INTERRELATIONS AMONG VARIOUS DEFINITIONS OF BIVARIATE POSITIVE DEPENDENCE

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In this paper—based on an extensive computer simulation—a detailed investigation and comparisons of seven types of positive dependence properties appearing in statistical and reliability literature is presented.

A numerical index of the "strength" of positive quadrant dependence (PQD) is proposed and compared with the correlation coefficient. This index can also be adopted to various other definitions of dependence.

1. Introduction. Dependence relations among variables constitute one of the basic topics of applied probability and statistics. This theory goes back to the classical investigations of Pearson in (1900) and (1904). While the concept of independence is mathematically defined by an equality relation, the violation of this equality by definition signifies dependence. Difficulties to provide an adequate measure of dependence are illustrated by the following statement of Cramér (1924): "Every attempt to measure a conception like this by a single number must necessarily contain amount of arbitrariness and suffer from certain inconveniences." In fact, Pearson (1900, 1904), Gini (1914), Fréchet (1951), Cramér (1924), Hoeffding (1940), Rényi (1959), Kolmogorov (1933), Lehmann (1966), and Lai and Robbins (1976) should be mentioned among the leading statisticians and probabilists who have studied this problem. The more recent work of Lehmann (1966) triggered an additional spurt of activity in this area after a certain period of dormancy and, in the last decade, we are witnessing a burgeoning awakening in this field which is closely associated with the renewed interest in statistical and probabilistic reliability methodology pioneered by the works of Barlow and Proschan (and summarized in their monograph (1981)).

A survey of results up to 1975 is presented in the paper by Kotz and Soong (1977) where some 10 properties of positive dependence have been discussed and

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interrelations among them have been analyzed in some details. See also Kotz (1980) for an updated but a shortened version. Concepts, properties, and measures of dependence are also discussed in Schweizer and Wolff (1976) and Yanagimoto (1972) among other sources.

In spite of substantial advances there still exists a certain confusion in the literature as to various implications of dependence definitions. The main purpose of this paper is to present a clear and self-contained definition of various dependence properties as proposed by various authors, in the last two decades and to provide an empirical study (to the best of our knowledge for the first time in the literature) which will indicate the presence or absence of the particular dependence properties as classified in this paper. These examples indicate the interrelation among these definitions is more delicate than it seems from the first glance. A Monte Carlo simulation was also carried out which may indicate the frequency of presence of particular dependence properties in the so called "typical" or natural models. These simulations are supplemented by appropriate graphical representations. The paper also examines, in some detail, the relationship between various measures of positive dependence and the classical measure of *linear* dependence—the coefficient of correlation.

Finally we propose a new *numerical index* of positive quadrant dependence and compare it with the correlation coefficient. A related extensive investigation for positive quadrant dependence has been conducted by Metry and Sampson (1988) using graph-theoretic methods. A general classification framework for positive dependence was recently developed by Kimeldorf and Sampson (1989).

2. Seven Types of Definitions of Non-negative Dependence Between Two Random Variables X and Y.

1) Non-negativity of the covariance:

$$\text{Cov}(X,Y) = E(XY) - E(X) * E(Y) \geq 0. \tag{1}$$

2) PQD(X,Y) - positive quadrant dependent.

The pair of variables (X,Y) satisfies the PQD(X,Y) property if:

$$P(X \leq x, Y \leq y) \geq P(X \leq x) * P(Y \leq y) \quad \text{for any } x \text{ and } y. \tag{2}$$

3) A(X,Y) - association.

The pair of variables (X,Y) satisfies the $A(X,Y)$ property if:

$$\text{Cov}(f(X,Y), g(X,Y)) \geq 0 \tag{3}$$

where f and g are non-decreasing functions of X and Y.

4) LTD($Y \mid X$) - Y is left tail decreasing in X.

The variables X, Y satisfy the LTD($Y \mid X$) property if:

(4) $P(Y \leq y \mid X \leq x_1) \geq P(Y \leq y \mid X \leq x_2)$ when $x_1 < x_2$ for all y.

5) RTI($Y \mid X$) - Y is right tail increasing in X.

The variables X, Y satisfy the RTI($Y \mid X$) property if:

(5) $P(Y > y \mid X > x_1) \leq P(Y > y \mid X > x_2)$ when $x_1 < x_2$ for all y.

6) a) CR$s(X, Y)$ property.

The pair of variables (X, Y) is said to be column regression dependent of order s (CRs) if for every $t = 1, 2, \ldots, s$, all $x_1 < x_2 < \ldots < x_t$ and all $y_1 < y_2 < \ldots < y_{t+1}$, the $(t+1) * (t+1)$ determinant

$$(6) \qquad \begin{vmatrix} P(X \leq x_1 \mid Y = y_1) & \ldots & P(X \leq x_1 \mid Y = y_{t+1}) \\ \vdots & & \vdots \\ P(X \leq x_t \mid Y = y_1) & \ldots & P(X \leq x_t \mid Y = y_{t+1}) \\ 1 & \ldots & 1 \end{vmatrix} \geq 0.$$

b) RR$s(X, Y)$ property.

The pair of variables (X, Y) is said to be row regression dependent of order s (RR$s(X, Y)$) if (Y, X) is CR$s(Y, X)$.

c) DR$s(X, Y)$ property.

If (X, Y) is both RRs and CRs dependent of order s, we call (X, Y) to be double regression dependent of order s (DRs).

These definitions are due to Schriever (1985).

RR1(X, Y) is also called PRD($Y \mid X$) $-$ Y is positively regression dependent on X.

The variables X, Y satisfy the PRD($Y \mid X$) property if:

(7) $P(Y \leq y \mid X = x_1) \geq P(Y \leq y \mid X = x_2)$ when $x_1 < x_2$ for all y.

7) TP$s(X, Y)$ property.

The pair of variables (X, Y) is said to be total positive dependent of order s if their joint density (or the mass function) $f(x, y)$ satisfies the following condition:

for any $t = 1, 2, \ldots, s$, all $x_1 < x_2 < \ldots < x_t$ and $y_1 < y_2 < \ldots < y_t$, the determinant

$$(8) \qquad \begin{vmatrix} f(x_1,y_1) & f(x_1,y_2) & \dots & f(x_1,y_t) \\ \vdots & \vdots & \dots & \vdots \\ f(x_t,y_1) & f(x_t,y_2) & \dots & f(x_t,y_t) \end{vmatrix} \geq 0.$$

The TP2(X,Y) and TP3(X,Y) property will be considered in this paper.

Exchanging X and Y in (4), (5), and (7), the definitions of LTD($X \mid Y$), RTI($X \mid Y$) and PRD($X \mid Y$) are given, respectively.

For brevity below we will use the term positive instead of non-negative.

3. A Network of Relationships Among Seven Types of Positive Dependence Properties. It has been proved by various authors (see citations in References) that the following implications among these 7 types of positive dependence properties between X and Y are valid (provided the appropriate covariances exist).

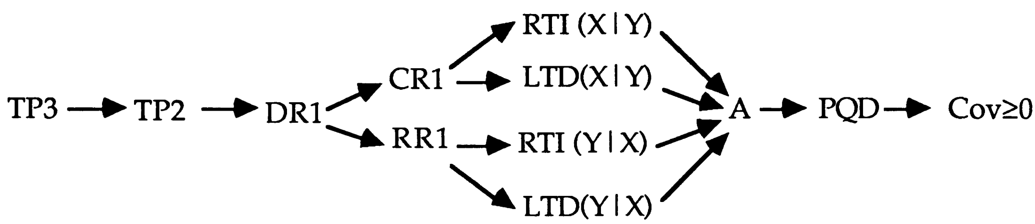


Figure 1

(See e.g. Lehmann (1966); Esary, Proschan, and Walkup (1967); Esary and Proschan (1970); Schriever (1985); and Bilodeau (1989)).

Schriever provides the theorem and a proof that TPs dependence implies DRs−dependence. In our Figure 1 we have the particular case TP2−>DR1.

All of the above implications are strictly held. Some numerical examples will be given in Section 5.

It should be noted that the properties TP3, TP2, DR1, A, PQD, and Cov$\geq$0 (on the middle line of the network) are symmetrical about X and Y, while the remaining ones are non-symmetrical.

4. Computer Simulation. To study the relations among these positive dependence properties, a Monte Carlo simulation was carried out using the following algorithm.

(1) Let g_{ij} be a uniform random number in (0,1), obtained from the BASIC subroutine, for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

(2) Let $P_{ij} = P(X = j, Y = i) = g_{ij}/[\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}\, g_{ij}]$ for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

It is obvious that $P_{ij} \geq 0$ and $\Sigma_{i=1}^{m}\Sigma_{j=1}^{n} P_{ij} = 1$.

Based on this algorithm a computer program was compiled. After running 30 times with different random seeds, 3000 cases with $m = 3$ and $n = 3$ have been generated. Frequencies of occurrences of the seven types of positive dependence properties stated above are presented in the Table 1 and the Figure 2. For brevity we shall denote LTD($Y \mid X$) by LTDx and similarly RTI($Y \mid X$) by RTIx as well as LTD($X \mid Y$) by LTDy and RTI($X \mid Y$) by RTIy.

Several remarks are in order.

1) In the above 3000 cases we did not find a single case which has the PQD property but not the A(X, Y) property. However, one such case was obtained in another simulation run (of *one million* cases related to Table 2).

2) There was not a single case observed such that both LTDx and RTIx were valid but not RR1 or both LTDy and RTIy were valid but not CR1. This is indeed impossible whenever $m = 3$ and $n = 3$. (See e.g. Esary and Proschan (1972).)

3) The frequencies of occurrences of the six one-sided tail properties are almost equal in pairs; a rather substantial decrease in frequencies of two-sided tail properties was observed.

4) Note the sharp decrease of the total number of cases from Cov(X, Y) ≥ 0 to PQD; from LTD and RTI to DR1 and from DR1 to TP2.

5) It is interesting to compare the PQD column of Table 1 with the second row of Metry and Sampson's (1988) Table 6.2 where the exact value 1/6 is obtained by enumeration. This shows that our simulation and their theoretical derivation yield very similar results.

Table 1. Total Numbers (T) and Percentages (P) of Occurrences of Various Dependence Properties in 3000 Simulations

	Cov≥ 0	PQD	A	LTDx	RTIx	LTDy
T	1494	503	503	314	320	308
P(%)	49.8	16.8	16.8	10.5	10.7	10.3
	RTIy	RR1	CR1	DR1	TP2	TP3
T	327	167	163	78	32	22
P(%)	10.9	5.6	5.4	2.6	1.1	0.7

In order to investigate the relation between the coefficent of correlation ρ and the positive dependence properties, we first subdivided the range of ρ from 0 to 0.75 into 15 intervals. Using the same algorithm as for the 3000 cases, additional one million cases were generated. For comparison purposes the first 1000 cases were selected for each of the first 13 intervals of ρ and over 100 cases were obtained for

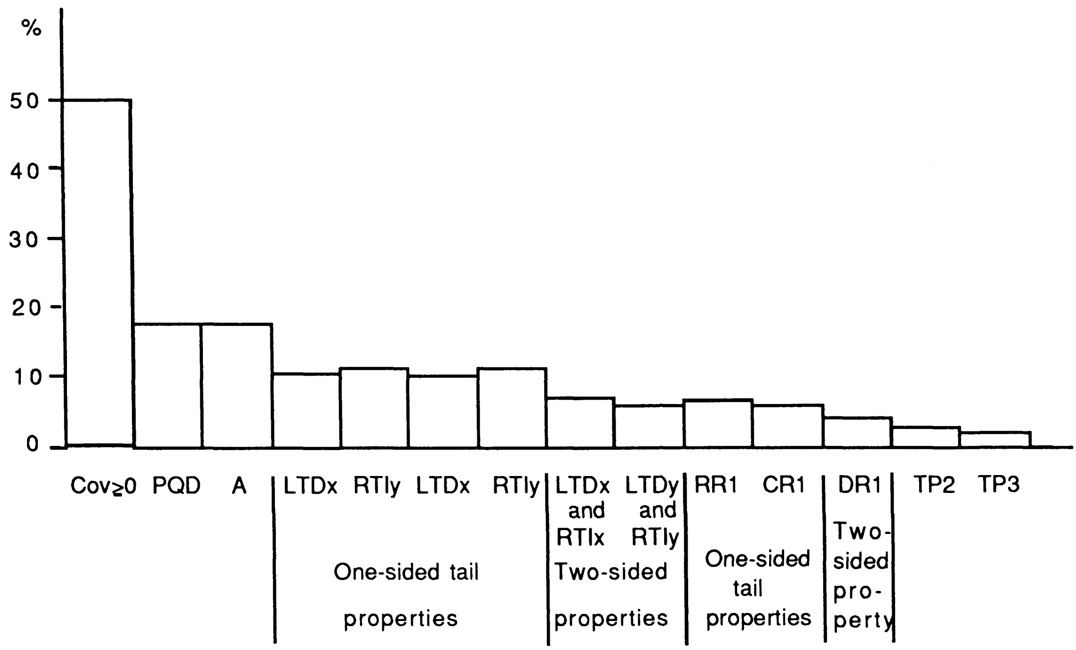


Figure 2. Percentages of Occurrences of Various Positive Dependence Properties

Table 2. Relations Between Frequencies Positive Dependence Properties and the Coefficient of Correlation

$\rho(X,Y)$	Mid-point	No. of CASES	PQD	LTDx	RTIx	RR1	DR1	TP2	TP3
(.00,.05)	.025	1000	5	3	2	1	1	1	1
(.05,.10)	.075	1000	69	32	38	13	3	1	0
(.10,.15)	.125	1000	199	102	99	33	8	4	2
(.15,.20)	.175	1000	368	183	206	69	30	13	9
(.20,.25)	.225	1000	535	316	313	137	65	28	16
(.25,.30)	.275	1000	705	439	440	204	91	50	24
(.30,.35)	.325	1000	810	551	531	305	160	57	32
(.35,.40)	.375	1000	892	636	615	378	210	109	66
(.40,.45)	.425	1000	941	723	692	485	295	152	104
(.45,.50)	.475	1000	962	736	720	505	323	181	136
(.50,.55)	525	1000	980	758	753	544	385	243	187
(.55,.60)	.575	1000	989	806	808	634	453	284	244
(.60,.65)	.625	1000	994	834	825	672	497	310	265
(.65,.70)	.675	407	407	342	338	277	220	141	135
(.70,.75)	.725	134	133	112	118	99	83	55	53
R-square			.838	.924	.946	.980	.968	.937	.881
Intercept			.135	.013	.008	-.085	-.110	-.083	-.088
Slope			1.496	1.352	1.356	1.188	.946	.611	.558

the last two intervals. The results are shown in Table 2. Since the frequency of the property $A(X,Y)$ is almost the same as that of PQD(X,Y) and the frequencies of LTDy, RTIy, and CR1 are very close to that of LTDx, RTIx, and RR1, respectively, only seven properties are included into Table 2. As ρ increases the frequency of virtually every property listed in Table 2 (except TP3) increases, but the effect of the increase of ρ is much more noticeable for the PQD, LTD, and RTI properties and is the least noticeable for TP2 and TP3. As a first approximation, simple linear regression models have been fitted setting ρ as an independent variable (using mid-points of the intervals) and the property as the dependent binary variables (if the property is valid, its value will be 1, and zero otherwise). Table 2 presents the intercept and slope of the regression equations, and the corresponding R-squares. The values of the R-square indicate that there exists a strong linear relationship between the correlation coefficient and the positive regression dependence. Note also a sharp sensitivity in the range (0.075–0.175) and a relative robustness in the range (0.425–0.625) of the values of $\rho(X,Y)$.

5. Ten Numerical Examples. In order to show that all of the implications in

Figure 1 are strict, we present ten numerical examples. Eight of them are obtained from the above 3000 cases and the remaining two involve additional computations. These examples seem to be the simplest for our purposes.

EXAMPLE 1.

$Y \setminus X$	1	2	3	
1	0.25	0.10	0.01	0.36
2	0.01	0.03	0.22	0.26
3	0.02	0.26	0.10	0.38
	0.28	0.39	0.33	

The table gives the probabilities $P(X = j, Y = i)$ and the marginal distributions $P(X = j)$ and $P(Y = i)$ for $j = 1,2,3$ and $i = 1,2,3$. For example, $P(X = 2, Y = 3) = 0.26$, $P(X = 1) = 0.28$, $P(Y = 2) = 0.26$. In this case, we have the coefficient of correlation = 0.476. However, the distribution does not possess PQD property since

$$P(X \leq 2, Y \leq 2) = 0.39 \text{ while } P(X \leq 2) * P(Y \leq 2) = 0.4154.$$

EXAMPLE 2.

$Y \setminus X$	1	2	3	
1	0.23	0.01	0.11	0.35
2	0.01	0.28	0.01	0.30
3	0.12	0.01	0.22	0.35
	0.36	0.30	0.34	

To show that the distribution possesses PQD(X,Y) property we form the table:

(i,j)	(1,1)	(1,2)	(2,1)	(2,2)
$P(X \leq j, Y \leq i)$	0.23	0.24	0.24	0.53
$P(X \leq j) * P(Y \leq i)$	0.126	0.231	0.234	0.429

For $i = 1,2$ and $j = 1,2$ we have strict inequality in (2) whereas equality holds for $i = 3$ or $j = 3$.

Hence the PQD property is valid. But there is no A(X,Y) property.

Indeed, let

$$f(X,Y) = \begin{cases} 0 & \text{if } X < 3 \text{ and } Y < 3 \\ 1 & \text{if } X = 3 \text{ or } Y = 3 \end{cases}$$

$$g(X,Y) = \begin{cases} 0 & \text{if } X = 1 \text{ or } Y = 1 \\ 1 & \text{if } X > 1 \text{ and } Y > 1 \end{cases}$$

f and g are non-decreasing functions of X or Y.

The distribution of $f(X,Y)$ and $g(X,Y)$ is then

$g \setminus f$	0	1
0	0.25	0.23
1	0.28	0.24

Since $\mathrm{Cov}(f(X,Y), g(X,Y)) = -0.0044$, the property $\mathrm{A}(X,Y)$ is not valid.

EXAMPLE 3.

$Y \setminus X$	1	2	3	
1	0.07	0.13	0.16	0.36
2	0.06	0.15	0.11	0.32
3	0.05	0.09	0.18	0.32
	0.18	0.37	0.45	

To show that $\mathrm{A}(X,Y)$ is valid we use the following theorem given in Esary, Proschan, and Walkup (1967).

Theorem. If $Cov(f(X,Y), g(X,Y)) \geq 0$ for all binary *non-decreasing functions f and g, then the pair of variables (X,Y) are associated.*

There are 20 different binary non-decreasing functions for this example (the case $m = n = 3$) which result in 400 different covariances. A computer program has been run and it was verified that all of the 400 covariances are non-negative.

However, neither $\mathrm{LTD}(Y \mid X)$ nor $\mathrm{RTI}(Y \mid X)$ are true in this case. Indeed: $P(Y \leq 2 \mid X \leq 1) = 0.722$, $P(Y \leq 2 \mid X \leq 2) = 0.745$ (so it is not LTDx) and $P(Y > 1 \mid X > 1) > P(Y > 1 \mid X > 2)$ (so it is not RTIx.)

Note that among our 3000 cases only 36 possess the properties of Example 3.

EXAMPLE 4.

$Y \setminus X$	1	2	3	
1	0.10	0.13	0.15	0.38
2	0.04	0.12	0.10	0.26
3	0.07	0.13	0.16	0.36
	0.21	0.38	0.41	

The corresponding conditional probabilities $P(Y \leq i \mid X \leq j)$, $i = 1,2,3$ and $j = 1,2,3$, indicate that LTDx property is valid. However, since $P(Y > 1 \mid X > 1) = 0.646$ and $P(Y > 1 \mid X > 2) = 0.634$ the RTIx does not hold.

EXAMPLE 5.

$Y \setminus X$	1	2	3	
1	0.37	0.11	0.02	0.50
2	0.04	0.12	0.16	0.32
3	0.05	0.02	0.11	0.18
	0.46	0.25	0.29	

The corresponding conditional probabilities show that RTIx property is valid. However, LTDx is not valid, since $P(Y \leq 2 \mid X \leq 1) = 0.89$ and $P(Y \leq 2 \mid X \leq 2) = 0.90$.

EXAMPLE 6.

$Y \setminus X$	1	2	3	4	
1	0.06	0.17	0.19	0.05	0.47
2	0.03	0.19	0.19	0.12	0.53
	0.09	0.36	0.38	0.17	

For this example, the values of $P(Y < 1 \mid X > j)$ and $P(Y > 1 \mid X > j)$, $j = 1,2,3,4$, indicate that both LTDx and RTIx are true.

However, PRD$(Y \mid X)$ is not valid since $P(Y \leq 1 \mid X = 2) = 0.47$ and $P(Y \leq 1 \mid X = 3) = 0.5$.

To avoid confusion we remind the reader that PRD$(Y \mid X)$ property is the same as RR1(X, Y) and below we shall use the later terminology.

EXAMPLE 7.

$Y \setminus X$	1	2	3	
1	0.19	0.15	0.09	0.43
2	0.02	0.02	0.01	0.05
3	0.21	0.19	0.12	0.52
	0.42	0.36	0.22	

The values of $P(Y \leq i \mid X = j)$, $i = 1,2,3$, $j = 1,2,3$, show that RR1 pattern exists.

However, the values of $P(X \leq j \mid Y = i)$, $(i = 1, 2, 3, j = 1, 2, 3)$, indicate that $\mathrm{CR1}(X, Y)$ property is not valid in this case.

Notice that neither $\mathrm{LTD}(X \mid Y)$ nor $\mathrm{RTI}(X \mid Y)$ are valid in this case.

EXAMPLE 8.

$Y \setminus X$	1	2	3	
1	0.10	0.04	0.06	0.20
2	0.25	0.01	0.24	0.50
3	0.04	0.10	0.16	0.30
	0.39	0.15	0.46	

The values of $P(X \leq j \mid Y = i)$ and $P(Y \leq i \mid X = j)$ indicate that this distribution is CR1, but not RR1.

EXAMPLE 9.

$Y \setminus X$	1	2	3	
1	0.14	0.13	0.09	0.36
2	0.11	0.13	0.10	0.34
3	0.09	0.12	0.09	0.30
	0.34	0.38	0.28	

The values of $P(Y \leq i \mid X = j)$ indicate that $\mathrm{CR1}(X, Y)$ is valid. The values of $P(Y \leq i \mid X = j)$ also indicate the existence of the $\mathrm{RR1}(X, Y)$ pattern. Therefore in this case the $\mathrm{DR1}(X, Y)$ property is valid. However, since

$$\begin{vmatrix} P(X = 2, Y = 2) & P(X = 2, Y = 3) \\ P(X = 3, Y = 2) & P(X = 3, Y = 3) \end{vmatrix} = -0.0003 < 0$$

the TP2 property is not valid.

EXAMPLE 10.

$Y \setminus X$	1	2	3	
1	0.09	0.16	0.11	0.36
2	0.04	0.08	0.08	0.20
3	0.08	0.17	0.19	0.44
	0.21	0.41	0.38	

The TP2(X,Y) property is valid since nine 2×2 related determinants are positive. However, the determinant

$$\begin{vmatrix} 0.09 & 0.16 & 0.11 \\ 0.04 & 0.08 & 0.08 \\ 0.08 & 0.17 & 0.19 \end{vmatrix} = -0.000004 < 0$$

indicates that TP3(X,Y) is not valid.

6. Pitfalls in the Relationship Between the Coefficient of Correlation and Other Six Types of Positive Dependence Properties. The implications of the seven types of positive dependence properties as presented in Figure 1 presumably indicates the degree of dependence between variables. However this relationship is somewhat more complex than it was originally envisioned.

In Example 1, Cov$(X,Y) = 0.319$ and the coefficient of correlation is the second largest among 10 examples ($\rho = 0.476$); however the ("weak") PQD property is not valid. At the same time in Example 7, $\rho = 0.037$ (less than 8% of the value obtained in Example 1), while here PRD$(Y \mid X)$ property holds. Example 10 provides the second strongest positive dependence as indicated by Figure 1, however the value of $\rho(0.114)$ is very small. If we have used ρ as an indicator of dependence, we would have concluded that X and Y are more "independent" in the situation of the Example 7 and 10 than in Example 1 and some other examples.

Our last example chosen from the 3000 generated cases illustrates a situation in which TP3(X,Y) property is valid.

EXAMPLE 11.

$Y \setminus X$	1	2	3	
1	0.07	0.06	0.04	0.17
2	0.14	0.14	0.11	0.39
3	0.13	0.16	0.15	0.44
	0.34	0.36	0.30	

The TP3(X,Y) property is valid. The verification is left to readers.

Again, we have a very low value of the correlation coefficient ($\rho = 0.104$) coupled with the presence of the strongest positive dependence TP3 property.

In conclusion we present two extreme examples that are not based on a computer simulation.

Let X and Y be independent and uniform:

$$P(X = x_j, Y = y_i) = 1/(mn) \quad \text{for} \quad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

Here we have $\rho = 0$, and moreover TP3(X,Y) and all of the seven types of positive dependence properties hold with the equality sign.

At the other extreme consider the family distribution parameterized by t:

$Y \setminus X$	1	2	3	
1	0	t	0	t
2	t	0.5-t	0	0.5
3	0	0	0.5-t	0.5-t
	t	0.5	0.5-t	

where t takes values in the interval (0,0.5).

Here $\text{Cov}(X,Y) = 0.25 + t - 4t^2$, $\text{Var}(X) = \text{Var}(Y) = 0.25 + 2t - 4t^2$, and $\rho = (0.25 + t - 4t^2)/(0.25 + 2t - 4t^2)$. As $t \to 0$, the probability mass concentrates at $(X = 2, Y = 2)$ and $(X = 3, Y = 3)$ and $\rho \to 1$, hence X and Y possess a very high linear relation. However, in this case even the PQD property is not valid, since

$$P(X < 1, Y < 1) = 0 < t^2 = P(X < 1)P(Y < 1).$$

7. Surface Representation of Bivariate Distributions With Selected Types of Positive Dependence. In this section we present several graphs which depict characteristic structures of the joint probability mass distributions for selected types of bivariate dependence.

1) Characteristic structure of distributions with a high positive correlation lacking most of the other positive dependence properties is presented in Graph 1 (corresponding to Example 1). The largest values of P_{ij} are on the main diagonal ($P_{11} = 0.25$) or near the main diagonal ($P_{23} = 0.22$ and $P_{32} = 0.26$), while P_{ij}'s that are far away from the main diagonal (at the corners) have low values ($P_{13} = 0.01$ and $P_{31} = 0.02$). The correlation coefficient $\rho = 0.476$ is "moderately high" (the second largest among all the examples). However, the probability P_{22} on the main diagonal at the center is only 0.03, which is too small to possess PQD and other positive dependence properties.

2) Graph 2 possesses the strongest positive dependence property TP3, while the correlation coefficient is very small ($\rho = 0.104$). The graph indicates that this distribution is closer to the uniform distribution than those in other examples.

3) The comparison between RR1 (but not CR1 or LTDy and RTIy) and CR1 (but not RR1 or LTDx and RTIx) is presented in Graphs 3 and 4 respectively. Note that the "horizontal" edges are prominent in Graph 3 while the "vertical" ones contain a substantial amount of probability mass in Graph 4.

8. A Proposed Modification of Definitions of Dependence Between Random Variables. The main difference between covariance (or the coefficient of correlation) and the other six properties of positive dependence discussed in this paper is that the latter simply provide a "Yes" or "No" answer to a certain probabilistic relationship, while the former presents a numerical measure of a linear dependence. Basically, the coefficient of correlation deals with an overall appraisal of "whole forest," while the other dependence properties attempt to check "the trees" individually.

It would seem appropriate to generalize the definitions of dependence between two variables by introducing a probabilistic component within the specified relationship. As an example, we propose an index of positive quadrant dependence defined below.

DEFINITION. An index of positive quadrant dependence of two random variables X and Y is given by:

$$\mathrm{IPQD}\phi(X \mid Y) = \Sigma_{(x,y)\epsilon R\phi} P(X = x, Y = y)/[1 - \Sigma_{(x,y)\epsilon R_b} P(X = x, Y = y)]$$

where $x = 1, 2, \ldots, n$ and $y = 1, 2, \ldots m$; $R\phi = \{(x, y) \mid P(X \leq x, Y \leq y) > P(X \leq x)P(Y \leq y) + \phi\}$, ϕ is a non negative parameter; and the boundary $R_b = \{(x, y) \mid x = n \text{ or } y = m)\}$, in this case.

Table 3 provides the index IPQDϕ for the examples presented in this paper for $\phi = 0$, 0.005, 0.01, 0.015, 0.02 and 0.05.

Note that in the column corresponding to $\phi = 0$ we obtain essentially "Yes" (IPQD$\phi = 1$) or "No" (IPQD$\phi < 1$) answers to the question "is the PQD property valid?". The other columns present values which successively indicate the strength of validity of the PQD property when ϕ ranges from 0.005 to 0.05.

An alternative—perhaps even more important—justification for the introduction of parameter ϕ is to assess the robustness of the properties as far as possible random errors are concerned. If our data is subject to random errors, the situation as presented in Examples 7, 9, and 11 may not indicate the presence of the PQD property while the values appearing in Examples 5, 8, 2, and 1 will very likely provide a "significant" PQD distribution.

Evidently the other definitions of positive dependence properties can be extended along these lines.

REFERENCES

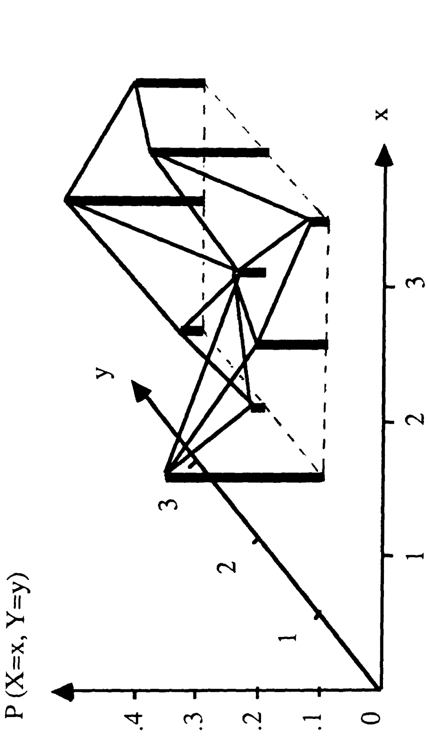
BARLOW, R. and PROSCHAN, F. (1981). *Statistical Theory of Reliability and Life Testing.* To Begin With, Silver Spring, Md.

BILODEAU, M. (1989). On the monotone regression dependence for Archimedian bivariate uniform. *Commun. Statist.-Theor. Meth.* **18(3)** 981–988.

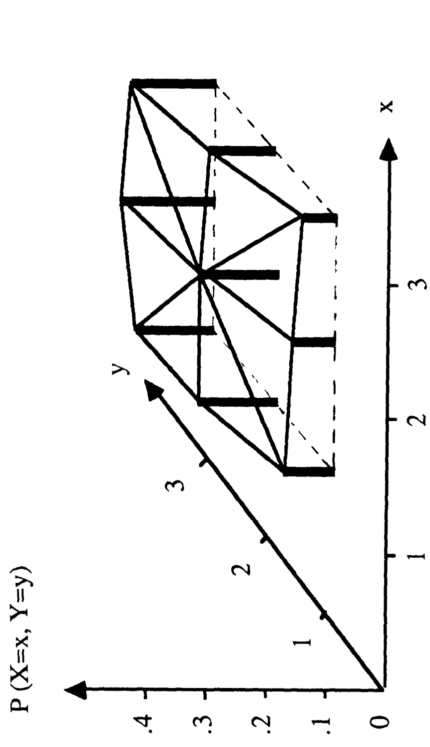
CRAMÉR, H. (1924). Remarks on correlation. *Skand. Aktuarietidskr.* **7** 220–240.

ESARY, J.D. and PROSCHAN, F. (1972). Relationships among some concepts of bivariate dependence. *Ann. Math. Statist.* **43** 651–655.

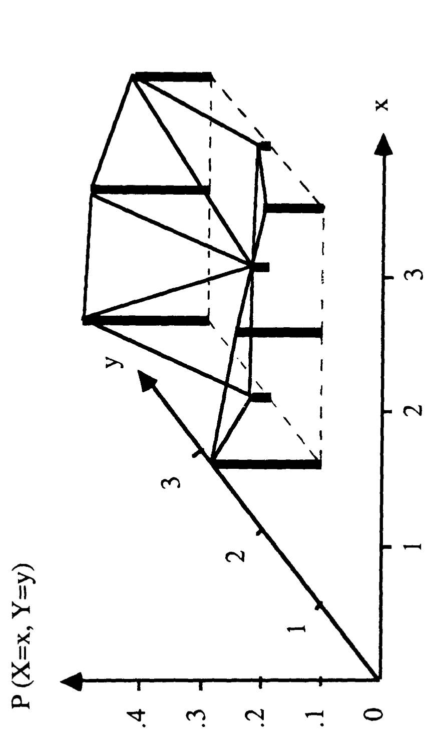
ESARY, J.D., PROSCHAN, F. and WALKUP, D.W. (1967). Association of random variables with applications. *Ann. Math. Statist.* **38** 1466–1474.



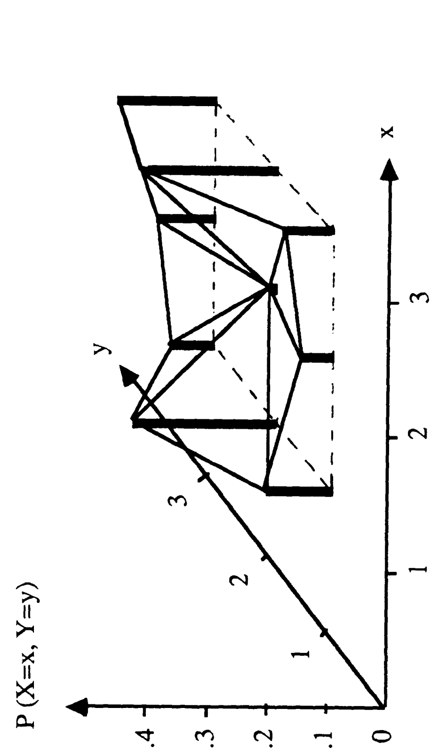
Graph 1 (Example 1) ρ = 0.478, PQD=0



Graph 2 (Example 11) ρ = 0.104, TP3=1



Graph 3 (Example 7) ρ = 0.037, RR1=1, CR1=0



Graph 4 (Example 8) ρ = 0.238, CR1=1, RR1=0

Table 3. Values of IPQDϕ for selected values of

Exmp.	Cov	ρ	$\phi = 0$	0.005	0.010	0.015	0.020	0.050
1	0.319	0.476	0.923	0.923	0.923	0.923	0.923	0.923
2	0.220	0.314	1	1	0.962	0.962	0.962	0.962
3	0.051	0.083	1	0.683	0.366	0.366	0.366	0
4	0.044	0.067	1	1	0.564	0.256	0.256	0
5	0.356	0.551	1	1	1	1	1	0.938
6	0.066	0.153	1	1	1	1	0.452	0
7	0.028	0.037	1	0.605	0	0	0	0
8	0.153	0.238	1	1	1	1	1	0.625
9	0.046	0.073	1	1	0.745	0.275	0	0
10	0.076	0.114	1	1	1	0.649	0.649	0
11	0.061	0.104	1	1	1	0.683	0	0

FRÉCHET, M. (1951). Sur les tableaux de correlation dont les marges sont donnees. *Ann. Univ. Lyon*, Sect. A, Ser. 3, **14** 53–57.

GINI, C. (1914). Di una misura della dissomiglianza tra due gruppi di quantita e delle due applicazioni allo studio delle relazioni statistiche. *Atti R. Ist. Veneto Sci. Lett. Arti.* (8) **74** 185–213.

HOEFFDING, W. (1940). Masstabinvariante korrelations-theorie. *Schriften Math. Inst. Univ. Berlin* **5** 181–233.

JENSEN, D.R. (1971). A note on positive dependence and the structure of bivariate distributions. *SIAM J. Appl. Math.* **20**, No. 4, 749–753.

JOHNSON, N.L. and KOTZ, S. (1975). On some generalized Farlie-Gumbel-Morgenstern distributions. *Commun. Statist.* **4** 415–427.

JOHNSON, N.L. and KOTZ, S. (1977). On some generalized Farlie-Gumble-Morgenstern distributions–II. Regression, correlation and further generalizations. *Commun. Statist. Theor. Meth.* **A6(6)** 485–496.

KARLIN, S. (1968). *Total Positivity, Vol. I.* Stanford University Press, Stanford, Calif.

KIMELDORF, G. and SAMPSON, A.R. (1989). A framework for positive dependence. *Ann. Inst. Statist. Math.* **41** 31–45.

KOLMOGOROV, A.N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Springer-Verlag, Berlin (English translation: Chelsea, New York, 1950).

KOTZ, S. and JOHNSON, N.L. (1977). Propriétés de dépendance des distributions itérées, généralisées â deux variables Farlie-Gumbel-Morgenstern. *C.R. Acad. Sc. Paris* **285**, Ser. A, 277–280.

KOTZ, S. and SOONG, C. (1975). On the measures of dependence, a survey of recent developments. Technical Report, Department of Mathematics, Temple University.

KOTZ, S. (1980). Dependence concepts and their application in probabilistic modeling. *Transact. of the 34-th ASQC Techn. Conference*, Atlanta, 245–251.

LAI, T.L. and ROBBINS, H. (1976). Maximally dependent random variables. *Proc. Nat. Acad. Sci.* U.S.A. **73** 286–288.

LEHMANN, E.L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

METRY, M.H. and SAMPSON, A.R. (1988). Characterizing and generating bivariate empirical rank distributions satisfying certain positive dependence concepts. Technical Report 88–04, Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, PA.

PEARSON, K. (1900). *Mathematical Contributions to the Theory of Evolution.* VII: On the correlation of characters not quantitatively measurable. *Philos. Trans. Roy. Soc.* A **195** 1–47.

PEARSON, K. (1904). *Mathematical Contributions to the Theory of Evolution.* XIII: On the theory of contingency and its relation to association and normal correlation. Drapers' Company Research Memoirs, *Biometric Series 1.*

RÉNYI, A. (1959). On measures of dependence. *Acta Math. Acad. Sci. Hungar.* **10** 441–451.

SCHRIEVER, B.F. (1985). *Order Dependence.* Vrije Universiteit te Amsterdam, Centrum voor Wiskunde en Informatica, Amsterdam, Holland.

SCHWEIZER, B. and WOLFF, E.F. (1976). Sur une measure dedependance pour les variables aleatoire. *C.R. Acad. Sci. Paris*, Ser. A, **283** 609–611.

SEEGER, J. (1978). Dependence with fixed marginals from a density viewpoint. Ph.D Dissertation, Department of Mathematics, Temple University, Philadelphia, Pa.

YANAGIMOTO, T. (1972). Families of positively dependent random variables. *Annals Inst. Statist. Math.* **24** 559–573.

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