ISING MODELS AND DEPENDENT PERCOLATION

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An intimate relation between Ising and certain dependent percolation models was discovered some twenty years ago by Kasteleyn and Fortuin and developed more recently by Swendsen and Wang. We review this relation and the role of stochastic domination within it. When the Ising model is not ferromagnetic (i.e., not positively dependent), the related percolation model is more complicated but still of interest.

1. Introduction. Although percolation and more recently Ising models have been of interest to probabilists and statisticians for some time, they have been largely unaware of the beautiful and useful relation which exists between the two types of systems. This relation, originally discovered by Kasteleyn and Fortuin (1969) was clarified by recent work of Swendsen and Wang (1987) on Ising simulation methods, and further explained by Edwards and Sokal (1988). Our purpose here is to describe the relation (Section 2), explain how it yields certain stochastic monotonicity properties (Section 3) of Fortuin (1972) and then mention some applications due to Aizenman, Chayes, Chayes and Newman (1987, 1988). We also discuss (Section 5) the situation when one goes beyond the case of ferromagnetic (i.e., positively dependent) Ising models. It should be noted that most (all?) of what is presented in this paper satisfies one or more of the following descriptions: old, already published, known to the experts. For more details related to Sections 3 and 4 (and much of Section 2), see Aizenman, Chayes, Chayes and Newman (1988).

2. Random Colored Graphs – Positively Dependent Case. Let Λ be a finite set of sites (or vertices) and $\mathcal{B}$ the corresponding set of all bonds (or edges); i.e., $\mathcal{B}$ is the set of pairs $b = \{x, y\}$ of sites. (For many applications Λ is a subset of some regular d-dimensional lattice, say $\mathbf{Z}^d$, and one takes $\Lambda \uparrow \mathbf{Z}^d$.) We will consider bond random variables n_b taking values 0 or 1 and site random variables T_x taking values in $\{1, \cdots, q\}$. The n_b's define a random graph with vertex set Λ in which

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a bond b is occupied (i.e., the edge b occurs) when $n_b = 1$; connected components of the graph are called clusters. The T_x's define a coloring of the sites from a set of q allowed colors. Our focus throughout this paper will be on certain natural randomly colored random graphs and on the corresponding marginal distributions of the T_x's alone and the n_b's alone. The colorings we consider will always be symmetric (i.e., the joint distribution will be invariant under exchanges of colors) but clearly generalizations to nonsymmetric situations are possible. Another type of generalization we will avoid (but which can be made in order to deal with Ising models with other than "pair interactions") is to consider, in addition to edges, randomly occupied faces, etc.

We begin by simply taking all the T_x's and n_b's to be jointly independent with the T_x's symmetric and with

$$Pr(n_b = 1) = p_b \quad \text{for each } b \in \mathcal{B}; \tag{1}$$

where we assume $0 \leq p_b < 1$. Such n_b's describe independent bond percolation; two important examples are nearest neighbor models (with $\Lambda \subset \mathbf{Z}^d$), where

$$p_{\{x,y\}} = \begin{cases} p, & \text{if } x \text{ and } y \text{ are nearest neighbors} \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

and $1/r^2$ models (with $\Lambda \subset \mathbf{Z}^1$), where

$$p_{\{x,y\}} = p_{|y-x|} \quad \text{with} \quad \lim_{x \to \infty} x^2 p_x = \beta. \tag{3}$$

We next introduce some dependence into this simple model by conditioning on the event

$$\{T_x = T_y \ \text{for every} \ b = \{x,y\} \ \text{with} \ n_b = 1\}; \tag{4}$$

i.e., *the sites in each cluster must have the same color.* Let μ^q denote the resulting joint (conditional) distribution of the T_x's and n_b's for some given Λ and p_b's. We leave it as an exercise for the reader to verify that the marginal distributions μ_v^q for the T_x's and μ_e^q for the n_b's, have densities (relative to uniform distributions),

$$\mu_v^q : \text{const. } \exp\left(\sum_{\{x,y\}} K_{xy} 1_{T_x = T_y}\right), \quad \text{where } 1 - e^{-K_{xy}} = p_{\{x,y\}}, \tag{5}$$

$$\mu_e^q : \text{const. } q^{\mathcal{C}} \prod_{b: n_b = 0} (1 - p_b) \prod_{b': n_{b'} = 1} p_{b'}, \quad \text{where } \mathcal{C} = \text{no. of clusters.} \tag{6}$$

When $q = 2$, μ_v^q is the Gibbs distribution of an Ising model; this may look more familiar when expressed in ± 1 valued variables S_x ($S_x = +1$ for $T_x = 1$, $S_x = -1$ for $T_x = 2$):

$$\mu_v^2: \ \text{const.} \exp\left(\frac{1}{2}\sum_{\{x,y\}} K_{xy}S_xS_y\right). \tag{7}$$

For $q = 3, 4, \cdots, \mu_v^q$ is the distribution of a Potts model. The random graph corresponding to (6) (Fortuin and Kasteleyn (1972)) is known as a Fortuin-Kasteleyn random cluster model (or simply FK model); it is a dependent percolation model which is perfectly well defined for non-integer values of q, even though μ_v^q and μ^q are not. Note that μ_e^q for $q = 1$ is just independent bond percolation.

Fortuin and Kasteleyn (1969, 1972) focused on the distribution μ_e^q (and not on the joint distribution μ^q as was implicitly done by Swendsen and Wang (1987)) and on how various Ising/Potts quantities can be expressed in terms of μ_e^q. For example, the reader may easily derive for $q = 2$:

$$\text{Cov}(S_x, S_{x'}) = \mu_e^2(x \text{ and } x' \text{ are in the same cluster}). \tag{8}$$

Such identities allow one to study Ising/Potts behavior by percolation methods.

3. Stochastic Domination. Among the most useful of percolation theoretic techniques are inequalities. Recall that the earliest example of associated random variables (variables such that increasing functions of them are positively correlated) is independent bond percolation. This result of Harris (1960) extends to FK models with $q > 1$ as shown by Fortuin (1972):

$$\text{for } q \geq 1, \text{ the } n_b\text{'s with distribution } \mu_e^q \text{ are associated.} \tag{9}$$

(9) can be easily proved by the "standard FKG method" of Fortuin, Kasteleyn and Ginibre (1971) and Sarkar (1969). We remark that for $q = 2$, the association of the n_b's is different from the association of the S_x's, which follows from the FKG method applied to μ_v^2.

To motivate the following stochastic monotonicity properties we note the elementary calculation,

$$\mu_e^q(n_b = 1 \mid \{n_{b'} : b' \neq b\}) = \ p_b \ \text{ or } \ \hat{p}_b \equiv \frac{p_b/q}{1 - p_b + p_b/q} \tag{10}$$

where the first value is taken if x and y are in the same cluster even with $n_b = 0$ and the second value otherwise. It was shown by Fortuin (1972) that

$$\mu_e^q \text{ is stochastically decreasing in } q \geq 1 \text{ for fixed } p_b\text{'s} \tag{11}$$

(i.e., the expectation of an increasing function is decreasing) and

$$\mu_e^q \text{ is stochastically increasing in } q \geq 1 \text{ for fixed } \hat{p}_b\text{'s} \tag{12}$$

These domination results follow from (9) and the fact that $\mathcal{C}$, the number of clusters, is a decreasing function while $\mathcal{C}$ plus the number of occupied bonds is increasing; the reader can supply the details.

4. Applications of Domination. In Ising or Potts models (in which the limit $\Lambda \uparrow \mathbf{Z}^d$ has already been taken) one is typically interested in the absence or presence of long range order. For our purposes, we will take this as synonymous, in the case of Ising models, with the vanishing or nonvanishing of $\lim_{\|x\| \to \infty} \mathrm{Cov}(S_0, S_x)$. A phase transition is said to occur if the change of a parameter (e.g., the p in a nearest neighbor model) switches the model from absence to presence of long range order.

By using identities such as (8) (note that its right-hand side is the expectation of an increasing function) together with both monotonicity results ((11) and (12)), one can generally conclude that a phase transition occurs for every real $q \geq 1$, if and only if it occurs for *some* real $q \geq 1$. By taking the special q to be 1, one can thus reduce the occurrence of Ising/Potts phase transitions to an issue of *independent* percolation! This approach is presented in great detail by Aizenman, Chayes, Chayes and Newman (1988), where it is applied to $1/r^2$ models and to situations where Λ tends to a logarithmic wedge in $\mathbf{Z}^d$; applications to dilute Ising/Potts models are given in Aizenman, Chayes, Chayes and Newman (1987). We remark that because the special q is 1, this approach does not even utilize the association of the FK models except for Harris' original independent percolation result. We conclude this section with a little more detail about one application to $1/r^2$ Ising models.

Consider fan $1/r^2$ independent percolation model (see (3)). If $\beta \equiv \lim_{x\to\infty} x^2 p_x = 0$ (in fact if $\beta \leq 1$), it was shown by Aizenman and Newman (1986) that (after $\Lambda \uparrow \mathbf{Z}^1$) there are a.s. no infinite clusters regardless of the individual p_x values. Next consider the related Ising model with $K_{xy} = K_{|x-y|}$ (see (5)). As a consequence of the stochastic domination of μ_e^2 by μ_e^1 and the identity (8) relating Ising and FK models, it follows from the independent percolation result that

(13) $$\lim_{x\to\infty} x^2 K_x = 0 \text{ implies absence of (Ising) long range order.}$$

We note that this result, a long standing conjecture in Ising model theory, was also independently proved (by other methods) by Berbee (1989).

5. Random Colored Graphs – General Case. The K_{xy}'s appearing in (5) (or (7)) are automatically non-negative. Let us consider more generally distributions with density

(14) $$\text{const. } \exp\left(\sum_{\{x,y\}} J_{xy} 1_{T_x = T_y}\right), \quad J_{xy} \in \mathbf{R}.$$

When $J_{xy} \geq 0$ for all x, y the Ising or Potts model is called ferromagnetic and when $J_{xy} \leq 0$ it is antiferromagnetic. It is quite fashionable these days to consider models with both positive and negative J_{xy}'s; spin glasses are of this type with the J_{xy}'s themselves random. In this section, we describe the generalization of FK models to the non-ferromagnetic context. The first published discussion of non-ferromagnetic FK models we are aware of is by Kasai and Okiji (1986). (We thank A. van Enter for informing us of this reference.) The most general extension of FK models is that of Edwards and Sokal (1988). Here, we only consider Ising or Potts models as in (14).

First partition the set of all bonds $\mathcal{B}$ into $\mathcal{B}_F$, the ferromagnetic ones (or F-bonds) corresponding to $J_{xy} \geq 0$, and $\mathcal{B}_A$ the antiferromagnetic ones (or A-bonds) where $J_{xy} < 0$. Define $p_b = 1 - \exp(- \mid J_{xy} \mid)$ for $b = \{x, y\}$ and again begin with independent T_x's and n_b's as before, but this time condition on the event

$$\begin{aligned} (15) \qquad \{T_x = T_y \quad &\text{for every} \quad b = \{x,y\} \in \mathcal{B}_F \text{ with } n_b = 1 \text{ and} \\ T_x \neq T_y \quad &\text{for every} \quad b \in \mathcal{B}_A \text{ with } n_b = 1\}. \end{aligned}$$

We denote the resulting joint distribution $\tilde{\mu}^q$ and the two marginals $\tilde{\mu}^q_v$ and $\tilde{\mu}^q_e$. $\tilde{\mu}^q_v$ can readily be seen to be precisely the Ising model (14). The density of $\tilde{\mu}^q_e$ can be expressed in the form,

$$(16) \qquad \text{const.} \prod_C R_q(C) \prod_{b:n_b=0} (1 - p_b) \prod_{b':n_{b'}=1} p_{b'}$$

where the first product is over all clusters C of the n_b's and $R_q(C)$ is the number of allowed colorings of C according to the rules that endsites of F-bonds (respectively A-bonds) have the same (respectively different) colors. Here C is thought of as a connected subgraph in which each edge is labelled F or A.

In the purely ferromagnetic case, R_q is of course simply q and we are back to the usual FK model measure (6). In the purely antiferromagnetic case, $R_q(C)$ is the number of q-colorings of the graph C in the usual sense of graph colorings where adjacent vertices must have different colors. For a given cluster C, $R_q(C)$ may vanish if q is not sufficiently large; for example for C the complete graph on k vertices in the purely antiferromagnetic case,

$$R_q(C) = \frac{q(q-1)\cdots(q-k+1)}{k!}.$$

This formula shows something besides the vanishing of R_q for certain integer values of q; it shows that q cannot in general be taken nonintegral (as can be done in the standard FK models) since that can lead to negative values of R_q.

In addition to clusters which are ruled out for small q, there are ones ruled out for all q—namely if there is an occupied A-bond between two sites connected by a path of occupied F-bonds. If C is not one of these totally prohibited clusters, then $R_q(C)$ can be calculated as the ordinary number of q-colorings of a "reduced graph" whose vertices are the connected components of C obtained by only using F-bonds and in which an edge occurs whenever there is at least one A-bond in C between the two corresponding components.

For the remainder of our discussion, we restrict attention to Ising models ($q = 2$). Here it is clear that for any C, $R_2(C)$ either vanishes or else equals exactly 2. Borrowing terminology from spin glass theory, we will call any configuration of the n_b's with a cluster C having $R_2(C) = 0$ a *frustrated* configuration; i.e., a configuration of the n_b's is frustrated if there is *no* 2-coloring of the sites which satisfies the *occupied* F-bonds and A-bonds. Let us write U for the collection of unfrustrated configurations of the n_b's. Then the density of $\tilde{\mu}_e^2$ may be expressed as

$$\tilde{\mu}_e^2 : \text{ const. } 1_U 2^C \prod_{b:n_b=0} (1-p_b) \prod_{b':n_{b'}=1} p_{b'}. \tag{17}$$

The Ising model covariance may be expressed in terms of the $\tilde{\mu}_e^2$ expectation (which we denote $\tilde{E}_e^2$) as (compare (8))

$$|\ \mathrm{Cov}(S_x, S_{x'})\ | = |\ \tilde{E}_e^2(\eta(x,x'))\ | \leq \ \tilde{\mu}_e^2\ (x \text{ and } x' \text{ are in the same cluster}) \tag{18}$$

where

$$\eta(x,x') = \begin{cases} 0, & \text{if } x \text{ and } x' \text{ are not in the same cluster} \\ -1, & \text{if there is a path of occupied bonds between } x \text{ and } x' \text{ with an odd number of } A\text{-bonds} \\ +1, & \text{otherwise.} \end{cases} \tag{19}$$

Although this measure seems rather difficult to work with, it can at least be used to obtain some modest results. We present these now primarily as an exercise in using $\tilde{\mu}_e^2$ rather than because of their intrinsic merit. Better results will presumably require an analysis of $\tilde{E}_e^2(\eta(x,x'))$ deeper than the trivial inequality of (18). (We remark that a formula similar to (18) appearing as Equation (A·7) in Kasai and Okiji (1988) appears to be seriously incorrect in that η is not averaged.)

For given J_{xy}'s, we let μ_e^2 denote the ($q = 2$) ferromagnetic FK measure, (6); its related Ising variables will be denoted S_x while those from $\tilde{\mu}_v^2$ will be denoted $\tilde{S}_x$. Any frustrated configuration of n_b's remains frustrated when an $n_{b'}$ is changed from 0 to 1 (whether b' be an F-bond or an A-bond); hence 1_U is a decreasing function on the n_b's. Since μ_e^2 is associated, it follows that

$$\tilde{\mu}_e^2 \text{ is stochastically dominated by } \mu_e^2. \tag{20}$$

We thus have from (18) (with tildes) and (8) that

$$|\ \mathrm{Cov}(\tilde{S}_x, \tilde{S}_{x'})\ | \leq \ \mathrm{Cov}(S_x, S_{x'}). \tag{21}$$

We conclude that absence of long range order in the Ising model with J_{xy} replaced by $|\ J_{xy}\ |$ implies its absence in the original model. For example, (13) remains valid with K_x replaced by J_x regardless of the signs of the J_x's.

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