APPLICATIONS OF RANDOM WALKS ON FINITE GRAPHS

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Abstract

This is a brief account of different contexts in which random walks on graphs occur, and of techniques used in their analysis. It is an expanded version of talks given in several places during 1986-89.

1. INTRODUCTION. Just as games of chance provide an easily comprehended setting for the elementary mathematics of probability, so random walks provide a simple illustration of stochastic processes. *Drunkard's walk*, with successive steps taken right or left at random, and *gambler's ruin*, the successive fortunes of a gambler staking $1 on red at roulette, are readily described to a non-mathematician. Generalizations to the study of sums of independent identically distributed random variables in 1 or d dimensions form a classic field of probability theory (usually called random walk, but here called *random flight*): see [40]. A different generalization is to consider a graph G and a particle stepping randomly around the vertices in the natural way: from a vertex v it chooses uniformly at random an edge e incident at v, and steps to the vertex v' at the other end of e. Such *random walks on graphs* are our subject.

The professional probabilist, vintage 1960-80, is liable to regard this topic as a dull special case of Markov theory. The author would prefer to draw an analogy with (say) branching processes, another special case of Markov theory which has long been recognized to have a distinctive flavor. The author, admittedly partisan, feels that "random walks on graphs" has at least as many applications, and a greater variety of mathematical tools and results, than does "branching processes", and deserves similar recognition as a subject in its own right.

Because these random walks arise in different settings, many elementary (and not-so-elementary) results have been repeatedly rediscovered. The purpose of this talk is to publicize the fact that a lot is known about these random walks, even though the subject has not been well-organized.

2. Contexts This section is devoted to a list of applications of random walks on graphs. "Applications" is a much-abused word: what I mean is "contexts in which random walks arise, in settings where the original question is not about random walks".

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Context 1. *Analogy with electric networks.* The delightful elementary monograph by Doyle and Snell [25] treats this analogy in detail. Consider the edges of a graph as a network of resistors of unit resistance. Apply unit voltage at vertex υ_1 and 0 voltage (ground) at vertex υ_1. Then some current $i_{\upsilon w}$ flows along each edge (υ,w), each vertex υ has some voltage $f(\upsilon)$ and some total current I flows through the network. These quantities are related to quantities defined in terms of the random walk on the graph. Write T_υ for first hitting time on υ, and T_υ^+ for first return time to υ.

Write $E_\upsilon(\cdot)$ and $P_\upsilon(\cdot)$ for expectation, probability for the walk started at υ. Write $d(\upsilon)$ for the degree of υ (= number of edges at υ). Then

$$f(x) = P_x(T_{\upsilon_1} < T_{\upsilon_2})$$

$$I = d(\upsilon_1) P_{\upsilon_1}\left(T_{\upsilon_0} < T^+_{\upsilon_1}\right)$$

$$i_{\upsilon w} = E_{\upsilon_1}((\#\text{ steps } \upsilon \to w) - (\#\text{ steps } w \to \upsilon) \text{ before } T_{\upsilon_0})$$

Facts about electric networks, such as

- Removing a wire can only decrease current flow
- actual current flow minimizes energy dissipation amongst all possible flows can be re-interpreted as facts abour random walks on graphs, and lead to results such as conditions for recurrence/transience for walks on infinite graphs.

Context 2. *Leveling networks.* Maps show heights of mountain-tops, etc., which are ultimately derived from measurements of height differences between points with a clear line of sight. Consider a graph whose vertices are the physical points whose heights are desired, and where an edge (υ,w) denotes that a measurement $a(\upsilon,w)$ of "height at υ - height at w" has been made. Let υ_0 be a reference point of known height $h(\upsilon_0)$. Assuming the measurements contain errors, an estimate of the true heights h(v) could be made by least squares: choose the function h which minimizes

$$\sum_{edges} \sum (h(\upsilon) - h(w) - a(\upsilon,w))^2$$

or more generally a weighted sum of this kind. It turns out that the least squares estimate $h^*(\upsilon)$ can be described in terms of the random walk X_n started at υ_0;

$$h^*(\upsilon) = E\sum_{n=1}^{T_\upsilon} a(X_n, X_{n-1})$$

In other words, each deterministic walk from υ_0 to υ yields an empirical height for υ as the sum of measured height differences along that walk; and the least squares estimate is the average of these empirical heights, where we use the random walk to do the averaging. These ideas are explored in a somewhat neglected monograph by Borre and Meissl [13].

Context 3. *Recreational problems.* Imagine a knight on a corner square of an otherwise empty chessboard. Move the knight by choosing at random from the legal knight-moves. What is the mean time

- until the knight first returns to the starting square?
- until the knight has visited all squares?

These look like messy problems for which no simple answer would be expected. It turns out that Markov chain theory gives a simple answer (168) to the first problem (this is a textbook exercise). The second problem really is messy, but some asymptotics suggest a rough approximation (500) which simulations reveal to be about right.

Of course, the knight is performing random walk on a graph. Random walks arise in other recreational problems (on polyhedra; in card-shuffling), and from time to time appear in the "math" section of electronic bulletin boards. Diaconis [21] contains a nice account of card-shuffling problems.

Context 4. *Universal traversal sequences.* It has long been realized that probabilistic methods are sometimes useful in combinatorics to establish the existence of objects which are hard to exhibit constructively: classical applications are in the monograph by Erdos and Spencer [27]. The following elegant application of random walk to a combinatorial problem was made by Aleliunas et al [6]. Let $S(N,d)$ be the set of all d-regular graphs G with N vertices and with the edges at each vertex labelled (1,2, ...,d). A *universal traversal sequence* $i_1, i_2, ..., i_U \in \{1, ..., d\}$ satisfies: for each $G \in S(N,d)$ the deterministic walk "at step n choose edge i_n" visits every vertex. What is the shortest length $U = U(N,d)$ of such a sequence? It can be proved that *most* sequences of length $O(dN^3 \log N)$ are universal traversal sequences.

To see why, choose $i_1, i_2, ...$ uniformly at random, so the walk is simple RW. On a fixed G, consider the *cover time* C = time to visit all vertices. It can be

shown that $EC = O(N^2)$ for G regular. Then by iterating, there is a constant K such that on fixed G,

$$P(C > KdN^3 \log N) \leq N^{-Nd}$$

But $|S(N,d)| = o(N^{Nd})$, so

$$C \leq KdN^3 \log N \text{ for all } G \in S(N,d)) \text{ -.}$$

Context 5. *Minimization algorithms.* One way in which probability enters the theory of algorithms is via randomized algorithms, as in the example above. Another way is that certain problems can be proved "hard" by showing that, when presented with random data, no algorithm has much chance of finding the solution quickly. An application of random walk was given by Aldous [3]. Consider a function h defined on the vertices of a graph G: for definiteness, the cube graph in d dimensions. One wants to find the vertex v at which $h(v)$ is minimized. Constrain h to have no local minima except the global minimum. Thus any deterministic "descent" algorithm will work, but it might work slowly. Could there be some more sophisticated algorithm which always works quickly? The answer is no. Consider random walk started at a uniform vertex v_0 and let h(v) be the first hitting time on v. Then h is a random function satisfying the constraint, but it can be proved that *every* algorithm requires expected time $\Omega\left(2^{\frac{d}{2}-\varepsilon}\right)$ to locate v_0.

Context 6. *Approximate counting, or simulating uniform distributions on combinatorial sets.* Consider finite combinatorial sets S_K, e.g. all permutations of 1,2,...,K satisfying some family of conditions. It is intuitively clear that finding an exact formula for $|S_K|$ is roughly equivalent to being able to write a program to simulate a uniform element of the set using a deterministic number of calls to a random number generator. But there is a classical "Markov chain" method of simulating approximately a desired distribution π, originally developed for Ising-type physics models [29]: define a chain whose stationary distribution is π and simulate the chain. In our setting, we make a graph G with vertex-set S_K and with edges (v,w) interpreted as "v can be obtained from w by a small change", for some specified notion of *small change*. If the graph is connected and regular (the non-regular case can be handled by minor modifications) then random walk has uniform stationary distribution, and so we can approxi-

mate the uniform distribution by simulating the random walk. For this method to be useful, the walk has to be "rapidly mixing": that is, the time for distributions to approach the stationary distribution should be small compared to the size of the graph. This is equivalent to G being an *expander* (i.e. highly- connected) graph.

Thus if S_K can be given a usable graph structure which gives an expander graph, we can simulate the uniform distribution approximately, and thus estimate quantities such as

$$h^* = \text{ave}_{s \in S} h(s)$$

for specified $h: S \to R$. A more interesting issue is to use such simulation to approximately count the size $|S_K|$. This is studied in detail by Sinclair and Jerrum [39], who develop a notion of "self-reducibility" for families (S_K) which permit approximate counting. To specify a simple example, for a given graph G let S_G be the set of partitions

$$s = \{(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)\}; k \le |G|/2$$

into edges (v_i, w_i) where the vertices $v_1, w_1, v_2, w_2, \ldots$ are distinct. There is no usable exact formula for $|S_G|$, but we can make S_G into a graph with edges (s_1, s_2) such that the edges of G comprising s_2 and s_1 are identical except for some one edge (v_i, w_i). Performing random walk on S_G allows approximate counting of that set.

A continuous variant of the above is the problem of approximating the volume of a convex body $S \subset R^d$ in high dimensions d. By considering a fine grid, one can re-interpret this as the problem of approximate counting for a "convex" subset $S^* \subset Z^d$. One can then seek to do the counting by running random walk on S^*, considered as a subgraph of Z^d. This is studied in detail by Dyer et al [26].

Context 7. *Random spanning trees in regular graphs.* A connected finite graph G has a set of *spanning trees*, that is subgraphs on all vertices which are trees. The number of spanning trees is given by a classical expression, the *matrix-tree formula*, found in many textbooks on graph theory. Now consider the (uniform) random spanning tree in G. Studying its properties via combinatorial methods is not easy, since this involves counting the number of spanning trees with some extra property, and typically there are no nice formulas available. However, there is a probabilistic construction of the random spanning tree using random walk, as follows. Start the walk $X(n)$ from an arbitrary vertex $X(0)$. For

each $v \neq X(0)$ let T_v be the first hitting time on v. Then the graph with edges $(X(T_v - 1), v)$ is a random spanning tree, and it turns out (but seems non-obvious) that this random tree is indeed uniform on the set of all spanning trees. Thus one can use properties of random walk to study the random spanning tree. See Aldous [4] and Broder [16]}.

Context 8. *Planar graphs.* The usual square lattice in R^2 has edges of constant length and vertices of degree 4. The triangular lattice has edges of constant length and vertices of degree 6. There is a sense in which "6" is critical. Conklin [19], using results of Dodzuik [23] and Doyle and Snell [25], proves the following result.

Let G be an infinite planar graph with edge-lengths l_e. Suppose there exist $\varepsilon > 0, n_0 < \infty$ such that, for each connected finite set of vertices of size at least n_0, the average degree is at least $6 + \varepsilon$. Then $\sup_e l_e / \inf_e l_e = \infty$.

The proof can be phrased in terms of random walk. The "average degree > 6" hypothesis implies the walk is transient; whereas if G is planar and the edge lengths were bounded above and below then the walk would be recurrent.

Context 9. *Voter Models.* One of the major post-1970 developments in probability theory has been the study of interacting particle systems on infinite lattices: see the book of Liggett [35]. The following finite version has been studied by Donnelly and Welsh [24]. Consider a graph where each vertex is colored, initially with different colors. Each vertex from time to time (precisely, at times of independent Poisson processes) picks an adjacent vertex at random and changes its color to the color of the picked neighbor. Eventually, on a finite graph, all vertices will have the same color: how long does this take? This question turns out to be related (via a certain notion of *duality*) to the following question. Imagine particles, initially one at each vertex, which perform random walk on the graph, but which coalesce when they meet. Eventually they will all coalesce into one particle: how long does this take? Such questions are analogous to the cover time problems in Context 4.

These questions have been studied more in the setting of infinite graphs. There, the chance two walks coalesce may be less than 1, which turns out to be equivalent to the original voter model having more than one stationary distribution.

Context 10. *Diffusions on regular fractals.* The d-dimensional diffusion (= continuous-path, strong Markov) processes arising in most applications behave

locally like Brownian motion with some drift and covariance functions, plus some boundary behavior. Quite different diffusions can be constructed to live on regular fractal sets such as the Sierpinsky gasket (divide an equilateral triangle into 4 smaller triangles, remove the center one, and repeat this procedure on the 3 remaining regions.) One can regard the fractal set S as a limit of finite graphs S_K and the diffusion on S as the renormalized limit of random walks on S_K. The properties of the diffusion may be established by first studying properties of the finite random walks. See Kusuoka [34], Barlow and Perkins [7], Krebs [33].

Context 11. *Random fractals.* Physicists believe that at "critical points" there is typically some type of "fractal" behavior. In undergraduate probability theory, the word *critical* arises in one place, in the context of simple branching processes. Consider a non-trivial critical (mean number of offspring = 1) branching process, and condition on non- extinction (i.e. condition on non-extinction at generation n and take weak limits of the whole process as $n \to \infty$.) This process, considered as a tree in the natural way, is a prototype for a class of "random fractals occuring at critical points in random processes". More complicated and more interesting random fractals occur as critical percolation clusters. One method of studying these fractal sets has been to study random walk on them. For instance, on the critical branching tree the distance from the origin at time n grows at rate $n^{1/3}$: see Kesten [32].

3. Topics and Techniques. We now turn from the extrinsic to the intrinsic. If we agree that random walks are worth studying, what are natural mathematical questions to ask about them? Let us list some topics which have been studied, and mention some techniques used. My own particular interest has been the case of large, finite, unstructured graphs, and that will be the central theme. But many of the topics discussed will be one step removed from this area, in one of many possible directions.

Technique 12. *Reversible Markov chains.* Simple random walk is an example of a *reversible* Markov chain. Where the graph has no special symmetry, there are few special techniques available beyond the general techniques for treating reversible chains. Thus it is arguable that the "natural mathematical setting" for the study of random walk is within the larger class of general reversible Markov chains. From this viewpoint, one special property of our random walks is that there is a simple explicit formula for the stationary distribution

$$\pi(v) = \frac{d(v)}{\Sigma_w d(w)}; d(v) = \text{degree of } v \tag{1}$$

and that this stationary distribution is typically roughly uniform. This contrasts with other settings where reversible chains arise:

- queueing networks - Kelly [31]
- interacting particle systems - Liggett [35]
- physical systems - Whittle [43]

where useful explicit formulas are often hard to find, and where typically one gets stationary distributions of the form $\pi(x) = C \exp(-V(x))$.

Topic 13. *Exact formulas/ group representation theory.* Exact formulas for n-step distributions or first hitting times can only be expected for graphs with very special structure. Given a discrete group G and a subset H (typically, H is a small set of generators) with $H = H^{-1}$, we can construct the *Cayley graph* which has vertex-set G and edges $(g, gh) : g \in G, h \in H$. Such graphs satisfy one of a hierarchy of symmetry conditions (see e.g. Biggs [10].) Given enough symmetry, it becomes feasible to use group representation theory (the analog of Fourier analysis for random walk in R^d) to get more-or-less explicit formulas. Diaconis [21] gives a fine introduction.

Problem 14. *Weakening symmetry conditions.* For most applications it is unreasonable to assume any group-theoretic symmetry condition. On the other hand, if a graph is merely assumed to be regular, then the random walk may behave quite badly. As a specific instance, consider asymmetry of mean hitting times. Vertex-transitivity implies $E_v T_w = E_w T_v$, whereas for regular graphs the ratio may grow polynomially fast as the size of the graph increases. One would like some "niceness" condition which forced ratios $E_v T_w / E_w T_v$ to stay bounded or grow only slowly.

Technique 15. *Hard Analysis.* Random walks specialize reversible chains, which in turn may be considered as discretizations of d-dimensional reversible diffusions (without the geometry of d-dimensional space), which in turn generalize the physics of the heat equation and the mathematics of Brownian motion. This gives an indirect link to a large chunk of classical physics and analysis, and some of the ideas developed there carry over to our setting. One natural concept is the *Dirichlet form*

$$<f,f> \equiv \frac{1}{2|E|} \sum_{(i,j) \in E} (f(j) - f(i))^2 \tag{2}$$

where f is a function on the vertices and E denotes the set of edges. For instance, the dominant eigenvalue λ_2 controlling the asymptotic rate of convergence to the stationary distribution can be characterized as

$$\lambda_2 = 1 - \inf_f \frac{<f,f>}{\Sigma_i \pi(i) f^2(i)} \tag{3}$$

There are several ways to actually bound this quantity for specific graphs: an elegant way, involving specifying a path between each pair of vertices and seeking to minimize the number of such paths using any fixed edge, was introduced in [30] and is treated systematically by Diaconis and Stroock [22]. Deeper results relating the distributions at time t to properties of the Dirichlet form are given by Carlen et al. [17] in the more general setting of reversible chains.

Another powerful result whose proof comes from classical analysis is a universal large deviation bound due to Carne [18]. For random walk (X_n) on any regular graph,

$$P_v(X_n = w) \le P(|S_n| \ge d(v,w))$$

where S_n is simple symmetric random walk on the integers and d is graph-distance (= minimum path length). When $d(v,w) \gg n^{1/2}$, the elementary Binomial bounds make the right side small. In particular, for graphs of polynomial growth (i.e. the number of vertices within a ball of radius r grows only polynomially) we can deduce that $d(X_n, X_0) = O(n^{1/2} \log^a(n))$. Sharp results of this kind for subgraphs of Z^d are given by Barlow and Perkins [8].

Topic 16. *Infinite and finite graphs.* For random walk on an infinite graph, one can ask questions like: is it transient or recurrent? does it have non-trivial boundary? At first sight such questions have no relevance for finite graphs. But it turns out that they do relate to properties of *sequences* G_K of finite graphs with $|V_k| \to \infty$. Loosely, with an infinite graph G one can often associate finite graphs G_K which can be thought of as "finitizations" of G. For instance, with Z^d we associate Z^d modulo K. On the other hand there are many sequences of finite graphs (e.g. the cube graphs) with no natural limit infinite graph: in a sense, the notion of "sequence of finite graphs" is more general than "infinite graph".

For example, a sequence G_K of finite graphs may or may not have the property

$$\text{ave}_{vw} E_v T_w / |V_K| \text{ is bounded in K.} \tag{4}$$

When the G_K are finitizations of an infinite G, this property is equivalent to transience of the random walk on G. Thus in general we can regard this property as the finite analog of transience.

Technique 17. *Naive and threshold limits.* Classical probability theory was much concerned with the asymptotic behavior of a fixed process as time $\to \infty$. Such *naive limit theory* is rather trite for random walks on finite graphs. Instead, one can consider non-asymptotic properties of random walk as the size of the graph increases. Limit theorems in this setting are sometimes called *threshold* results, a term used for analogous results in the theory of random graphs (see Bollobas [12]}). As an example, under the boundedness assumption (4) one can show that normalized first hitting distributions converge to a limiting exponential distribution: see Aldous [1].

Topic 18. *Isoperimetric inequalities.* An inequality of the type: for all finite subsets A of vertices of a graph G,

$$\#\text{edges}\,(A \to A^c) \geq c \min(|A|, |A^c|)$$

is called an isoperimetric inequality. The largest c permitted (strictly positive, for a finite connected graph) is the isoperimetric constant for G. One can use c to bound λ_2 (recall (3)) and thence the time for convergence to the stationary distribution: see Jerrum and Sinclair [39] (this is one technique used in Context 6). Thus for finite G, the property that c is not close to 0 (such G's are loosely called *expanders*) is equivalent to the rapid mixing property of random walk on G.

For an infinite graph we have a rigorous definition: it is an expander iff c > 0. This graph-theoretic property corresponds sharply to some probabilistic properties of the random walk:

- The random walk has non-trivial boundary
- $P_v(X_n = v) \to 0$ exponentially fast.

See Gerl [28] for a survey. These results allow us to regard "rapid mixing" for walks on finite graphs as the finite analog of "non-trivial boundary" for walks on infinite graphs.

A "pure math" treatment of the connection between eigenvalues and expanders, in the context of graphs associated with groups, is in the forthcoming book by Lubotzky [36].

Technique 19. *Analytic and probabilistic parameters.* One can define parameters of a finite graph which measure typical values of quantities such as mean hitting times, cover times, or convergence times. Much of the theory of random walk deals explicitly or implicitly with inequalities between such parameters. It turns out there are two parallel theories for such parameters. One is an analytic theory, using eigenvalues such as λ_2, which can be interpretated as an "average-case" (over starting places) or L^2 theory. The other is a probabilistic theory, using techniques such as coupling (see e.g. Diaconis [21]; Aldous [5]}) to study the "worst-case" (over starting places) or L^∞ theory. The interplay between these two methods is a recurring theme.

Technique 20. *Pure graph theory.* The part of pure graph theory which relates most to our concerns is perhaps the spectral theory of graphs, treated in detail by Cvetkoviec et al [20]. [*Aside.* Actually [20] treat spectral theory of the adjacency matrix, which has no direct probabilistic relevance. More recently graph theorists have studied spectral theory of the Laplacian matrix Q, which is essentially the transition rate matrix for the continuous-time Markov chain with transition rate 1 along each edge: see e.g. Mohar [37]. Our concern is primarily with the transition matrix P, whose study reduces to that of Q iff the graph is regular. Of course, similar techniques can be used on all these matrices.]

It would be foolish to pretend that our random walk studies are central to pure graph theory. Many topics in graph theory (chromatic number, perfect graphs) involve qualitative or quantitative properties of graphs which do not seem to relate significantly to properties of random walk on the graph. Let us merely observe that *occasionally* random walk is useful in answering non-random questions about graphs (e.g. Contexts 4, 6, 8). And *occasionally* considering random walk on graphs will illuminate connections between properties of graphs. For example, the connection between eigenvalues and expanders is a little mysterious in itself, but become clear when you realize that both measure how rapidly random walk is converging to its stationary distribution.

Topic 21. *Exact extremal graphs.* The subject of extremal graphs - i.e. those for which general inequalities become equalities - is an established part of graph theory (see e.g. Bollobas [11]). As mentioned before, much of the theory of random walks involves bounds between parameters of the walk and parameters of the graph, so it is natural to ask for the exact extremal graphs where the bounds are attained. It is easy to show that the complete graph is extremal in various senses. For example, there is a general inequality

$$E_\pi T_v \geq (1-\pi(v))^2/\pi(v)$$

with equality iff the graph is complete. (π is the stationary distribution (1)). The only known hard extremal result about general graphs concerns $t^* \equiv \max_{v,w} T_v$. Brightwell and Winkler [15] show that t* is maximized, over n-vertex graphs, by a graph consisting of a complete graph on $k \approx 2n/3$ vertices, with an attached path of length n-k.

On *trees* there are simple explicit formulas (apparently due to Moon [38]) for means and variances of hitting times, so it is not surprising that extremal results are somewhat easier. Results concerning hitting times are given by Yaron [44], and results concerning cover times by Brightwell and Winkler [14]}.

Topic 22. *Fractional dimension.* For a subgraph G of the lattice Z^d, or more generally for a graph in which the number of vertices within a ball of radius r is $O(r^d)$ for some d, it is natural to try to assign some "fractional dimension" $\dim(G) \leq d$ to G. As mentioned in Context 11, one can try to relate definitions of dimension to properties of random walk on G : for recent work see Barlow and Taylor [9], Telcs [41, 42].

Technique 23. *Martingale methods.* Martingales occur naturally in some questions about random walk on *infinite* graphs, e.g. boundary theory. But the key results about martingales are the general inequalities, which are purely finitistic; one would expect these to be useful in establishing inequalities concerning finite random walks. Surprisingly, very little use of martingales is made in the existing literature (Aldous [2] is one exception): perhaps they will be exploited more in future.

A more extensive bibliography concerning random walks on finite graphs is available from the author.

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