5.19}$$

Proof: If $\beta^{j+1} < u \leq \beta^j$, then, for any $x \in \mathbf{X}$, $g(x,u) < ln^{1/2}(1/m_{j+1}(x))$. Thus, again letting $j_\delta = \{j : \beta^{j+1} < \delta \leq \beta^j\}$,

$$\begin{aligned}\int_0^\delta g^{1/2}(x,u)\,du &< \sum_{j\geq j_\delta} \int_{\beta^{j+1}}^{\beta^j} g^{1/2}(x,u)\,du \\ &< \sum_{j\geq j_\delta} (\beta^j - \beta^{j+1})\, lu^{1/2}(1/m_{j+1}(x)) \\ &= ((1-\beta)/\beta) \sum_{j\geq j_\delta+1} \beta^j ln^{1/2}(1/m_j(x)).\end{aligned}$$ □

Fix $\delta \in (0,1]$, and for each $x \in \mathbf{X}$ and $k \geq 0$, choose $\delta_k(x)$ to satisfy

$$\delta_k(x)/g^{1/2}(x,\delta_k(x)) = \delta\beta^k/g^{1/2}(x,\delta). \tag{5.20}$$

Lemma 5.5 For each x, the sequence $\{\delta_k : k \geq 0\}$ is a strictly decreasing sequence. In addition, if $\gamma(x,1) < \infty$, then $\lim_{k\to\infty} \delta_k(x) = 0$.

Proof: Since for each fixed x the sequence $\delta\beta^k/g^{1/2}(x,\delta)$ is strictly decreasing in k and the function $u/g^{1/2}(x,u)$ is strictly increasing in u, it is clear that $\delta_k(x)$ is a strictly decreasing sequence. From (5.18) we see that

$$g^{1/2}(x,u) \leq (1-\beta)\gamma(x,1)/\beta u,$$

so

$$\delta_k(x) = \delta\beta^k g^{1/2}(x, \delta_k(x))/g^{1/2}(x, \delta) \le \delta(1-\beta)\beta^k\gamma(x, 1)/\beta\delta_k(x) g^{1/2}(x, \delta).$$

Thus $\delta_k(x) \le (\delta(1-\beta)\gamma(x, 1)\ /\beta g^{1/2}(x, \delta))^{1/2}\beta^{k/2}$, so $\lim_{k\to\infty}\delta_k(x) = 0$ if $\gamma(x, 1) < \infty$. □

It is easy to see that $\delta_k(x)$ decreases more slowly than β^k.

Lemma 5.6. For each $x \in \mathbf{X}$ and $k \ge 0$,

$$\beta\delta_k(x) < \delta_{k+1}(x). \tag{5.21}$$

Proof: For any $x \in \mathbf{X}$

$$\beta\delta_k(x)/g^{1/2}(x,\delta_k(x)) = \delta_{k+1}(x)/g^{1/2}(x,\delta_{k+1}(x)) < \delta_{k+1}(x)/g^{1/2}(x,\delta_k(x)). \square$$

The following summability result is quite important.

Lemma 5.7. For $\delta_k(x)$ as defined in (5.20) above

$$\sum_{k\ge 0}\delta_k(x) g^{1/2}(x, \delta_k(x)) \le (2/\beta)\gamma(x, \delta). \tag{5.22}$$

Proof: For any $x \in \mathbf{X}$ and $K > 0$,

$$(1-\beta)\sum_{0\le k\le K}\delta_k(x) g^{1/2}(x, \delta_k(x))$$

$$= (1-\beta)\delta g^{1/2}(x, \delta) + \sum_{0<k\le K}\delta_k^2(x)\,(g^{1/2}(x, \delta_k(x)))/\delta_k(x)$$

$$-g^{1/2}(x, \delta_{k-1}(x))/\delta_{k-1}(x))$$

$$= (1-\beta)\delta g^{1/2}(x, \delta) + \sum_{0<k\le K}\delta_k(x) g^{1/2}(x, \delta_k(x))$$

$$-\sum_{0\le k<K}\delta_{k+1}^2(x) g^{1/2}(x, \delta_k(x))/\delta_k(x)$$

$$= -\beta\delta g^{1/2}(x, \delta) + \sum_{0\le k<K}(\delta_k(x) - \delta_{k+1}^2(x)/\delta_k(x)\ ) g^{1/2}(x, \delta_k(x))$$

$$+\delta_K(x) g^{1/2}(x, \delta_K(x))$$

$$\le 2\sum_{0\le k<K}(\delta_k(x) - \delta_{k+1}(x)) g^{1/2}(x, \delta_k(x)) + \delta_K(x) g^{1/2}(x, \delta_K(x))$$

$$\le 2\sum_{0\le k<K}\int_{\delta_{k+1}(x)}^{\delta_k(x)} g^{1/2}(x, u)\,du + \int_0^{\delta_K(x)} g^{1/2}(x, u)\,du$$

$$< 2\int_0^\delta g^{1/2}(x,u)\,du\ .$$

The first equality above follows from the recursion relation

$$\beta g^{1/2}(x,\delta_k(x))/\delta_k(x) = g^{1/2}(x,\delta_{k-1}(x))/\delta_{k-1}(x)\,.$$

The first inequality stems from dropping the leading negative term and observing that

$$\delta_k(x) - \delta_{k+1}^2(x)/\delta_k(x) = (\delta_k(x) - \delta_{k+1}(x))(\delta_k(x) + \delta_{k+1}(x))/\delta_k(x) \le$$

$$2(\delta_k(x) - \delta_{k-1}(x))\,.$$

To complete the proof use Lemma 5.4. □

Proof of Proposition 5.1: We partition the sample space using the following construction. Fix $j_0 \ge 0$, $\beta \in (0,1)$, and set $\delta = \beta^{j_0}$. We use the sequence $\{\delta_k(x) : k \geq 0\}$ defined in (5.20) and discussed above. Note that $\delta_0(x) = \delta = \beta^{j_0}$ for all $x \in \mathbf{X}$. Also define the geometrically decreasing sequence

$$a_k(x) = \delta\beta^k/g^{1/2}(x,\delta) = \delta_k(x)/g^{1/2}(x,\delta_k(x))\,.$$

For each $x \in \mathbf{X}$, set $j_k(x) = \{j : \beta^{j+1} < \delta_k(x) \leq \beta^j$. Again, note that $j_0(x) = j_0$ for all $x \in \mathbf{X}$. For $k \geq 0$ let

$$A_{n,i}^k(x) = [a_{k+1}(x) < \Delta_{n,i}^{(j_k(x))}(x) \le a_k(x)]\,,$$

$$\tilde{A}_{n,i}^k(x) = [a_{k+1}(x) < \Delta_{n,i}^{(j_k(x))}(x)]\,,$$

and

$$D_{n,i}^k(x) = (\bigcup_{0 \le j \le k} \tilde{A}_{n,i}^k(x))^c.$$

The following sets are defined iteratively as follows. Set

$$B_{n,i}^0(x) = \tilde{A}_{n,i}^0(x)$$

and for $k > 0$, set

$$B_{n,i}^k(x) = \tilde{A}_{n,i}^k(x) \setminus \bigcup_{0 \le j \le k-1} B_{n,i}^j(x)\,.$$

Notice that

$$B_{n,i}^k(x) = \tilde{A}_{n,i}^k(x) \setminus \bigcup_{0 \le j \le k-1} \tilde{A}_{n,i}^j(x)$$

$$= A^k_{n,i}(x) \setminus \bigcup_{0 \le j \le k-1} \tilde{A}^j_{n,i}(x)$$

$$\subset A^k_{n,i}(x),$$

$$B^k_{n,i}(x) = D^{k-1}_{n,i}(x) \setminus D^k_{n,i}(x), \tag{5.23}$$

and

$$D^k_{n,i}(x) \subset (\tilde{A}^k_{n,i}(x))^c. \tag{5.24}$$

Since the $Y_{n,i}(x)$ is bounded a.s. by $a_0(x)/2$ for all $n,i \ge 1$, each $\Delta^{(0)}_{n,i}(x)$ is bounded by $a_0(x)$ a.s.. Thus for any fixed x and $K \ge 0$, the collection of sets $\{D^k_{n,i}, B^k_{n,i} : 0 \le k \le K\}$ provide a disjoint partition of the sample space.

We now stratify S_n. For $k \ge 0$, set

$$S_{n,k}(x,t) = \sum_{1 \le i \le \tau(n,t)} \left(Y_{n,i}(x) 1_{B^k_{n,i}(x)} - E\left(Y_{n,i}(x) 1_{B^k_{n,i}(x)} \middle| F_{n,i-1} \right) \right)$$

and

$$T_{n,k}(x,t) = \sum_{1 \le i \le \tau(n,t)} \left(Y_{n,i}(x) 1_{D^k_{n,i}(x)} - E\left(Y_{n,i}(x) 1_{D^k_{n,i}(x)} \middle| F_{n,i-1} \right) \right).$$

Since $E(Y_{n,i}(x) \mid F_{n,i-1}) = 0$, for any $K \ge 0$,

$$S_n(x,t) = \sum_{0 \le k \le K} S_{n,k}(x,t) + T_{n,K}(x,t). \tag{5.25}$$

Each $S_{n,k}(x,t)$ is compared to its (approximately) $\delta_k(x)$-approximation $S_{n,k}(x_{j_k(x)}, t)$ as follows:

$$\left| S_{n,k}(x,t) - S_{n,k}(x_{j_k(x)}, t) \right| = \Big| \sum_{1 \le i \le \tau(n,t)} ((Y_{n,i}(x) - Y_{n,i}(x_{j_k(x)})) 1 B^k_{n,i}(x)$$

$$-E((Y_{n,i}(x) - Y_{n,i}(x_{j_k(x)})) 1 B^k_{n,i}(x) \mid F_{n,i-1}) \Big|$$

$$\le \sum_{1 \le i \le \tau(n,t)} E\left(\Delta^{(j_k(x))}_{n,i}(x) 1_{B^k_{n,i}(x)} \middle| F_{n,i-1} \right) + \sum_{1 \le i \le \tau(n,t)} \Delta^{(j_k(x))}_{n,i}(x) 1_{B^k_{n,i}(x)}$$

$$:= R^{(0)}_{n,k}(x,t) + R^{(1)}_{n,k}(x,t).$$

It is important to notice that, as both $\Delta^{(j_k(x))}_{n,i}(x)$ and , $B^k_{n,i}(x)$ depend on x only through $x_{j_k(x)}$, both $R^{(0)}_{n,k}$(x,t) and $R^{(1)}_{n,k}$(x,t) also depend on x only through $x_{j_k(x)}$. Likewise,

$$|T_{n,k}(x,t) - T_{n,k}(x_{j_k(x)},t)| \tag{5.27}$$

$$= |\sum_{1\le i\le\tau(n,t)} ((Y_{n,i}(x) - Y_{n,i}(x_{j_k(x)}))1_{D^k_{n,i}(x)}$$

$$-E((Y_{n,i}(x) - Y_{n,i}(x_{j_k(x)}))1_{D^k_{n,i}(x)}|F_{n,i-1}))$$

$$\le \sum_{1\le i\le\tau(n,t)} \left(\Delta^{j_k(x)}_{n,i}(x)1_{D^k_{n,i}(x)} + E\left(\Delta^{j_k(x)}_{n,i}(x)1_{D^k_{n,i}(x)}|F_{n,i-1}\right)\right)$$

$$\le 2\tau(n,t)\,a_{k+1}(x).$$

In turn we stratify $S_n(x_{j_0},t)$ and compare each stratum to $S_{n,k}(x_{j_k(x)},t)$ for a particular k. Let

$$S^{(0)}_{n,k}(x,t) = \sum_{1\le i\le\tau(n,t)} \left(Y_{n,i}(x_{j_0})1_{B^k_{n,i}(x)} - E\left(Y_{n,i}(x_{j_0})1_{B^k_{n,i}(x)}|F_{n,i-1}\right)\right),$$

$$T^{(0)}_{n,k}(x,t) = \sum_{1\le i\le\tau(n,t)} \left(Y_{n,i}(x_{j_o})1_{D^k_{n,i}(x)} - E\left(Y_{n,i}(x_{j_o})1_{D^k_{n,i}(x)}|F_{n,i-1}\right)\right),$$

and note that for any K ≥ 0, and $x \in \mathbf{X}$

$$S_n(x_{j_0},t) = \sum_{0\le k\le K} S^{(0)}_{n,k}(x,t) + T^{(0)}_{n,K}(x,t).$$

Then, for any K ≥ 0,

$$S_n(x_{j_0},t) - \left(\sum_{0\le k\le K} S_{n,k}(x_{j_k(x)},t) + T_{n,K}(x_{j_K(x)},t)\right) \tag{5.28}$$

$$= \sum_{0\le k\le K}\left(S^{(0)}_{n,k}(x,t) - S_{n,k}(x_{j_k(x)},t)\right) + \left(T^{(0)}_{n,K}(x,t) - T_{n,K}(x_{j_K(x)},t)\right)$$

$$= \sum_{0<k\le K}\ \sum_{1\le i\le\tau(n,t)} ((Y_{n,i}(x_{j_0(x)}) - Y_{n,i}(x_{j_k(x)}))1_{B^k_{n,i}(x)}$$

$$-E((Y_{n,i}(x_{j_0(x)}) - Y_{n,i}(x_{j_k(x)}))1_{B^k_{n,i}(x)}|F_{n,i-1}))$$

$$+ \sum_{1\le i\le\tau(n,t)} ((Y_{n,i}(x_{j_0(x)}) - Y_{n,i}(x_{j_K(x)}))1_{D^K_{n,i}(x)}$$

$$- E((Y_{n,i}(x_{j_0(x)}) - Y_{n,i}(x_{j_K(x)}))1_{D^K_{n,i}(x)}|F_{n,i-1}))$$

$$= \sum_{0<k\le K} \sum_{1\le i\le \tau(n,t)} ((Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))1_{D^{k-1}_{n,i}(x)\setminus D^K_{n,i}(x)}$$

$$- E((Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))1_{D^{k-1}_{n,i}(x)\setminus D^K_{n,i}(x)} \,|\, F_{n,i-1}))$$

$$+ \sum_{0<k\le K} \sum_{1\le i\le \tau(n,t)} ((Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))1_{D^K_{n,i}(x)}$$

$$- E((Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))1_{D^K_{n,i}(x)} \,|\, F_{n,i-1}))$$

$$= \sum_{0<k\le K} \sum_{1\le i\le \tau(n,t)} ((Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))1_{D^{k-1}_{n,i}(x)}$$

$$- E((Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))1_{D^{k-1}_{n,i}(x)} \,|\, F_{n,i-1})).$$

$$: = \sum_{0<k\le K} R^{(2)}_{n,k}(x,t)$$

In deriving the third equality, we interchange summation and note that

$$\sum_{0<k\le K} (y_0 - y_k)1_{B_k} = \sum_{0<k\le K} \sum_{0<j\le k} (y_{j-1} - y_j)1_{B_k}$$

$$= \sum_{0<j\le K} (y_{j-1} - y_j) \sum_{j\le k\le K} 1_{B_k}$$

$$= \sum_{0<j\le K} (y_{j-1} - y_j)1_{\bigcup_{j\le k\le K} B_k}.$$

We then apply (5.23) to see that $\bigcup_{j\le k\le K} B^k_{n,i}(x) = D^{j-1}_{n,i}(x)\setminus D^K_{n,i}(x)$. Now for $t\in \mathbf{I}$, set

$$\mathrm{K(n,t)} = \inf\{\mathrm{k} : \mathrm{k} \ge \sup_{x\in \mathbf{X}} ln(\gamma(x,\delta)\, g^{1/2}(x,\delta)\,/2\delta\beta\tau(n,t))\,/ln\beta\}.$$

K(n,t) is finite a.s., since condition (5.1) (through Lemmas 1.1 and 5.4) insures that both $\gamma(\cdot,\delta)$ and g $(\cdot,\delta)$ are bounded on **X**. Now combine the comparisons (5.25), (5.26), (5.27) and (5.28) derived above to arrive at the following:

$$|S_n(x,t) - S_n(x_{j_0},t)| \tag{5.29}$$

$$\le \Big|\sum_{0\le k\le K(n,t)} (S_{n,k}(x,t) - S_{n,k}(x_{j_k(x)},t))\Big|$$

$$+ |T_{n,K(n,t)}(x,t) - T_{n,K(n,t)}(x_{j_{K(n,t)}(x)},t)|$$

$$+\left|\sum_{0\le k\le K(n,t)} S_{n,k}(x_{j_k(x)},t)+T_{n,K(n,t)}(x_{j_{K(n,t)}},t)-S_n(x_{j_0},t)\right|$$

$$\le \sum_{0<k\le K(n,t)} (R^{(0)}_{n,k}(x,t)+R^{(1)}_{n,k}(x,t))$$

$$+2\tau(n,t)\,a_{K(n,t)+1}(x)+\sum_{0<k\le K(n,t)}\left|R^{(2)}_{n,k}(x,t)\right|$$

$$\le \sum_{0<k\le K(n,t)} (R^{(0)}_{n,k}(x,t)+R^{(1)}_{n,k}(x,t))+\sum_{0<k\le K(n,t)}\left|R^{2}_{n,k}(x,t)\right|+\gamma(x,\delta).$$

For $k \geq 0$, set

$$\eta_k(x) = (2e^2/\beta^4)\,\delta_k(x)\,g^{1/2}(x,\delta_k(x)).$$

Then, from Lemma 5.7,

$$\sum_{k\ge 0}\eta_k(x) < (4e^2/\beta^5)\,\gamma(x,\delta).$$

Thus, for any $b > 0$, from (5.29),

$$P^*([\sup_{t\in \mathbf{I}}|S_n(x,t)-S_n(x_{j_0},t)| > (1+12e^2/\beta^5)\,\gamma(x,\delta) \text{ for some } x\in \mathbf{X}] \cap \left(\Lambda^{(1)}_{j_0(Y_n)}\cup\Lambda^{(2)}_{j_0}(Y_n,b)\right)^c) \quad (5.30)$$

$$\le P^*([R^{(0)}_{n,k}(x) > \eta_k(x) \text{ for some } x\in \mathbf{X} \text{ and } 0\le k\le K(n,t)] \cap \left(\Lambda^{(1)}_{j_0}(Y_n)\right)^c)$$

$$+P^*([R^{(1)}_{n,k}(x) > \eta_k(x) \text{ for some } x\in \mathbf{X} \text{ and } 0\le k\le K(n,t)] \cap \left(\Lambda^{(1)}_{j_0}(Y_n)\right)^c)$$

$$+P^*([R^{(2)}_{n,k}(x) > \eta_k(x) \text{ for some } x\in \mathbf{X} \text{ and } 0\le k\le K(n,t)] \cap \left(\Lambda^{(2)}_{j_0}(Y_n,b)\right)^c).$$

The set where the first remainder is large is actually contained in $\Lambda^{(1)}_{j_0}$. For any $x\in \mathbf{X}$ and $t\in \mathbf{I}$, we have

$$R^{(0)}_{n,k}(x,t) \le a_k(x)\sum_{1\le i\le\tau(n,t)} P(\Delta^{(j_k(x))}_{n,i}(x) > a_{k+1}(x)\,|\,F_{n,i-1}),$$

so that

$$\bigcup_{0\le k\le K(n,t)} [R^{(0)}_{n,k}(x,t) > \eta_k(x)]$$

$$\subset \bigcup_{k\ge 0}\left[a^2_{k+1}(x)\sum_{1\le i\le\tau(n,t)} P(\Delta^{j_k(x)}_{n,i} > a_{k+1}(x)\,|\,F_{n,i-1}) > 2e^2\delta^2_k(x)\;/\beta^2\right]$$

$$\subset \bigcup_{k \geq 0} [\sup_{a>0} a^2 \sum_{1 \leq i \leq \tau(n,t)} P(\Delta_{n,i}^{j_k(x)}(x) > a | F_{n,i-1}) > 2e^2 \beta^{2j_k(x)}]$$

$$\subset \bigcup_{j \geq j_0} [\sup_{a>0} a^2 \sum_{1 \leq i \leq \tau(n,t)} P(\Delta_{n,i}^{j}(x_j) > a | F_{n,i-1}) > \beta^{2j}]$$

$$\subset \Delta_{j_0}^{(1)}(Y_n).$$

Bounding the second remainder term in probability is more complicated. For $j \geq j_0$, set $b_j(x) = \beta^j / g^{1/2}(x, \beta^j)$. From the definition of $j_k(x)$ and $a_k(x)$ observe that $\delta_k(x) \leq \beta^{j_k(x)}$ from Lemma 5.6 and $a_k(x) \leq b_{j_k(x)}(x)$, so for any fixed $x \in \mathbf{X}$ and $t \in \mathbf{I}$,

$$\bigcup_{k \geq 0} [R_{n,k}^{(1)}(x,t) > \eta_k(x)]$$

$$\subset \bigcup_{k \geq 0} [a_{k+1}(x) \sum_{1 \leq i \leq \tau(n,t)} \Delta_{n,i}^{(j_k(x))}(x) 1_{[a_{k+1}(x) < \Delta_{n,i}^{(j_k)(x)}(x) \leq a_k(x)]} > 2e^2 \delta_k^2(x)/\beta^3]$$

$$\subset \bigcup_{k \geq 0} [\sup_{a>0} a \sum_{1 \leq i \leq \tau(n,t)} \Delta_{n,i}^{(j_k(x))}(x) 1_{\left[a < \Delta_{n,i}^{(j_k(x))}(x) \leq b_{j_k(x)}(x)\right]} > 2e^2 \beta^{2j_k(x)-1}]$$

$$\subset \bigcup_{j \geq j_0} [\sup_{a>0} a \sum_{1 \leq i \leq \tau(n,t)} \Delta_{n,i}^{(j)}(x_j) 1_{[a < \Delta_{n,i}^{(j)}(x_j) \leq b_j(x_j)]} > 2e^2 \beta^{2j-1}].$$

Thus, using the above, Lemma 4.3, and the definitions of the functions $g(\cdot,\cdot)$ and $m(\cdot,\cdot)$, we see that

$$P^*([R_{n,k}^{(1)}(x,t) > \eta_k(x) \text{ for some } x \in \mathbf{X}, t \in \mathbf{I}, \text{ and } k \geq 0] \cap \left(\Lambda_{j_0}^{(1)}(Y_n)\right)^c) \tag{5.32}$$

$$\leq \sum_{j \geq j_0} \sum_{x \in \mathbf{X}_j} P^*([\sup_{a>0} a \sum_{1 \leq i \leq \tau(n,t)} \Delta_{n,i}^{(j)}(x) 1_{[a < \Delta_{n,i}^{(j)}(x) \leq b_j(x)]}$$

$$> 2e^2 \beta^{2j-1} \text{ for some } t \in \mathbf{I}] \cup \left(\Lambda_{j_0}^{(1)}(Y_n)\right)^c)$$

$$\leq \sum_{j \geq j_0} \sum_{x \in \mathbf{X}_j} (\beta/e)^{e^2 g(x,\beta^j)/\beta} / (1-\beta/e)$$

$$\leq \sum_{j \geq j_0} \sum_{x \in \mathbf{X}_j} m(x, \beta^j) / (1-\beta/e)$$

$$\leq \beta^{j_0} / (1-\beta/e).$$

Finally, $R^{(2)}_{n,k}(x,t)$ is made up of the conditionally centered increments

$$(Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))\ |\ 1_{D^{k-1}_{n,i}(x)}$$
$$-E((Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}))1_{D^{k-1}_{n,i}(x)}\big| F_{n,i-1})$$

which depend on x only through $x_{j_k(x)}$. From (5.24), $D^{k-1}_{n,i}(x) \subset [\Delta^{(k-1)}_{n,i}(x) \le a_k(x)]$, and from Lemma 5.6, we know $\delta_k(x) > \beta\delta_{k-1}(x)$, so either $j_{k-1}(x) = j_k(x)+1$ or $j_{k-1}(x) = j_k(x)$. Thus

$$Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)}) = \begin{cases} Y_{n,i}(x_{j_k(x)-1}) - Y_{n,i}(x_{j_k(x)}) & \text{if } j_{k-1}(x) \ne j_k(x) \\ 0 & ow \end{cases}. \tag{5.33}$$

Furthermore, if $j_{k-1}(x) \ne j_k(x)$, on the set $D^{k-1}_{n,i}(x)$ we have

$$|Y_{n,i}(x_{j_{k-1}(x)}) - Y_{n,i}(x_{j_k(x)})| \le a_k(x) = \delta_k(x)/g^{1/2}(x,\delta_k(x)) \le \beta^{j_k(x)}/g^{1/2}(x,\beta^{j_k(x)}).$$

Set $k_j(x) = \{k : \delta_k(x) \le \beta^j < \delta_{k-1}(x)\}$, so $k_j(x) = k_j(x_j)$, and notice that again from Lemma 5.6 we have $\delta_{k_j(x)}(x) \ge \beta\delta_{k_j(x)-1}(x) > \beta^{j+1}$, so that

$$\eta_{k_j(x)} = (2e^2/\beta^4)\delta_{k_j(x)}(x)g^{1/2}(x,\delta_{k_j(x)}(x)) \ge 2e^2\beta^{j-3}g^{1/2}(x_j,\beta^j), \tag{5.34}$$

as well as $a_{k_j(x)} \le \beta^j/g^{1/2}(x,\beta^j)$. Then, for any fixed $x \in \mathbf{X}, t \in \mathbf{I}$, from (5.33) and (5.34),

$$\bigcup_{k>0} [|R^{(2)}_{n,k}(x,t)| > \eta_k(x)] = \bigcup_{j>j_0} \left[|R^{(2)}_{n,k_j(x)}(x,t)| > \eta_{k_j(x)}(x)\right]$$
$$\subset \bigcup_{j>j_0} \left[|R^{(2)}_{n,k_j(x)}(x_j,t)| > 2e^2\beta^{j-3}g^{1/2}(x,\beta^j)\right]$$

and

$$[|R^{(2)}_{n,k}(x,t)| > \eta_k(x) \text{ for some } x \in \mathbf{X}, t \in \mathbf{I}, \text{and } k > 0]$$

$$\subset \bigcup_{j>j_0} \bigcup_{x\in\mathbf{X}_j} [|R^{(2)}_{n-k_j(x)}(x,t)| > 2e^2\beta^{j-3}g^{1/2}(x,\beta^j) \text{ for some } t \in \mathbf{I}]. \tag{5.35}$$

Set

$$E_{n,j}(x,t) = [$$

$$\sum_{1 \le i \le \tau(n,t)} E\big((Y_{n,i}(x_{j-1}) - Y_{n,i}(x_j))^2 1_{[|Y_{n,i}(x_{j-1}) - Y_{n,i}(x_j)| \le a_j(x)]}$$

$$| F_{n,i-1}) \le 2(e^2 - \beta^3) e^2 \beta^{2j-6}]$$

and note that $E_{n,j}(x,t) \supset (\Lambda^{(2)}(Y_n, b))^c$ for $b \le 2e^2(e^2-\beta^3)/\beta^4$. Then, from (5.35), Proposition 4.1, and the definition of g(·,·), we have

$$P^*([|R_{n,k}^{(2)}(x,t)| > \eta_k(x) \text{ for some } x \in \mathbf{X}, t \in \mathbf{I}, \text{ and } k > 0] \cap (\Lambda^2(Y_n, b))^c) \qquad (5.36)$$

$$\le \sum_{j > j_0} \sum_{x \in \mathbf{X}_j} P^* \Big([|R_{n,k_j(x)}^{(2)}(x,t)| > 2e^2 \beta^{j-3} g^{1/2}(x, \beta^j)] \quad \cap E_{n,j}(x,t)$$

$$\text{for some } t \in \mathbf{I})$$

$$\le \sum_{j > j_0} \sum_{x \in \mathbf{X}_j} 2 \exp\{-4e^4 \beta^{2j-6} g(x, \beta^j) / 2(2e^2\beta^{2j-3} + 2(e^2 - \beta^3) e^2 \beta^{2j-6}) \}$$

$$\le \sum_{j > j_0} \sum_{x \in \mathbf{X}_j} 2 \exp\{-g(x, \beta^j)\}$$

$$\le 2\beta^{j_0+1}/(1-\beta).$$

Combining (5.30), (5.31), (5.32) and (5.36), we have, for $b \le 2e^2(e^2-\beta^3)/\beta^4$,

$$P^*([\sup_{t \in \mathbf{I}} |S_n(x,t) - S_n(x_{j_0}, t)| > (1 + 12e^2/\beta^5)\gamma(x, \beta^{j_0}) \text{ for some } x \in \mathbf{X}]$$

$$\cap \left(\Lambda_{j_0}^{(1)}(Y_n) \cup \Lambda_{j_0}^{(2)}(Y_n, b) \right)^c \Big)$$

$$\le \beta^{j_0}(1 + \beta - 2\beta^2/e)/(1-\beta)(1-\beta/e),$$

which completes the proof of (5.17). □

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