A LOOK AT PERTURBATION APPROXIMATIONS FOR EPIDEMICS

H.E. Daniels
University of Cambridge

Abstract

Perturbation-type approximations have been derived by many authors for epidemics and related problems. Here we use the standard perturbation technique familiar to physicists to approximate the mean and variance of paths of some epidemic processes. For epidemics with no removals the saddlepoint approximation to the distribution of infectives is highly accurate for quite small populations and can be used to assess the accuracy of perturbation approximations. For epidemics with removals no saddlepoint approximation is available and simulations have to be used for comparison. The technique turns out to be equivalent to Daley and Kendall's method of 'diffusion of arbitrary constants'.

1. Introduction. Approximations of the perturbation type have been around in epidemic theory for a considerable time. Bartlett (1956) used such an approximation to study the fluctuations about the endemic equilibrium in a recurrent epidemic. Bailey (1968) and Weiss (1971) obtained by different routes essentially the same approximations for the mean and variance of a simple epidemic in large populations. Barbour (1972) developed the method of 'diffusion of arbitrary constants' introduced by Daley and Kendall (1965) to extend these results to more general epidemics. McNeil and Schach (1973) considered diffusion approximations of the Ornstein-Uhlenbeck type for epidemic and similar processes. Daniels (1960) described what is essentially a perturbation technique for approximating to the distribution and moments of processes of the epidemic type. Other references may be found in Bailey (1972) and Renshaw (1986).

There is, however, a standard procedure familiar to physicists and engineers for deriving perturbation approximations which can with advantage be used in problems of this kind. My interest in the method was first aroused by Bellman's elegant little monograph (Bellman, 1964) though he treats there only deterministic problems. In the present paper the procedure is applied systematically to obtain perturbation approximations for the mean and variance of some epidemic processes.

2. The Univariate Case. We first consider the univariate birth process which includes the so-called simple epidemic with no removals and its generalizations. At time t there are $N(t)$ infectives and the probability of a new infection in time $(t,t+dt)$ is $\lambda(N)dt$, ignoring terms which are $o(dt)$. If m is the total population size the number of susceptibles in the population is $m-N(t)$, and $\lambda(m)=0$. On a suitable time scale the simple epidemic has $\lambda(N)=N(m-N)$.

The process evolves according to the equation (Bartlett, 1960)

$$dN = \lambda(N)\ dt + dZ(t) \tag{2.1}$$

where $dZ(t)$ is a centred Bernoulli variable such that

$$P\{dZ(t) = 1-\lambda(N)\ dt\} = \lambda(N)\ dt$$

$$P\{dZ(t) = -\lambda(N)\ dt\} = 1-\lambda(N)\ dt$$

Initially $N(0) = a$. Then

$$E\{dZ(t)|N(t) = n\} = 0,\quad E\{[dZ(t)]^r|N(t)= n\} = \lambda(n)dt \text{ when } r \geq 2,$$

and

$$E\{dZ(t_1)\,dZ(t_2)\,|\,N(t_1) = n_1, N(t_2) = n_2\} = 0,\quad t_1 \neq t_2.$$

Since the moments of $dZ(t)$ involve n implicitly it is convenient to write

$$dZ(t) = \sqrt{\lambda(n)}\,dW(t)$$

in which case $E(dW(t)|n) = 0$, $E[(dW(t))^2|n] = dt$, but $E[(dW(t))^2|n] = dt/[\lambda(n)]^{\frac{1}{2}r-1}$, $r > 2$ still involves n. Since we are concerned only with the mean and variance of $N(t)$ we shall use a diffusion approximation where $W(t)$ is regarded as Brownian motion whose higher moments are free of n.

The basic idea underlying perturbation expansions is to embed (2.1) in the family of models

$$\begin{aligned} dN &= \lambda(N)\,dt + \varepsilon dZ(t) \\ &= \lambda(N)\,dt + \varepsilon\sqrt{\lambda(N)}\,dW(t)\,. \end{aligned} \tag{2.2}$$

When $\varepsilon = 0$ the model is deterministic with a known solution; when $\varepsilon = 1$ we have the stochastic model (2.1). It is then assumed that the solution of (2.2) can be formally expanded as a power series in ε

$$N = N_0 + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots \tag{2.3}$$

which is substituted in (2.2). Thus

$$d(N_0 + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots) = \lambda(N_0 + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots)\,dt$$

$$+\varepsilon\sqrt{\lambda\,(N_0+\varepsilon N_1+\varepsilon^2 N+\ldots)}\,dW(t)$$

$$=\lambda\,(N_0)\,dt+\varepsilon\lambda'\,(N_0)\,N_1 dt\quad+\varepsilon^2\left\{\lambda'\,(N_0)\,N_2+\frac{1}{2}\lambda''\,(N_0)\,N_1^2\right\}dt$$

$$+\ \{\varepsilon\sqrt{\lambda\,(N_0)}+\frac{1}{2}\varepsilon^2\frac{\lambda'\,(N_0)}{\sqrt{\lambda\,(N_0)}}N_1+\ldots\ \}\,dW(t)$$

Equating coefficients of powers of ε we obtain the following equations

$$dN_0 = \lambda\,(N_0)\,dt$$

$$dN_1 = \lambda'\,(N_0)\,N_1 dt+\sqrt{\lambda\,(N_0)}\,dW(t)$$

$$dN_2 = \lambda'\,(N_0)\,N_2 dt+\frac{1}{2}\lambda''\,(N_0)\,N_1^2 dt$$

$$+\frac{1}{2}\frac{\lambda'\,(N_0)}{\sqrt{\lambda\,(N_0)}}N_1 dW(t)\,.$$

The first is the deterministic equation. Once its solution is found the others can be solved recursively.

The success of the procedure does not depend on ε being small. All that is needed is that the series (2.3) is a convergent or an asymptotic expansion for some range of ε which include $\varepsilon = 1$. Bellman refers to ε as a 'book keeping' variable allowing terms of comparable degrees of approximation to be marshalled in an orderly way. However, in the epidemic application the magnitude of successive terms is more clearly exhibited in terms of proportions $R = N/m$ rather than N. Initially $R = a/m = \alpha$. For simplicity we shall assume that $\lambda(N)$ has the form

$$\lambda\,(N) = m^c\beta\,(R)\,. \tag{2.4}$$

This includes the simple epidemic with $\lambda(N) = m^2R(1 - R)$ and Severo's more general form $\lambda(N) = N^j(m - N)^k = m^{j+k}R^j(1 - R)^k$ (Severo, 1969). Let $T = m^{c-1}t$, $dB\,(T) = m^{(c-1)/2}dW(t)$ and $\varepsilon = \omega\sqrt{m}$. Then (2.2) becomes

$$dR\,(T) = \beta\,(R)\,dT+\omega\sqrt{\beta\,(R)}\,dB\,(T) \tag{2.5}$$

where $EdB(T)=0$ $E\{\,dB(T)\}^2=dT$. The original stochastic epidemic is recovered on putting

$\omega = 1/\sqrt{m}$ which is now small.

As before, write $R = R_0 + \omega R_1 + \omega^2 R_2 + \ldots$ and equate coefficients of powers of w, obtaining

$$\begin{aligned} dR_0 &= \beta(R_0)\,dT \\ dR_1 &= \beta'(R_0)\,R_1 dT + \sqrt{\beta(R_0)}\,dB(T) \\ dR_2 &= \beta'(R_0)\,R_2 dT + \frac{1}{2}\beta''(R_0)\,R_1^2 dT \\ &\qquad + \frac{1}{2}\frac{\beta'(R_0)}{\sqrt{\beta(R_0)}} R_1 dB(T). \end{aligned}$$

The first equation has the deterministic solution

$$T = \int_\alpha^{r_0} \frac{dx}{\beta(x)} \tag{2.6}$$

where r_0 has been written in place of R_0 since it is deterministic. The second equation can then be solved using the integrating factor

$$e^{-\int \beta'[r_0(T)]\,dT} = e^{-\int \frac{\beta'(r_0)}{\beta(r_0)} dr_0} = \frac{1}{\beta(r_0)}$$

which yields

$$R_1(T) = \beta[r_0(T)] \int_0^T \frac{dB(u)}{\sqrt{\beta[r_0(u)]}}\,. \tag{2.7}$$

It follows that $E\{R_1(T)\} = 0$ and

$$E\{R_1^2(T)\} = \beta^2[r_0(T)] \int_0^T \frac{du}{\beta[r_0(u)]}$$

or, in terms of r_0,

$$E(R_1^2) = \beta^2(r_0) \int_\alpha^{r_0} \frac{dx}{\beta^2(x)}\,. \tag{2.8}$$

Using the same integrating factor the third equation yields

$$\begin{aligned} R_2(T) &= \frac{1}{2}\beta[r_0(T)] \int_0^T \beta''[r_0(u)]\,R_1^2(u)\,du \\ &\quad + \frac{1}{2}\beta[r_0(T)] \int_0^T \frac{\beta'[r_0(u)]\,R_1(u)\,dB(u)}{\beta^{\frac{3}{2}}[r_0(u)]} \end{aligned} \tag{2.9}$$

Notice that in the second term $dB(u)$ is an innovation and should strictly be written $dB(u+)$. Hence $E\{R_1(u)dB(u)\} = 0$ and the second term vanishes on taking expectations. After some manipulation it is found that

$$E(R_2) = \frac{1}{2}\beta(r_o)\beta'(r_0)\int_\alpha^{r_0}\frac{dx}{\beta^2(x)} + \frac{1}{2}\left\{1-\frac{\beta(r_0)}{\beta(\alpha)}\right\} \tag{2.10}$$

which can be written as

$$E(R_2) = \frac{1}{2}\frac{\beta'(r_0)}{\beta(r_0)}E(R_1^2) + \frac{1}{2}\left\{1-\frac{\beta(r_0)}{\beta(\alpha)}\right\} . \tag{2.11}$$

Finally, on inserting $\omega = 1/\sqrt{m}$ we obtain

$$E(R) = r_0 + \frac{1}{m}E(R_2) + 0\left(m^{-\frac{3}{2}}\right) \tag{2.12}$$

$$\mathrm{var}R = \frac{1}{m}E(R_1^2) + 0\left(m^{-\frac{3}{2}}\right) \tag{2.13}$$

where $E(R_1^2)$ and $E(R_2)$ are given by (2.8) and (2.11). It follows from the general theory of birth processes that, necessarily, $E(R_2) < 0$.

A useful consequence of the fact that $E\{R_1(u)\,dB(u)\} = 0$ in (2.9) is that as far as terms in ω^2 we could equally well have taken $E\{(dZ)^2|\,n\}$ to be $\lambda(n_0)\,dt$, thus avoiding the need to expand $\sqrt{\lambda(N)}$ in the last term of (2.2). This fact is used later when discussing the general epidemic with removals.

In terms of the actual number of susceptibles we have

$$n_0 = mr_0, \quad N_1 = \sqrt{m}R_1, \quad N_2 = R_2 \tag{2.14}$$

and the corresponding formulae are

$$t = \int_\alpha^{n_0}\frac{dx}{\lambda(x)}$$

$$E(N_1^2) = \lambda^2(n_0)\int_\alpha^{n_0}\frac{dx}{\lambda^2(x)} \tag{2.15}$$

$$E(N_2) = \frac{1}{2}\frac{\lambda'(n_0)}{\lambda(n_0)}E(N_1^2) + \frac{1}{2}\left\{1 - \frac{\lambda(n_0)}{\lambda(\alpha)}\right\}.$$

The approximations are then

$$\begin{aligned} E(N) &\sim n_0 + E(N_2) + 0\left(m^{-\frac{1}{2}}\right) \\ \text{var}N &\sim E(N_1^2) + 0\left(m^{-\frac{1}{2}}\right) \end{aligned} \quad . \qquad (2.16)$$

3. Two Examples. We now calculate the perturbation approximations to the mean and variance for two particular models. The first is the simple epidemic with $\lambda(n) = n(m-n)$ where the results are found to agree with those obtained by other methods. The second is that suggested by McNeil (1972) with $\lambda(n) = n^{1/2}(m-n)^{1/2}$ as a simplified model for a spatial epidemic.

Except in special cases, or for very small populations, the exact results are difficult to calculate for comparison. However, it was shown (Daniels, 1982) that the saddlepoint approximation to the distribution of the number of infectives, even for quite small population sizes, is almost indistinguishable from the exact distribution. The corresponding means and variances have therefore been used for comparison. The paper by Daniels (1982) may be consulted for details of the saddlepoint method.

The simple epidemic $\lambda(n) = n(m-n)$

The computations are most easily carried out in terms of R, with $\beta(r) = r(1-r)$. The deterministic solution is $r_0/(1-r_0) = \{\alpha/(1-\alpha)\}e^t$ and it is found from (2.8) that

$$E(R_1^2) = r_0^2(1-r_0)^2\left\{2T - \frac{1}{r_0} + \frac{1}{1-r_0} + \frac{1}{\alpha} - \frac{1}{1-\alpha}\right\},$$

from which $E(R_2)$ and $E(R)$ can be calculated using (2.11) and (2.12). The formulae agree with those given by Daniels (1960, p.398) and Bailey (1968).

The corresponding results for $E(N)$ and $SD(N) = \sqrt{\text{var}N}$ are compared with the exact values (based on the saddlepoint approximations) in Table 1 for m=20 and a=1,5,10. The deterministic values n_0 are also shown. Notice first that when a=1, excluding the initial stages when the process behaves like a linear birth process, the perturbation approximation to the mean and SD goes seriously wrong. This is a well known feature of diffusion approximations, which cannot properly account for very small initial values. When a=5 and a=10 the values are in good agreement. There is a tendency for the approximate values of $SD(N)$ to drop towards the completion of the epidemic but as the distribution is then very skew the SD is perhaps not very appropriate.

The McNeil model $\lambda(n) = n^{1/2}(m-n)^{1/2}$.

A simplified spatial epidemic is modelled by considering the infections to take place on the circumference of an expanding circle within which all are infected. In terms of R, $\beta(r) = r^{1/2}(1-r)^{1/2}$ and the deterministic solution is

$$r_0 = \sin^2(\sin^{-1}\sqrt{\alpha} + \tfrac{1}{2}T).$$

With this model the epidemic is completed in a finite time $T(\max) = \pi - 2\sin^{-1}\sqrt{\alpha}$. From (2.8)

$$E(R_1^2) = r_0(1-r_0)\log\left\{\frac{r_0(1-r_0)}{\alpha(1-\alpha)}\right\}$$

$$E(R_2) = \frac{1}{4}\frac{(1-2r_0)}{r_0(1-r_0)}ER_1^2 + \frac{1}{2}\left\{1 - \left[\frac{r_0(1-r_0)}{\alpha(1-\alpha)}\right]^{\frac{1}{2}}\right\}$$

from which $E(R)$ and var$R$, and hence $E(N)$ and $SD(N)$ are calculated as before.

Comparisons with the exact mean and SD, and the deterministic value n_0 are shown in Table 2 for m=20, a=1 and 5. In this case even the results for a=1 are quite good.

4. The Bivariate Case A similar technique is next applied to the bivariate process

$$\left.\begin{aligned} dX &= \phi(X,Y)\,dt + \varepsilon dG(t) \\ dY &= \psi(X,Y)\,dt + \varepsilon dH(t) \end{aligned}\right\} \qquad (4.1)$$

which becomes the required process when $\varepsilon = 1$. We shall not specify $dG(t)$, $dH(t)$ in detail at the moment. In the next section the results are applied to the so-called general epidemic with removal of infectives where the appropriate $dG(t)$ and $dH(t)$ are defined.

As before, the solution of (4.1) is formally expanded as

$X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 \tau \ldots,\ Y = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \ldots$ and the coefficients of powers of ε are equated. From ε^0 we obtain

$$\frac{dX_0}{dt} = \phi(X_0, Y_0),\ \frac{dY_0}{dt} = \psi(X_0, Y_0). \tag{4.2}$$

It is assumed (which is often not the case) that (4.2) has a known deterministic solution $x_0(\xi, \eta, t),\ y_0(\xi, \eta, t)$ where ξ, η are the initial values of X_0, Y_0, i.e.

$$x_0(\xi, \eta, 0) = \xi, \qquad y_0(\xi, \eta, 0) = \eta \quad .$$

The terms in ε^1 are

$$\left.\begin{aligned} dX_1 &= \phi'(x_0, y_0) X_1 dt + \dot{\phi}(x_0, y_0) Y_1 dt + dG(t) \\ dY_1 &= \psi'(x_0, y_0) X_1 dt + \dot{\psi}(x_0, y_0) Y_1 dt + dH(t) \end{aligned}\right\} \tag{4.3}$$

when $\phi' = \dfrac{\partial\phi(x,y)}{\partial x}$, $\dot{\phi} = \dfrac{\partial\phi(x,y)}{\partial y}$, and so on. In the univariate case the equivalent equation was solved by using an integrating factor. Here we use the method of variation of parameters.

The homogeneous equations

$$\left.\begin{aligned} \frac{dX_1}{dt} &= \phi'(x_0, y_0) X_1 + \dot{\phi}(x_0, y_0) Y_1 \\ \frac{dY_1}{dt} &= \psi'(x_0, y_0) X_1 + \dot{\psi}(x_0, y_0) Y_1 \end{aligned}\right\} \tag{4.4}$$

are the linearised form of (4.3). They have two independent solutions

$$\left(\frac{\partial x_0}{\partial\xi}, \frac{\partial y_0}{\partial\xi}\right) \text{ and } \left(\frac{\partial x_0}{\partial\eta}, \frac{\partial y_0}{\partial\eta}\right),$$

as may be discovered on substituting $x_0(\xi + \delta\xi, \eta + \delta\eta, t)$, $y_0(\xi + \delta\xi, \eta + \delta\eta, t))$ in (4.2), expanding and extracting the coefficients of $\delta\xi$ and $\delta\eta$.

The complete solution of (4.4) is therefore

$$\left(a_1\frac{\partial x_0}{\partial\xi} + a_2\frac{\partial x_0}{\partial\eta}\right),\ \left(a_1\frac{\partial y_0}{\partial\xi} + a_2\frac{\partial y_0}{\partial\eta}\right).$$

The solution of the nonhomogeneous equations (4.3) is then found by allowing a_1 and a_2 to be functions of t in

$$\left.\begin{aligned} X_1(t) &= a_1(t)\frac{\partial x_0}{\partial \xi} + a_2(t)\frac{\partial x_0}{\partial \eta} \\ Y_1(t) &= a_1(t)\frac{\partial y_0}{\partial \xi} + a_2(t)\frac{\partial y_0}{\partial \eta} \end{aligned}\right\}. \tag{4.5}$$

On substituting in (4.3) it is found that $a_1(t)$, $a_2(t)$ must satisfy

$$\left.\begin{aligned} \frac{\partial x_0}{\partial \xi}da_1(t) + \frac{\partial x_0}{\partial \eta}da_2(t) &= dG(t) \\ \frac{\partial y_0}{\partial \xi}da_1(t) + \frac{\partial y_0}{\partial \eta}da_2(t) &= dH(t) \end{aligned}\right\}. \tag{4.6}$$

The solution of (4.6) is

$$a_1(t) = \int_0^t \frac{\left\{\dfrac{\partial y_0(u)}{\partial \eta}dG(u) - \dfrac{\partial x_0(u)}{\partial \eta}dH(u)\right\}}{\dfrac{\partial[x_0(u), y_0(u)]}{\partial(\xi, y)}}$$

$$a_2(t) = \int_0^t \frac{\left\{-\dfrac{\partial y_0(u)}{\partial \xi}dG(u) + \dfrac{\partial y_0(u)}{\partial \xi}dH(u)\right\}}{\dfrac{\partial[x_0(u), y_0(u)]}{\partial(\xi, \eta)}}. \tag{4.7}$$

Hence

$$X_1(t) = \int_0^t \frac{\dfrac{\partial[x_0(t), y_0(u)]}{\partial(\xi, \eta)}dG(u) - \dfrac{\partial[x_0(t), x_0(u)]}{\partial(\xi, \eta)}dH(u)}{\dfrac{\partial[x_0(u), y_0(u)]}{\partial(\xi, \eta)}}$$

$$Y_1(t) = \int_0^t \frac{-\dfrac{\partial[y_0(t), y_0(u)]}{\partial(\xi, \eta)}dG(u) + \dfrac{\partial[y_0(t), x_0(u)]}{\partial(\xi, \eta)}dH(u)}{\dfrac{\partial[x_0(u), y_0(u)]}{\partial(\xi, \eta)}} \tag{4.8}$$

where all the Jacobians are known functions calculated from the deterministic solution.

The terms in ε^2 lead to the equations

$$dX_2 = \phi'(x_0, y_0) X_2 dt + \dot{\phi}(x_0, y_0) Y_2 dt$$

$$+ \left\{ \frac{1}{2}\phi''(x_0, y_0) X_1^2 + \dot{\phi}'(x_0, y_0) X_1 Y_1 + \frac{1}{2}\ddot{\phi}(X_0, y_0) Y_1^2 \right\} dt \tag{4.9}$$

$$dY_2 = \psi'(x_0, y_0) X_2 dt + \dot{\psi}(x_0, y_0) Y_2 dt$$

$$+ \left\{ \frac{1}{2}\psi''(x_0, y_0) X_1^2 + \dot{\psi}'(x_0, y_0) X_1 Y_1 + \frac{1}{2}\ddot{\psi}(x_0, y_0) Y_1^2 \right\} dt \ .$$

They are solved in precisely the same way, the solution for $X_2(t)$, $Y_2(t)$ being (4.8) with $dG(u) = \{ \frac{1}{2}\phi''(x_0,y_0)X_1^2 + \dot{\phi}'(x_0,y_0)X_1Y_1 + \frac{1}{2}\ddot{\phi}(x_0,y_0)Y_1^2 \} dt$ and $dH(u)$ the corresponding expression with ψ for ϕ. Then $E(X_1^2)$, $E(X_1Y_1)$, $E(Y_1^2)$ and $E(X_2)$, $E(Y_2)$ are found on taking expectations. Just as in the univariate case, as far as terms in ε^2 it can without loss be assumed that $E\{[dG(u)]^2 | X,Y\} = x_0y_0dt$ and so on.

We now apply these results to the general epidemic.

5. The General Epidemic With Removals. The population consists of $X(t)$ susceptibles and $Y(t)$ infectives. The infection rate is taken to be $X(t)Y(t)$ as in the simple epidemic, but infectives are removed at rate $\rho Y(t)$ during the course of the epidemic. The appropriate equations are

$$\begin{aligned} dX &= -XYdt - dZ_X \\ dY &= (X-\rho)Ydt + dZ_X - dZ_Y \end{aligned} \tag{5.1}$$

where

$$E(dZ_X | X,Y) = 0, \quad E(dZ_Y | X,Y) = 0, \quad E\{(dZ_X)^2 | X,Y\} = XYdt,$$

$$E\{(dZ_Y)^2 | X,Y\} = \rho Ydt, \quad E[dZ_X dZ_Y | X,Y] = 0.$$

Initially $X(0) = \xi$, $Y(0) = \eta$. Here $\phi(X,Y) = -XY, \psi(X,Y) = (X-\rho)Y$ and $dG = -dZ_X$, $dH = dZ_X - dZ_Y$.

The deterministic solution is x_0, y_0 where

$$\left. \begin{aligned} t &= \int_{x_0}^{\xi} \frac{dx}{x\{\rho\log(x/\xi)\ \ x\ \ \xi\ \ \eta\}} \\ y_0 &= \rho\log(x_0/\xi) - x_0 + \xi + \eta \end{aligned} \right\} . \tag{5.2}$$

It is convenient to rearrange (4.8) in the form

$$\left.\begin{aligned} X_1(t) &= \int_0^t \{-A_x dZ_x(u) + B_X dZ_Y(u)\} \\ Y_1(t) &= \int_0^t \{A_Y dZ_X(u) - B_Y dZ_Y(u)\} \end{aligned}\right\} \tag{5.3}$$

where

$$\begin{aligned} A_X(t,u) &= \frac{\partial[x_0(t), x_0(u)+y_0(u)]}{\partial(\xi,\eta)} \Big/ \frac{\partial[x_0(u), y_0(u)]}{\partial(\xi,\eta)} \\ B_X(t,u) &= \frac{\partial[x_0(t), x_0(u)]}{\partial(\xi,\eta)} \Big/ \frac{\partial[x_0(u), y_0(u)]}{\partial(\xi,\eta)} \\ A_Y(t,u) &= \frac{\partial[y_0(t), x_0(u)+y_0(u)]}{\partial(\xi,\eta)} \Big/ \frac{\partial[x_0(u), y_0(u)]}{\partial(\xi,\eta)} \\ B_Y(t,u) &= \frac{\partial[y_0(t), x_0(u)]}{\partial(\xi,\eta)} \Big/ \frac{\partial[x_0(u), y_0(u)]}{\partial(\xi,\eta)}. \end{aligned} \tag{5.4}$$

Then

$$\begin{aligned} E(X_1^2) &= \int_0^t \{A_X^2 x_0(u) + B_X^2 \rho\} y_0(u)\, du \\ E(X_1Y_1) &= \int_0^t \{A_X A_Y x_0(u) + B_X B_Y \rho\} y_0(u)\, du \\ E(Y_1^2) &= \int_0^t \{A_Y^2 x_0(u) + B_Y^2 \rho\} y_0(u)\, du \end{aligned} \tag{5.5}$$

The solution for X_2, Y_2 is obtained by taking

$$dG(u) = -X_1(u)Y_1(u)du, \qquad dH(\mathrm{u}) = X_1(u)Y_1(u)du$$

in (4.8). Then

$$X_2 = -\int_0^t A_X X_1(u) Y_1(u)\, du, \quad Y_2 = \int_0^t A_Y X_1(u) Y_1(u)\, du \tag{5.6}$$

from which $E(X_2)$, $E(Y_2)$ can be found using $E(X_1Y_1)$ as described at the end of section 9. The expectations have to be evaluated by numerical integration.

From (5.2), $\dfrac{\partial x_0}{\partial \xi} = x_0 y_0 \{1/\xi\eta - (1-\rho/\xi) w(x_0)\}$, $\dfrac{\partial x_0}{\partial \eta} = -x_0 y_0 w(x_0)$, where

$$w(x_0) = \int_{x_0}^{\xi} \frac{dx}{x\{\rho \log(x/\xi) \;\; x \;\; \xi \;\; \eta\}^2} \tag{5.7}$$

Also $\dfrac{\partial y_0}{\partial \xi} = (1-\rho/\xi) - (1-\rho/x_0)\dfrac{\partial x_0}{\partial \xi}, \dfrac{\partial y_0}{\partial \eta} = 1 - (1-\rho/x_0)\dfrac{\partial x_0}{\partial \eta}$. These can be used to evaluate the Jacobians. In particular the denominator of A_X etc. has the simple form $\partial[x_0(u),y_0(u)]/\partial(\xi,\eta) = x_0(u)y_0(u)/\xi\eta$. As in the univariate case it was also found easier to work in terms of x_0 rather than t in the integrations using $dx_0(u) = x_0(u)y_0(u)\,du$.

Calculations were carried out for $\xi = 100$, $\eta = 5$, $\rho = 25$. With these values the probability of an epidemic failing to develop initially is $(\rho/\xi)^\eta = 0.001$ which can be neglected. Unfortunately no saddlepoint approximation is available for the exact values, which have therefore had to be estimated by simulation. Table 3 compares these with the perturbation approximations for $E(X)$, $SD(X)$ and $E(Y)$, $SD(Y)$.

In practice, the number of infectives Y at any time in the population cannot be directly observed. What can be observed is the number of removals $Q = \xi + \eta - X - Y$. A table has therefore been included of $E(Q)$ and $SD(Q) = \{\mathrm{var}X + \mathrm{var}Y + 2\mathrm{cov}XY\}^{1/2}$, and their perturbation approximations.

There is reasonably good agreement with the exact values for $E(X)$, $E(Y)$ and $E(Q)$, but as in the case of the simple epidemic the standard deviations do not match up so well.

6. General Comments. We have chosen to deal with the particularly simple case where a diffusion approximation suffices for the first and second moments. If higher moments are needed, for example, in the univariate case, powers of $m^{-1/2}$ appear in the expectations which are not accounted for by the powers of ω. The method can still be used but care is needed in deciding which powers of $m^{-1/2}$ have to be retained at each stage. Similar considerations arise when $\lambda(N)$ does not have the simple form (2.4).

This observation also applies to an approach like that of McNeil and Schach (1973) which is based on the characteristic function but still employs a diffusion approximation. On the other hand the methods used by Bailey (1968), Weiss (1971) and Daniels (1960) do not require diffusion approximations.

The application of the method to the bivariate equation in Section 4 will be recognised as essentially the same as the method of 'diffusion of arbitrary constants' introduced by Daley and Kendall (1965). Professor David Williams has also pointed out to me that the treatment of the linearised equations (4.3) is formally equivalent to the Malliavin calculus currently employed in the study of large deviations.

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TABLE 1

Simple epidemic. $\lambda(n) = n(m - n)$, $T = mt$

$m = 20$, $a = 1$

Deterministic		Exact		Perturbation	
T	n_0	$E(N)$	$SD(N)$	$E(N)$	$SD(N)$
0.2	1.21	1.21	0.49	1.21	0.50
0.5	1.61	1.59	0.93	1.59	0.96
1.0	2.50	2.44	1.71	2.43	1.84
2.0	5.60	5.05	3.42	4.90	4.11
3.0	10.28	8.61	4.74	8.06	5.59
4.0	14.84	12.30	5.08	11.96	4.65
5.0	17.73	15.33	4.53	15.75	2.72
6.0	19.10	17.42	3.53	18.13	1.39
7.0	19.66	18.77	2.54	19.25	0.71
8.0	19.87	19.25	2.17	19.87	0.39

$m = 20$, $a = 5$

Deterministic		Exact		Perturbation	
T	n_0	$E(N)$	$SD(N)$	$E(N)$	$SD(N)$
0.5	7.09	7.07	1.55	7.06	1.58
1.0	9.51	9.38	2.27	9.37	2.36
1.5	11.98	11.70	2.63	11.69	2.74
2.0	14.22	13.82	2.67	13.80	2.71
2.5	16.05	15.59	2.47	15.57	2.40
3.0	17.40	16.96	2.14	16.96	1.97
3.5	18.34	17.96	1.76	17.98	1.55
4.0	18.96	18.66	1.40	18.69	1.18
4.5	19.36	19.13	1.10	19.16	0.90

$m = 20, a = 10$

Deterministic		Exact		Perturbation	
T	n_0	$E(N)$	$SD(N)$	$E(N)$	$SD(N)$
0.1	10.50	10.50	0.71	10.50	0.71
0.2	11.00	10.99	0.98	10.99	0.99
0.4	11.97	11.96	1.35	11.95	1.37
0.6	12.91	12.88	1.58	12.87	1.61
0.8	13.80	13.74	1.72	13.73	1.76
1.0	14.62	14.53	1.80	14.53	1.83
1.5	16.35	16.21	1.78	16.21	1.80
2.0	17.62	17.46	1.59	17.46	1.57
2.5	18.48	18.33	1.35	18.33	1.30
3.0	19.05	18.93	1.10	18.93	1.03

Table 2

McNeil model. $\lambda(n) = \sqrt{n(m-n)}$, $T = t$

$m = 20$, $a = 1$

Deterministic		Exact		Perturbation	
T	n_0	$E(N)$	$SD(N)$	$E(N)$	$SD(N)$
0.25	2.36	2.31	1.33	2.30	1.39
0.50	4.19	4.02	2.20	3.99	2.32
0.75	6.39	6.04	2.94	6.01	3.08
1.00	8.81	8.26	3.51	8.25	3.65
1.25	11.30	10.56	3.85	10.56	3.97
1.50	13.71	12.77	3.94	12.80	4.01
1.75	15.87	14.77	3.75	14.83	3.75
2.00	17.71	16.45	3.33	16.52	3.18
2.25	19.05	17.73	2.77	17.71	2.32
2.50	19.82	18.64	2.18	18.23	1.17

$m = 20$, $a = 5$

Deterministic		Exact		Perturbation	
T	n_0	$E(N)$	$SD(N)$	$E(N)$	$SD(N)$
0.25	7.28	7.28	1.57	7.28	1.59
0.50	9.76	9.69	2.25	9.69	2.29
0.75	12.24	12.09	2.64	12.09	2.72
1.00	14.59	14.32	2.76	14.33	2.87
1.25	16.64	16.23	2.59	16.26	2.75
1.50	18.28	17.70	2.20	17.74	2.33
1.75	19.41	18.72	1.70	18.64	1.62
2.00	19.96	19.34	1.21	18.61	0.57

Table 3

General epidemic. $X(t)$ susceptibles, $Y(t)$ infectives

$X(0) = \xi = 100$, $Y(0) = \eta = 5$, Removal rate $\rho = 25$

		Susceptibles			
Deterministic		Simulation		Perturbation	
t	x_0	$E(X)$	$SD(X)$	$E(X)$	$SD(X)$
1	92.90	92.99	4.06	92.98	4.13
2	80.48	81.10	8.85	81.14	9.73
3	62.84	64.99	13.36	65.13	13.92
4	43.90	47.85	15.52	47.93	14.92
5	28.46	33.19	14.86	32.82	12.40
6	18.14	22.45	12.68	21.63	9.05
7	11.89	15.27	10.21	14.35	6.44
8	8.41	10.70	8.10	9.87	4.74
9	5.99	7.80	6.53	7.16	3.68
10	4.62	5.95	5.44	5.47	3.01
11	3.74	4.75	4.73	4.38	2.58
12	3.15	3.94	4.28	3.71	2.29
13	2.74	3.41	3.99	3.20	2.09
14	2.46	3.04	3.80	2.92	1.96

Table 3 continued

Infectives

	Deterministic	Simulation		Perturbation	
t	y_0	$E(X)$	$SD(X)$	$E(X)$	$SD(X)$
1	10.26	10.18	4.04	10.19	4.10
2	19.09	18.54	7.93	18.50	8.24
3	30.55	28.75	10.85	28.63	11.22
4	40.52	37.55	11.29	37.50	10.63
5	45.12	42.15	9.62	42.52	7.94
6	44.19	42.25	7.72	42.95	6.43
7	39.88	39.19	6.70	39.84	6.31
8	34.29	34.53	6.42	34.92	6.30
9	28.64	29.42	6.28	29.58	6.03
10	23.51	24.55	6.06	24.51	5.58
11	19.09	20.16	5.67	20.03	5.05
12	15.40	16.40	5.17	16.54	4.52
13	12.36	13.23	4.64	13.29	4.01
14	9.90	10.64	4.13	11.25	3.55

Table 3 continued

Removals $Q = \xi + \eta - X - Y$

	Deterministic	Simulation		Perturbation	
t	q_0	$E(X)$	$SD(X)$	$E(X)$	$SD(X)$
1	1.84	1.83	1.25	1.84	1.26
2	5.43	5.36	2.39	5.36	2.43
3	11.62	11.25	4.21	11.24	4.39
4	20.58	19.60	6.39	19.57	6.61
5	31.41	29.67	8.26	29.66	8.17
6	42.67	40.30	9.41	40.43	8.73
7	53.23	50.53	9.76	50.22	8.54
8	62.31	59.77	9.53	60.21	7.93
9	70.36	67.78	8.98	68.27	7.16
10	76.87	74.51	8.20	75.02	6.37
11	82.17	80.09	7.48	80.58	5.64
12	86.45	84.65	6.72	84.74	4.99
13	89.89	88.36	6.09	88.50	4.53
14	92.64	91.32	5.54	90.83	3.96