DEPENDENCE RELATIONSHIPS FOR RAYS IN A CONVEX BODY IN $\Re^n$

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Line segments within a convex body, one of whose endpoints lies on the surface of the body are called rays. A forward and backward ray are complementary extensions of one another to form a secant of the body. This article illustrates that when these rays are generated by ν-randomness they display a negative orthant dependence when the convex body is a sphere in $\Re^n$. However, the forward and backward rays in any convex body in $\Re^n$ do not always display this dependence.

1. Introduction. In the field of geometrical probability, one of the topics which is extensively discussed is that of the distributions of the lengths of random rays and secants within and through convex bodies. See, for example, Coleman (1969, 1973), Enns and Ehlers (1978, 1980, 1981), Kellerer (1971, 1984), Kingman (1969), and Santalo (1976, 1986).

In most cases one is concerned with the distribution of a single quantity. This article will illustrate that for a sphere in $\Re^n$, the bivariate distribution of two related rays displays a negative orthant dependence (see Block, Savits, and Shaked (1982) and Joag-Dev and Proschan (1983)). It will also be illustrated that this dependence relationship does not hold for these rays in all convex bodies in $\Re^n$.

2. Definitions and Notation. In the notation of Enns and Ehlers (1978), the normalized average overlap volume of a convex body K with its translate in $\Re^n$ is defined as:

$$\Omega(\ell) = \mathcal{E}_\theta[V(K \cap K(\ell,\theta))]/V(K) \tag{1}$$

where $K(\ell,\theta)$ is the translate of a convex body K a distance ℓ in direction θ. Also $V(\cdot)$ and $\mathcal{E}_\theta(\cdot)$ are respectively the volume and expected value of $(\cdot)$ when uniformly averaged over all θ.

Line segments may be entirely in the interior of K or they may have one endpoint (rays) or both endpoints (secants) in the surface of K. These rays and

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secants may be generated by a variety of measures or types of randomness, for example, Coleman (1969) or Ehlers and Enns (1981).

The measure considered here is that of ν-randomness, namely a point P is chosen at random in K, according to the uniform distribution (Lebesgue measure) on K. A direction θ is then selected independently of P from a uniform distribution over all possible directions in $\Re^n$. The length of the forward ray L_1 is the distance from P to the surface of K in the direction θ. Similarly, the backward ray extends from P to the surface of K in the opposite direction to θ and has length L_2. Obviously L_1 and L_2 have the same marginal distributions and from Enns and Ehlers (1980) these are:

$$P(L_1 > \ell) = P(L_2 > \ell) = \Omega(\ell). \tag{2}$$

The secant traverses the whole body K and has length $L = L_1 + L_2$. It has been shown in Kingman (1969) and Enns and Ehlers (1978) that the probability density function of L is:

$$f(\ell) = \ell \frac{d^2\Omega(\ell)}{d\ell^2}.$$

The bivariate distribution of L_1 and L_2 derived in Enns and Ehlers (1980) is:

$$P(L_1 > \ell_1, L_2 > \ell_2) = \Omega(\ell_1 + \ell_2). \tag{3}$$

3. The Negative Dependence Condition. The bivariate distribution of the lengths of the forward and backward rays (3) displays a negative orthant dependence (NOD) if (2) has an increasing hazard rate. This may be shown by considering the form of the bivariate distribution (3). For NOD one requires:

$$P(L_1 > \ell_1, L_2 > \ell_2) \leq P(L_1 > \ell_1)P(L_2 > \ell_2), \tag{4}$$

see, for example, Block, Savits, and Shaked (1982) or Joag-Dev and Proschan (1983). Incorporating (2) and (3) into (4) yields our condition for NOD, namely:

$$\Omega(\ell_1 + \ell_2) \leq \Omega(\ell_1) \cdot \Omega(\ell_2). \tag{5}$$

If $r(\ell)$ is the hazard rate of the distribution of L_1 or L_2, then:

$$\Omega(\ell) = \left[\exp - \int_0^\ell r(x)dx\right]. \tag{6}$$

Condition (5) then implies that

$$\int_0^{\ell_1+\ell_2} r(x)dx \geq \int_0^{\ell_1} r(x)dx + \int_0^{\ell_2} r(x)dx$$

or equivalently:

$$\int_{\ell_1}^{\ell_1+\ell_2} r(x)dx \geq \int_0^{\ell_2} r(x)dx, \text{ for all positive } \ell_1 \text{ and } \ell_2. \tag{7}$$

Now (7) is the condition for an increasing hazard rate average (IHRA). Hence if L_1 and L_2 have an increasing hazard rate (IHR) then they are (IHRA) and hence the pair (L_1, L_2) is NOD.

In the following section it will be shown that if L_1 and L_2 are ν-random rays in a sphere in $\Re^n$ (n-sphere), then (L_1, L_2) are NOD. It is intuitive that if a forward ray in a sphere is large, then it is likely that the backward ray is small. However (L_1, L_2) are not NOD for all convex bodies K. An example where NOD breaks down is a sufficiently elongated rectangle. Here it is intuitive that a long forward ray will have to be in the direction of the elongation and will hence most likely have a correspondingly long backward ray.

4. Negative Orthant Dependence of the Forward and Backward Rays in an n-Sphere. From Enns and Ehlers (1978), it has been shown that for a unit n-sphere

$$P(L_1 > \ell) = \Omega(\ell) = \frac{2C_{n-1}}{C_n} \int_{\ell/2}^{1} (1-x^2)^{\frac{n-1}{2}} dx, \quad 0 \leq \ell \leq 2, \tag{8}$$

where $C_n = \pi^{n/2} / \left(\frac{n}{2}\right)! =$ the volume of a unit n-sphere.

From Section 3 it is now sufficient to show that the hazard rate of $\Omega(\ell)$, namely $r(\ell) = -\frac{d}{d\ell} \ell n \Omega(\ell)$ is increasing in ℓ. This then implies (L_1, L_2) are NOD.

Let

$$\xi(\ell) = 2\Omega^2(\ell) \left[C_n / C_{n-1}\right]^2 \frac{dr(\ell)}{d\ell}. \tag{9}$$

If $\xi(\ell) > 0$, then $r'(\ell) > 0$ and the result is shown.

From (8) one obtains

$$\xi(\ell) = 2\left[1 - \frac{\ell^2}{4}\right]^{n-1} - (n-1)\ell \left[1 - \frac{\ell^2}{4}\right]^{\frac{n-3}{2}} \int_{\ell/2}^{1} (1-x^2)^{\frac{n-1}{2}} dx$$

or equivalently:

$$\xi(\ell) = \left[1 - \frac{\ell^2}{4}\right]^{\frac{n-3}{2}} \int_{\ell/2}^{1} (1-x^2)^{\frac{n-1}{2}} \{2(n+1)x - (n-1)\ell\} dx. \tag{10}$$

Let the partial integrand $\alpha(\ell, x) = 2(n+1)x - (n-1)\ell$.

For x in the range of the integral one obtains:

$$\alpha(\ell, x) \geq 2\ell \geq 0$$

and it follows that $\xi(\ell) > 0$ and the result is proved.

A recent article by Enns and Ehlers (1988) shows that the dimensional moment of the extended ray length does not depend on the generating body K. The extended ray is the ray projected to the boundary of the convex body G where the convex body $K \subset G$. Redefine L_1 and L_2 as the extended forward and backward rays when K is any convex body and G is an n-sphere. I conjecture that (L_1, L_2) will be NOD. Again intuitively it does not matter where the generating point is chosen within G. If the forward ray is long, then the backward ray will tend to be short. Hence the generating point may be chosen in any interior region K.

Negative orthant dependence between the two rays does not always hold for some convex bodies. It can be shown for a sufficiently elongated rectangle, that NOD is violated. This is a messy but straightforward algebraic exercise and will not be reproduced here.

REFERENCES

BLOCK, H.W., SAVITS, T.H., and SHAKED, M. (1982). Some concepts of negative dependence. *Ann. Prob.* **10** 765–772.

COLEMAN, R. (1969). Random paths through convex bodies. *J. Appl. Prob.* **6** 430–441.

COLEMAN, R. (1973). Random paths through rectangles and cubes. *Metallography* **6** 103–114.

ENNS, E.G. and EHLERS, P.F. (1978). Random paths through a convex region. *J. Appl. Prob.* **15** 144–152.

ENNS, E.G. and EHLERS, P.F. (1980). Random paths originating within a convex region and terminating on its surface. *Austral. J. Statist.* **22**(1) 60–68.

ENNS, E.G. and EHLERS, P.F. (1988). Chords through a convex body generated from within an embedded body. *J. Appl. Prob.* **25** 700–707.

EHLERS, P.F. and ENNS, E.G. (1981). Random secants of a convex body generated by surface randomness. *J. Appl. Prob.* **18** 157–166.

JOAG-DEV, K. and PROSCHAN, F. (1983). Negative association of random variables with applications. *Ann. Statist.* **11** 286–295.

KELLERER, A.M. (1971). Considerations on the random transversal of convex bodies and solutions for general cylinders. *Radiation Research* **47** 359–376.

KELLERER, A.M. (1984). Chord-length distributions and related quantities for spheroids. *Radiation Research* **98** 425–437.

KINGMAN, J.F.C. (1969). Random secants of a convex body. *J. Appl. Prob.* **6** 660–672.

SANTALO, L.A. (1976). *Integral Geometry and Geometric Probability.* Addison-Wesley, Reading, MA.

SANTALO, L.A. (1986). On the measure of line segments entirely contained in a convex body. *Aspects of Mathematics and Its Applications*, North-Holland.

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