CALCULATING IMPROVED BOUNDS AND APPROXIMATIONS FOR SEQUENTIAL TESTING PROCEDURES

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Recently developed product type bounds are utilized for calculating improvements of the expected stopping time, the variance of the stopping time and the power of various sequential testing procedures. These are presented for testing the mean of a normal distribution, but the results and techniques apply more generally. These bounds have been easily calculated to 5th order by taking advantage of the dependence structure. It should be noted that these improvements are most useful when the distribution for the stopping time has its probability mass shifted towards later stopping times. Testing procedures that are developed with an early stopping time are easily handled with lower order bounds such as 3rd order, which are faster to calculate. This will be demonstrated in the examples.

1. Introduction. Recently, there has been interest in obtaining improvements in bounds for multivariate probabilities (Games (1977), Glaz and Johnson (1984), Miller (1981), and Worsley (1982)). It has been shown that first order bounds such as the usual Bonferroni bound are often not sharp enough to be useful when there are many events, B_i, and the $P(B_i)$ are not "small", or when there is a strong dependence structure in the multivariate distribution (Glaz and Johnson (1984), Miller (1981), Schwager (1984), and Worsley (1982)). Let $Y_1, Y_2, \ldots$ denote independent and identically distributed random variables with mean θ and variance σ^2. Define $\tau = \inf\{k \geq 1 \mid S_k \notin I_k\}$, where $S_k = \sum_{i=1}^{k} Y_i$ and I_k are intervals of the same type: $(-\infty, a_k)$, (b_k, ∞) or $(-c_k, c_k)$. The quantity τ is the stopping time of the sequential testing procedure that is specified by the intervals, I_k. Recall that:

$$\begin{aligned} P(\tau > n) &= P(S_k \in I_k;\ k = 1, \ldots, n), \\ E(\tau) &= \sum_{n=0}^{\infty} P(\tau > n), \end{aligned}$$

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and

$$\text{Var}(\tau) = E(\tau)(1 - E(\tau)) + 2\sum_{n=1}^{\infty} nP(\tau > n).$$

Additionally, the power function, $\beta(\theta)$, is given by: $\beta(\theta) = P_\theta(S_\tau \in \text{Rejection Region}) = \sum_{n=1}^{\infty} P_\theta\{(\cap_{k=1}^{n-1} S_k \in I_k) \cap (S_n \in \text{ Rejection Region})\}$.

Power calculations will be performed under the assumption that crossing the boundary results in rejecting the null hypothesis. In the examples of Section 3, this occurs when the upper boundary is crossed.

The difficulty here is in calculating $P_\theta(S_k \in I_k;\ k = 1, \ldots, n)$, or the similar term for $\beta(\theta)$, particularly as n increases. By calculating improved bounds for these probabilities, improvement of bounds for $E(\tau)$, and approximations for $\text{Var}(\tau)$ and $\beta(\theta)$ are obtained.

We consider the following bounds to $\cup_{i=1}^{n} A_i$, where A_i are arbitrary events:

(1) first order upper Bonferroni bound

$$P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i),$$

(2) second order lower Bonferroni bound

$$P(\cup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j),$$

(3) Hunter's upper Bonferroni-type bound (A)

$$P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - 2/n \sum_{i<j}^{n} P(A_i \cap A_j),$$

(4) Hunter's upper Bonferroni-type bound (B)

$$P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n-1} P(A_i \cap A_{i+1}),$$

(5) Kwerel-Galambos lower bound (for $k \geq 2$)

$$P(\cup_{i=1}^{n} A_i) \geq 2/k \sum_{i=1}^{n} P(A_i) - 2/(k(k-1)) \sum_{i<j}^{n} P(A_i \cap A_j),$$

$$(\text{optimal } k = \ \text{INT}(2\sum_{i\leq j}^{n} P(A_i \cap A_j) / \sum_{i=1}^{n} P(A_i)) + 2,$$

where $\text{INT}(x)$ is the greatest integer $\leq x$),

(6) Sidak's first order product-type approximation

$$P(\cup_{i=1}^{n} A_i) \approx (\leq) 1 - \Pi_{i=1}^{n} P(A_i^c),$$

(7) Glaz-Johnson higher order product-type approximations

$$P(\cup_{i=1}^{n} A_i) \approx (\leq) 1 - P(\cap_{h=1}^{k-1} A_h^c) \Pi_{i=k}^{n} P(A_i^c \mid P(\cap_{j=1}^{k-1} A_{i-j}^c), \text{ for } k \in \{1, 2, \dots, n\}$$

(Note $k = 1$ is Sidak's bound)

We note that the approximations (6) and (7) are bounds in the noted parenthetical directions under certain positive dependence conditions (see Glaz and Johnson (1984)).

Since bounds (1)–(6) did not perform as well as (7), for these and other types of problems, attention has been focused on the bounds of Glaz and Johnson (1984). Also, these bounds exploit the dependence structure of the partial sums. Thus, it is not unexpected that they performed more favorably here. More recently, Glaz (1990) has shown that the product type bounds are superior to the Bonferroni type bounds under positive dependence conditions which are present here.

As observed, the conditional approximations of Glaz and Johnson are not always guaranteed to be bounds. A sufficient, but not necessary condition for these approximations to be bounds, is that the multivariate distribution be multivariate totally positive of order 2, MTP_2, (Karlin and Rinott (1980)).

In some cases, the bounds for $E(\tau)$ and approximations for $\mathrm{Var}(\tau)$ calculated by Glaz and Johnson (1986) for $k = 1, 2$, and 3, were not as good as one might desire. Here further conditioning, $k = 5$, will be employed to improve the bounds and approximations.

2. Calculation Methods. It is expected that evaluation of $P_\theta(S_k \in I_k)$ is available as well as the probability density function of S_k. The conditional probabilities above will be obtained from probability of the joint events and the probability of the conditioning event [e.g., $P(A \mid B) = P(A \cap B)/P(B)$]. For this problem, it should be noted that high dimensional numerical integration is not required. By conditioning on the middle term and using conditional independence we obtain the following formulas, which will be useful in calculating multivariate probabilities.

$$P(S_{k-1} \in I_{k-1}, S_k \in I_k) = \int_{I_{k-1}} P(S_k \in I_k \mid S_{k-1} = s) f_{S_{k-1}}(s) ds$$

$$P(S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k) =$$

$$\int_{I_{k-1}} P(S_{k-2} \in I_{k-2} \mid S_{k-1} = s) f_{S_{k-1}}(s) P(S_k \in I_k \mid S_{k-1} = s) ds$$

$$P(S_{k-3} \in I_{k-3}, S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k)$$

$$= \int_{I_{k-2}} P(S_{k-3} \in I_{k-3} \mid S_{k-2} = s) f_{S_{k-2}}(s) P(S_{k-1} \in I_{k-1}, S_k \in I_k \mid S_{k-2} = s) ds$$

$$= \int_{I_{k-2}} P(S_{k-3} \in I_{k-3} \mid S_{k-2} = s) f_{S_{k-2}}(s)$$

$$\times \int_{I_{k-1}} P(S_k \in I_k \mid S_{k-1} = t, S_{k-2} = s) f_{S_{k-1}|S_{k-2}=s}(t) dt$$

$$P(S_{k-4} \in I_{k-4}, S_{k-3} \in I_{k-3}, S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k)$$

$$= \int_{I_{k-2}} P(S_{k-4} \in I_{k-4}, S_{k-3} \in I_{k-3} \mid S_{k-2} = s)$$

$$\times f_{S_{k-2}}(s) P(S_{k-1} \in I_{k-1}, S_k \in I_k \mid S_{k-2} = s) ds$$

$$= \int_{I_{k-2}} f_{S_{k-2}}(s) \int_{I_{k-4}} P(S_{k-3} \in I_{k-3} \mid S_{k-4} = t, S_{k-2} = s) f_{S_{k-4}|S_{k-2}=s}(t) dt$$

$$\times \int_{I_{k-1}} P(S_k \in I_k \mid S_{k-1} = u, S_{k-2} = s) f_{S_{k-1}|S_{k-2}=s}(u) du\, ds$$

The sequential probability ratio test (SPRT), the triangular boundary test and the asymptotic optimal Bayes sequential test are well known and utilized sequential tests for testing the mean of a normal distribution (e.g., Glaz and Johnson (1986)). These consider the case where the Y_i are independent, normal random variables with mean θ and variance $\sigma^2 = 1$ (or σ^2 known). Consequently, the S_k are normal with mean $k\theta$ and variance k (or $k\sigma^2$), but are not independent. $\mathrm{Cov}(S_k, S_h) = \min(k,h)\sigma^2$. For the multivariate normal distribution, MTP_2 is equivalent to all the partial correlations being ≥ 0 or equivalently $-(\Sigma)^{-1} \geq 0$, where Σ is the variance-covariance matrix (Karlin and Rinott (1980)). This condition is satisfied here (Glaz and Johnson (1986)).

Thus the probability density functions and the conditional distribution functions for the above are all normal with the appropriate mean and variance.

3. Examples for the Normal Distribution. For testing $H_o : \theta = 0$ vs $H_a : \theta = \theta_1 > 0$, consider $X_i = (Y_i - \theta_1/2)$. The transformed hypothesis is $H_o : \theta = -\theta_1/2$ vs $H_a : \theta = \theta_1/2$ and the intervals determining the test boundaries will be symmetric about 0 since it is usually desired that $\alpha = \beta$. The boundaries for the various test procedures are as follows:

Test	Intervals I_k (a, θ_1 such that $\alpha = \beta$)
Wald's SPRT	$(-a/\theta_1,\ a/\theta_1)$
Triangular	$(-a + k/4\theta_1,\ a - k/4\theta_1)$ $k \leq M < \infty$
Optimal Bayes	$(-c_k + k/2\theta_1,\ c_k - k/2\theta_1)$ $c_k = \sqrt{2ak}, a > 0, k \leq M < \infty$.

Now, consider the Gauss-Laguerre formulas:

$$\int_0^\infty e^{-x} f(x) dx \approx \sum_{i=1}^n B_i f(x_i)$$

and the Gauss-Legendre formulas:

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} B_i[f(x_i) + f(-x_i)]$$
$$= \sum_{i=1}^{n} 2B_i f(x_i), \text{ when } f(x) \text{ is symmetric about 0.}$$

Since some variables of integration have semi-infinite limits, the Gaussian methods do have a natural advantage. Other methods require either truncation of the integral besides performing an approximation of this truncated integral, or usage of a transformation of the variable of integration which yields finite limits of integration. In addition, Stroud and Secrest (1966) give a comparison of Gaussian quadrature with other methods. For an equal number of points the Gaussian quadrature error is comparable to the other methods, even for cases where it is not believed the "best". They also compare different approaches to calculating some specific integrals with semi-infinite limits, including transformations to integrals with finite limits. All these results appear to indicate this approach for these particular densities. Additionally, the Gaussian methods do have some optimal properties for numerical integration.

These two numerical integration methods can be used for all three cases of I_k. This is accomplished as follows.

For $I_k = (-\infty, a_k)$:

$$\int_{-\infty}^{a_k} f(x)dx = \int_{0}^{\infty} f(a_k - s)e^{-s+s}ds \approx \sum_{i=1}^{n} B_i f(a_k - s_i)e^{s_i}$$

or

$$\int_{-\infty}^{a_k} f(x)e^{x}dx = \int_{0}^{\infty} f(a_k - s)e^{-s}e^{a_k}ds \approx e^{a_k}\sum_{i=1}^{n} B_i f(a_k - s_i),$$

or any appropriate transformation to obtain the desired integral in the form, $\int_0^\infty e^{-t}g(t)dt$.

For $I_k = (b_k, \infty)$:

$$\int_{b_k}^{\infty} f(x)dx = \int_{0}^{\infty} f(b_k + s)e^{-s+s}ds \approx \sum_{i=1}^{n} B_i f(b_k + s_i)e^{s_i}$$

or

$$\int_{b_k}^{\infty} f(x)dx = 1 - \int_{-\infty}^{b_k} f(x)dx$$

which then becomes the case of $I_k = (-\infty, a_k)$.

For $I_k = (-c_k, c_k)$:

$$\int_{-c_k}^{c_k} f(x)dx = c_k \int_{-1}^{1} f(s\, c_k)ds \approx c_k \sum_{i=1}^{n} B_i[f(s_i\, c_k) + f(-s_i\, c_k)].$$

We now apply these Gaussian integration formulas, which will be useful in calculating multivariate probabilities. Let $\Phi(\theta, \sigma^2, x)$ be the c.d.f. at x, and $\phi(\theta, \sigma^2, x)$ be the p.d.f. at x for a normal distribution with mean θ and variance σ^2. Then for $I_k = (-c_k, c_k)$:

$$\begin{aligned}
&P_\theta(S_k \in I_k) = \Phi(k\theta, k, c_k) - \Phi(k\theta, k, -c_k),\\
&P_\theta(S_{k-1} \in I_{k-1}, S_k \in I_k) \approx c_{k-1} \sum_{i=1}^{n} B_i\\
&\quad\times\quad [\Phi(\theta + c_{k-1}s_i, 1, c_k) - \Phi(\theta + c_{k-1}s_i, 1, -c_k)]\phi((k-1)\theta, k-1, c_{k-1}s_i),
\end{aligned}$$

$$\begin{aligned}
&P_\theta(S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k) \approx c_{k-1} \sum_{i=1}^{n} B_i\\
&\quad\times\quad [\Phi((c_{k-1}s_i)(k-2)/(k-1), (k-2)/(k-1), c_{k-2})\\
&\qquad\quad -\Phi((c_{k-1}s_i)(k-2)/(k-1), (k-2)/(k-1), -c_{k-2})]\\
&\quad\times\quad \phi((k-1)\theta, k-1, c_{k-1}s_i)[\Phi(\theta + c_{k-1}s_i, 1, c_k) - \Phi(\theta + c_{k-1}s_i, 1, -c_k)],
\end{aligned}$$

$$\begin{aligned}
&P_\theta(S_{k-3} \in I_{k-3}, S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k) \approx c_{k-2}c_{k-1} \sum_{i=1}^{n} B_i\\
&\quad\times\quad [\Phi((c_{k-2}s_i)(k-3)/(k-2), (k-3)/(k-2), c_{k-3})\\
&\qquad\quad -\Phi((c_{k-2}s_i)(k-3)/(k-2), (k-3)/(k-2), -c_{k-3})]\\
&\qquad\qquad \times\phi((k-2)\theta, k-2, c_{k-2}s_i)\\
&\quad\times\quad \sum_{j=1}^{m} B_j \phi(\theta + c_{k-2}s_i, 1, c_{k-1}t_j)[\Phi(\theta + c_{k-1}t_j, 1, c_k)\\
&\qquad\qquad -\Phi(\theta + c_{k-1}t_j, 1, -c_k)],
\end{aligned}$$

$$\begin{aligned}
&P_\theta(S_{k-4} \in I_{k-4}, S_{k-3} \in I_{k-3}, S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k)\\
&\quad\approx\quad c_{k-3}c_{k-2}c_{k-1} \sum_{i=1}^{n} B_i \phi((k-2)\theta, k-2, c_{k-2}s_i)\\
&\quad\times\quad \sum_{j=1}^{n} B_j \phi((c_{k-2}s_i)(k-3)/(k-2), (k-3)/(k-2), c_{k-3}t_j)\\
&\quad\times\quad [\Phi((c_{k-3}t_j)(k-4)/(k-3), (k-4)/(k-3), c_{k-4})\\
&\qquad\quad -\Phi((c_{k-3}t_j)(k-4)/(k-3), (k-4)/(k-3), c_{k-4})]
\end{aligned}$$

$$\times \sum_{h=1}^{m} B_h \phi(\theta + c_{k-2}s_i, 1, c_{k-1}u_h)[\Phi(\theta + c_{k-1} + u_h, 1, c_k) - \Phi(\theta + c_{k-1} + u_h, 1, -c_k)],$$

where m, and n are the number of points used in the Gauss-Legendre approximation.

For $I_k = (-\infty, a_k)$:

$$P_\theta(S_k \in I_k) = \Phi(k\theta, k, a_k),$$

$$P_\theta(S_{k-1} \in I_{k-1}, S_k \in I_k) \approx \sum_{i=1}^{n} B_i e^{s_i} \Phi(\theta + a_{k-1} - s_i, 1, a_{k-1}) \times \phi((k-1)\theta, k-1, a_{k-1} - s_i),$$

$$P_\theta(S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k) \approx \sum_{i=1}^{n} B_i e^{s_i} \Phi((a_{k-1} - s_i)(k-2)/(k-1), (k-2)/(k-1), a_{k-2}) \times \phi((k-1)\theta, k-1, a_{k-1} - s_i) \Phi(\theta + a_{k-1} - s_i, 1, a_k),$$

$$P(S_{k-3} \in I_{k-3}, S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k) \approx \sum_{i=1}^{n} B_i e^{s_i} \Phi((a_{k-2} - s_i)(k-3)/(k-2), (k-3)/(k-2), a_{k-3}) \times \phi((k-2)\theta, k-2, a_{k-2} - s_i) \times \sum_{j=1}^{m} B_j e^{t_j} \phi(\theta + a_{k-2} - s_i, 1, a_{k-1} - t_j) \Phi(\theta + a_{k-1} + t_j, 1, a_k),$$

$$P(S_{k-4} \in I_{k-4}, S_{k-3} \in I_{k-3}, S_{k-2} \in I_{k-2}, S_{k-1} \in I_{k-1}, S_k \in I_k) \approx \sum_{i=1}^{n} B_i e^{s_i} \phi((k-2)\theta, k-2, a_{k-2} - s_i) \times \sum_{j=1}^{m} B_j e^{t_j} \phi((a_{k-2} - s_i)(k-3)/(k-2), (k-3)/(k-2), a_{k-3} - t_j) \times \Phi((a_{k-3} - t_j)(k-4)/(k-3), (k-4)/(k-3), a_{k-4}) \times \sum_{h=1}^{m} B_h e^{u_h} \phi(\theta + a_{k-2} - s_i, 1, a_{k-1} - u_h) \Phi(\theta + a_{k-1} + u_h, 1, a_k),$$

where m, and n are the number of points used in the Gauss-Laguerre approximation.

The continuation region given by I_n is usually determined by selecting an alternative $\theta_1 > 0$ such that $\beta(0) = \alpha$, and $\beta(\theta_1) = 1 - \alpha$, where α is the desired significance level of the test. Notice that for this situation, P_0(Type I Error) $= P_{\theta_1}$(Type II Error). The power function, $\beta(\theta)$, for the transformed case with symmetric intervals using S_k and I_k is written as follows:

$$\beta(\theta) = \sum_{n=1}^{\infty} P_\theta \left(\cap_{k=1}^{n-1} (S_k \in I_k) \cap (S_n > c_n) \right).$$

This is similar to the evaluation of the $P(\tau > n)$ but, it should be noted that the important difference in this case is that the n^{th} event is not $S_n \in I_n$, but $S_n > c_n$. This requires calculation of an additional term, $P(S_n > c_n \mid \text{prior } S_k's \in I_k)$. As performed previously, the necessary joint probabilities involving the event $S_n > c_n$ will be calculated and used in calculating an approximation for $\beta(\theta)$. The calculated result is not necessarily a bound because the intervals are not all of the same type (Glaz and Johnson (1986)). The formulas for calculating the approximation to $\beta(\theta)$ are similar to those given earlier.

The following tables illustrate the cases where higher bounds are providing a needed improvement for the tests considered. This is particularly noticeable for $E(\tau) > 30$ (i.e., for the distribution of τ having more mass for larger values of τ). For all examples used here, the approximation for the power of the test did not have much room for improvement above the 3rd order calculations. In each table, the upper value is approximation (7) with $k = 3$ from Glaz and Johnson (1986), the middle value is approximation (7) with $k = 5$, the lower value is from a simulation with 10,000 trials also from Glaz and Johnson (1986), while starred values are exact.

Table I (SPRT)

Lower Bounds and Simulated Values of $E_\theta(\tau)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	31.39	46.22	59.46	9.05	14.13	19.04
	33.53	51.18	67.39	9.23	14.94	20.59
	36.16	61.31	84.90	9.23	15.25	21.54
0.025	24.70	34.49	41.53	7.13	10.47	13.01
	26.41	38.05	46.61	7.27	10.89	13.71
	28.19	43.32	54.83	7.27	11.03	13.98
0.050	19.51	25.77	29.53	5.62	7.70	8.99
	20.82	28.20	32.71	5.69	7.86	9.22
	22.32	30.58	36.20	5.72	7.99	9.25
0.100	14.12	17.39	19.02	4.02	5.02	5.53
	14.93	18.68	20.57	4.03	5.04	5.56
	15.36	19.49	21.72	4.02	5.10	5.62

Approximated and Simulated Values of $\sigma_\theta(\tau)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	16.13	26.93	37.31	5.31	9.47	13.86
	18.51	32.27	45.78	5.56	10.65	15.92
	23.02	46.09	68.66	5.70	11.35	17.69
0.025	13.95	21.60	27.37	4.47	7.45	9.79
	16.03	25.73	33.18	4.74	8.13	10.82
	18.83	33.25	44.48	4.77	8.14	11.33
0.050	11.95	17.13	20.35	3.77	5.70	6.94
	13.68	20.17	24.25	3.90	5.99	7.32
	15.86	24.08	29.61	3.97	6.20	7.43
0.100	9.46	12.34	13.80	2.85	3.82	4.32
	10.64	14.12	15.91	2.88	3.87	4.38
	11.55	15.22	17.60	2.86	3.93	4.34

Approximated and Simulated Values of $\beta(\theta)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	0.0099	0.0906	0.5000	0.0099	0.0905	0.5000
	0.0099	0.0908	0.5000	0.0099	0.0907	0.5000
	0.0097	0.0886	0.5000*	0.0103	0.0891	0.5000*
0.025	0.0244	0.1362	0.5000	0.0245	0.1361	0.5000
	0.0244	0.1364	0.5000	0.0244	0.1363	0.5000
	0.0219	0.1376	0.5000*	0.0230	0.1366	0.5000*
0.050	0.0474	0.1823	0.5000	0.0475	0.1823	0.5000
	0.0476	0.1826	0.5000	0.0476	0.1825	0.5000
	0.0445	0.1798	0.5000*	0.0488	0.1842	0.5000*
0.100	0.0906	0.2399	0.5000	0.0911	0.2401	0.5000
	0.0908	0.2401	0.5000	0.0912	0.2402	0.5000
	0.0884	0.2425	0.5000*	0.0938	0.2391	0.5000*

Table II (Triangular)

Lower Bounds and Simulated Values of $E_\theta(\tau)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	39.11	51.67	59.82	10.52	14.04	16.19
	40.44	53.89	62.32	10.60	14.19	16.37
	41.67	56.54	64.88	10.61	14.16	16.48
0.025	29.62	37.66	42.11	8.01	10.22	11.39
	30.63	39.17	43.78	8.05	10.29	11.47
	31.46	40.71	45.15	8.03	10.34	11.51
0.050	22.29	27.17	29.53	6.05	7.35	7.97
	23.01	28.13	30.58	6.06	7.37	7.99
	23.26	28.88	31.38	6.09	7.41	8.03
0.100	14.75	16.99	17.94	4.01	4.59	4.83
	15.11	17.42	18.40	4.01	4.59	4.83
	15.29	17.60	18.08	4.01	4.58	4.80

Approximated and Simulated Values of $\sigma_\theta(\tau)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	13.80	18.22	19.39	3.95	5.13	5.30
	14.97	19.57	20.47	4.05	5.23	5.38
	16.18	21.14	21.43	4.07	5.26	5.41
0.025	11.58	14.32	15.03	3.27	3.96	4.09
	12.48	15.25	15.83	3.32	4.01	4.13
	13.26	16.04	16.36	3.35	4.03	4.11
0.050	9.49	11.11	11.54	2.63	3.02	3.11
	10.12	11.72	12.09	2.65	3.04	3.12
	10.53	12.27	12.51	2.66	3.05	3.16
0.100	6.90	7.64	7.86	1.85	2.03	2.08
	7.21	7.94	8.14	1.85	2.03	2.08
	7.33	8.08	8.14	1.85	2.02	2.09

Approximated and Simulated Values of $\beta(\theta)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	0.0071	0.1019	0.5000	0.0094	0.1156	0.5000
	0.0084	0.1101	0.5000	0.0099	0.1182	0.5000
	0.0103	0.1177	0.5000*	0.0089	0.1119	0.5000*
0.025	0.0205	0.1470	0.5000	0.0244	0.1583	0.5000
	0.0229	0.1542	0.5000	0.0249	0.1597	0.5000
	0.0238	0.1585	0.5000*	0.0236	0.1616	0.5000*
0.050	0.0450	0.9135	0.5000	0.0496	0.2016	0.5000
	0.0481	0.1991	0.5000	0.0500	0.2023	0.5000
	0.0496	0.2022	0.5000*	0.0507	0.2034	0.5000*
0.100	0.0964	0.2546	0.5000	0.1000	0.2584	0.5000
	0.0991	0.2576	0.5000	0.1001	0.2585	0.5000
	0.0959	0.2573	0.5000*	0.1033	0.2541	0.5000*

Table III (Bayes)

Lower Bounds and Simulated Values of $E_\theta(\tau)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	33.86	51.48	65.65	9.61	14.48	17.95
	36.49	56.03	71.11	9.86	14.97	18.49
	40.02	62.54	79.90	9.89	15.10	18.78
0.025	26.69	38.75	47.03	7.60	10.82	12.78
	28.84	42.24	51.10	7.76	11.11	13.10
	31.51	47.50	56.30	7.89	11.07	13.20
0.050	21.12	29.13	33.91	5.98	8.02	9.10
	22.82	31.71	36.86	6.07	8.15	9.25
	24.30	34.62	40.36	6.09	8.12	9.33
0.100	15.32	19.70	21.93	4.22	5.21	5.67
	16.45	21.28	23.67	4.24	5.23	5.69
	17.36	22.50	25.13	4.24	5.23	5.69

Approximated and Simulated Values of $\sigma_\theta(\tau)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	19.02	28.31	33.62	5.40	7.73	8.63
	21.05	31.08	35.99	5.66	8.03	8.85
	24.62	35.68	39.32	5.74	8.15	9.00
0.025	16.61	23.63	27.33	4.63	6.25	6.88
	18.41	25.92	29.42	4.80	6.45	7.03
	21.06	29.30	31.78	4.89	6.51	7.06
0.050	14.39	19.53	22.03	3.87	4.95	5.37
	15.93	21.38	23.80	3.97	5.06	5.46
	17.56	23.63	25.79	3.96	5.01	5.50
0.100	11.62	14.78	16.18	2.87	3.44	3.65
	12.77	16.09	17.49	2.89	3.46	3.68
	13.63	17.09	18.77	2.87	3.50	3.65

Approximated and Simulated Values of $\beta(\theta)$

	$\theta_0 = 0.25$			$\theta_0 = 0.50$		
α/θ	-0.25	-0.125	0.0	-0.50	-0.25	0.0
0.010	0.0083	0.0791	0.5000	0.0086	0.0954	0.5000
	0.0084	0.0841	0.5000	0.0089	0.0991	0.5000
	0.0086	0.0908	0.5000*	0.0079	0.1031	0.5000*
0.025	0.0208	0.1204	0.5000	0.0216	0.1371	0.5000
	0.0211	0.1251	0.5000	0.0222	0.1399	0.5000
	0.0207	0.1279	0.5000*	0.0204	0.1431	0.5000*
0.050	0.0413	0.1645	0.5000	0.0432	0.1805	0.5000
	0.0419	0.1684	0.5000	0.0438	0.1821	0.5000
	0.0471	0.1760	0.5000*	0.0445	0.1822	0.5000*
0.100	0.0816	0.2230	0.5000	0.0854	0.2379	0.5000
	0.0824	0.2287	0.5000	0.0865	0.2382	0.5000
	0.0790	0.2283	0.5000*	0.0854	0.2379	0.5000*

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