CONVEX-ORDERING AMONG FUNCTIONS, WITH APPLICATIONS TO RELIABILITY AND MATHEMATICAL STATISTICS

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Hardy, Littlewood, and Pólya (1952) introduced the notion of one function being convex with respect to a second function and developed some inequalities concerning the means of the functions. We use this notion to establish a partial order called convex-ordering among functions. In particular, the distribution functions encountered in many parametric families in reliability theory are convex-ordered. We have formulated some inequalities which can be used for testing whether a sample comes from F or G, when F and G are within the same convex-ordered family. Performance characteristics of different coherent structures can also be compared with respect to this partial ordering. For example, we will show that the reliability of a $k+1$-*out-of-n* system is convex with respect to the reliability of a k-*out-of-n* system.

When F is convex with respect to G, the tail of the distribution F is heavier than that of G; therefore, our convex-ordering implies stochastic ordering. Convex-ordering is also related to total positivity and monotone likelihood ratio families. This provides us a tool to obtain some useful results in reliability and mathematical statistics.

1. Introduction. Notions of partial ordering among survival distributions have played a useful role in providing numerous inequalities in reliability. The notion of a random variable X with distribution F being stochastically larger than another random variable Y with distribution G is well known in the literature. Van

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Zwet (1964) defined F to be *convex-ordered* with respect to G (written $F \overset{<}{Z} G$) if $G^{-1}F(x)$ is a convex function in x on the support of F. When this ordering occurs, one can show that Y is a convex function of X. Barlow and Proschan (1966) have obtained tolerance limits for distributions satisfying this ordering.

Lee (1981) defined and analyzed another notion of convex-ordering: F is *convex-hazard ordered* with respect to G, written $F \overset{<}{CH} G$ if $R_F R_G^{-1}$ is convex, where $\bar{F} = 1 - F$ and $R_F = -\log \bar{F}$ is the hazard function of F. Lee used this ordering to generalize certain inequalities and preservation theorems in reliability. We give still another notion of convex-ordering in Definition 2.1, which is different from those proposed by Van Zwet (1964) and Lee (1981).

2. Convex-Ordering Among Distributions. Throughout this paper, we define the inverse function h^{-1} of a nondecreasing function h by $h^{-1}(t) = \inf\{x : h(x) \geq t\}$. When h is nonincreasing, we define $h^{-1}(t) = \inf\{x : h(x) \geq t\}$. We use "$\ll$" to symbolize "absolutely continuous."

DEFINITION 2.1. Let G be any continuous distribution and F be absolutely continuous with respect to G. We say that F is *more convex than* G, written $F \overset{c}{>} G$, if $FG^{-1}(t)$ is a convex function in the interval (0,1).

Throughout this paper, we refer to the above as convex-ordering. Other notions of convex-ordering will be referred to with their authors' names like Van Zwet, etc. This definition of convex-ordering checks directly whether one distribution function can be expressed as a convex transformation of another distribution function, in contrast to that of Van Zwet which checks if the random variables can be so transformed. Thus, the distribution function x^3 is more convex than the distribution function x^2 on the interval (0,1). This concept coincides with that of Hardy, Littlewood, and Pólya (1952, p. 65). Although the above definition applies to the class of all monotonic functions, we shall generally restrict our attention to life distributions. (For an exception, see Theorem 2.8.)

The following lemma gives useful properties of FG^{-1}.

LEMMA 2.2. *Let G be any continuous distribution and F be absolutely continuous with respect to G. Then*

(i) $FF^{-1}(t) = t, 0 < t < 1;$ $F^{-1}F(x) \leq x, x \geq 0.$

(ii) $FG^{-1}G(x) = F(x), x \geq 0.$

(iii) $(FG^{-1})^{-1} = GF^{-1}.$

(iv) FG^{-1} *is nondecreasing and continuous. If G is also absolutely continuous with respect to F, then*

(v) FG^{-1} *is strictly increasing.*

(vi) $F \overset{c}{>} G$ *implies that* GF^{-1} *is concave.*

PROOF.

(i) The facts that $F^{-1}F(x) \leq x$ and $FF^{-1}(t) \geq t$ are easily seen from the definition of F^{-1} and do not require the continuity of F. Suppose now that $FF^{-1}(t) > t$. Then by the continuity of F, there is an $x < F^{-1}(t)$ such that $F(x) > t$, which contradicts the definition of F^{-1}. Hence $FF^{-1}(t) = t$.

(ii) From (i), we have $FG^{-1}G(x) \leq F(x)$. Suppose $FG^{-1}G(x) < F(x)$, then $F\{(G^{-1}G(x), x)\} > 0$ which implies that $G(G^{-1}G(x)) < G(x)$, since $F << G$. This leads to a contradiction because $G(G^{-1}G(x)) = G(x)$.

(iii) $FG^{-1}(GF^{-1}(t)) = FF^{-1}(t) = t$ implies that $GF^{-1}(t) \geq (FG^{-1})^{-1}(t)$.

Conversely, if $FG^{-1}(x) \geq t$, then $G^{-1}(x) \geq F^{-1}FG^{-1}(x) \geq F^{-1}(t)$, which in turn implies that $x = GG^{-1}(x) \geq GF^{-1}(t)$. Thus $(FG^{-1})^{-1}(t) \geq GF^{-1}(t)$.

(iv) Let $t_n \to t$. Then $t_n = GG^{-1}(t_n) \to GG^{-1}(t) = t$. Since $F << G$, this implies that $FG^{-1}(t_n) \to FG^{-1}(t)$.

(v) For $t_1 < t_2$, $GG^{-1}(t_1) = t_1 < t_2 = GG^{-1}(t_2)$. Since $G << F$, this implies $FG^{-1}(t_1) < FG^{-1}(t_2)$.

(vi) Let $\phi = FG^{-1}$, then ϕ is convex and strictly increasing. We need to show that ϕ^{-1} is concave. Let $0 \leq \lambda \leq 1$, then for any x and Y, we have

$$\begin{aligned}
\phi\phi^{-1}[\lambda x + (1-\lambda)y] &= \lambda x + (1-\lambda)y \\
&= \lambda\phi\phi^{-1}(x) + (1-\lambda)\phi\phi^{-1}(y) \\
&\geq \phi(\lambda\phi^{-1}(x) + (1-\lambda)\phi^{-1}(y)).
\end{aligned}$$

Since ϕ is strictly increasing, this implies $\phi^{-1}(\lambda x + (1-\lambda)y) \geq \lambda\phi^{-1}(x) + (1-\lambda)\phi^{-1}(y)$. Therefore $\phi^{-1} = GF^{-1}$ is concave.

Convex-ordering represents a partial ordering in the class of continuous distributions as indicated below:

(a) Reflexivity: $F \overset{c}{>} F$. (Since F is continuous, $FF^{-1}(t) = t$ and this is a convex function.)

(b) Transitivity: $F \overset{c}{>} G$ and $G \overset{c}{>} H$ imply that $F \overset{c}{>} H$. (Since $FH^{-1} = FG^{-1}(GH^{-1})$ and FG^{-1}, GH^{-1} are convex nondecreasing functions on $(0, 1)$, it follows that FH^{-1} is convex.)

(c) Antisymmetry: $F \overset{c}{>} G$ and $G \overset{c}{>} F$ imply that $F = G$. (Since $FG^{-1} = (GF^{-1})^{-1}, FG^{-1}$ and GF^{-1} are both convex and concave. Thus $FG^{-1}(t) = t = GF^{-1}(t)$, and $F(x) = FG^{-1}[G(x)] = G(x)$.)

A very useful way to characterize convex-ordering is by using the Radon-Nikodym derivative $\frac{dF}{dG}$ and is given in the following theorem.

THEOREM 2.3. *$F \overset{c}{>} G$ if and only if $f = \frac{dF}{dG}$ is nondecreasing almost everywhere with respect to G.*

PROOF. Let λ denote the Lebesgue measure on (0,1). Since G is continuous, $GG^{-1}(t) = t$. For each measureable set E, we have $\lambda(E) = G(G^{-1}(E))$.

This shows that the condition f is nondecreasing almost everywhere with respect to G is equivalent to fG^{-1} being nondecreasing almost everywhere with respect to λ. Now

$$FG^{-1}(t) = \int_{-\infty}^{G^{-1}(t)} f(x)dG(x) = \int_0^t fG^{-1}(y)dy$$

since G is continuous.
Consequently, FG^{-1} is convex if and only if f is nondecreasing almost everywhere with respect to G. ||

REMARK. If G is absolutely continuous with respect to λ, then $f = \frac{dF}{dG} = \frac{dF}{d\lambda} / \frac{dG}{d\lambda}$ a.e. G. Thus $F \overset{c}{>} G$ is equivalent to the property of monotone increasing likelihood ratio of F with respect to G; i.e., $\frac{dF}{d\lambda} / \frac{dG}{d\lambda}$ is nondecreasing on the support of G. This is the likelihood ratio ordering of Ross (1983).

REMARK. If $f = \frac{dF}{dG}$ is continuous and G is a continuous distribution, then $F \overset{c}{>} G$ if and only if f is monotone nondecreasing.

We now define the notion of a convex-ordered family.

DEFINITION 2.4. A family of distributions $\{F_\alpha\}$ is said to be a *convex-ordered family*, or simply a *convex family* if $\alpha_2 > \alpha_1$ implies that $F_{\alpha_2} \overset{c}{>} F_{\alpha_1}$.

The following families of distributions are convex-ordered with respect to α for $\alpha > 0$.

EXAMPLES.

(1) Exponential: $F_\alpha(t) = 1 - e^{-t/\alpha}$, $t > 0$.

(2) Gamma: $F_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-x} dx$, $t > 0$.

(3) Truncated Normal: $F_\alpha(t) = \frac{1}{a_\alpha \sigma\sqrt{2\pi}} \int_0^t e^{-(x-\alpha)^2/2\sigma^2} dx$ for $t > 0$, where $\sigma > 0$ is fixed and $a_\alpha = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\alpha)^2/2\sigma^2} dx$.

(4) Weibull: $F_\alpha(t) = 1 - e^{-(t/\alpha)^\lambda}$ for $t > 0$, where $\lambda > 0$ is fixed.

(5) Proportional hazards: $F_\alpha(t) = 1 - e^{-\frac{1}{\alpha}R(t)}$, $t > 0$, where $R(t) = -\log \bar{F}(t)$ is the hazard function of some life distribution F.

DEFINITION 2.5. A nonnegative function $f_\alpha(x)$ on $R \times R$ is *totally positive of order 2* (TP_2) in (α, x) if

$$\begin{vmatrix} f_{\alpha_1}(x_1) & f_{\alpha_1}(x_2) \\ f_{\alpha_2}(x_1) & f_{\alpha_2}(x_2) \end{vmatrix} \geq 0$$

for all $\alpha_1 < \alpha_2$ and $x_1 < x_2$ (also called the monotone likelihood ratio property.)

The following theorems relate total positivity to convex-ordering. The theory of total positivity has been fruitful in obtaining many new results in reliability and life testing. We can also make use of this powerful tool in studying convex-ordered families. Karlin (1968) is an excellent source for results on total positivity.

THEOREM 2.6. *F_α is a convex family if and only if the corresponding density $f_\alpha(t)$ with respect to some dominating measure λ is TP_2 in (α, t).*

PROOF. By Theorem 2.3, we have that $F_{\alpha_2} \overset{c}{>} F_{\alpha_1}$ for $\alpha_2 > \alpha_1$ if and only if $\frac{f_{\alpha_2}}{f_{\alpha_1}}$ is increasing. Thus for $\alpha_2 > \alpha_1$, $t_2 > t_1$,

$$\begin{vmatrix} f_{\alpha_1}(t_1) & f_{\alpha_1}(t_2) \\ f_{\alpha_2}(t_1) & f_{\alpha_2}(t_2) \end{vmatrix} \geq 0,$$

which is the defining condition that $f_\alpha(t)$ is TP_2 in (α, t). $\|$

THEOREM 2.7. *If $\{F_\alpha\}$ is a convex family, then $F_\alpha(t)$ is TP_2 in (α, t).*

PROOF. For $\alpha_2 > \alpha_1$, $F_{\alpha_2}(F_{\alpha_1}^{-1})$ is convex on (0,1). This implies that $F_{\alpha_2}(F_{\alpha_1}^{-1}(t))/t$ is nondecreasing in t. Since f_{α_1} is continuous, $F_{\alpha_1}F_{\alpha_1}^{-1}(t) = t$. We have $F_{\alpha_2}(F_{\alpha_1}^{-1}(t))/F_{\alpha_1}\ (F_{\alpha_1}^{-1}(t))$ is nondecreasing in t. By noting that $F_{\alpha_1}^{-1}$ is increasing, we conclude that $F_\alpha(t)$ is TP_2 in (α, t). $\|$

Another characterization of convex-ordering is given by:

THEOREM 2.8. *$F \overset{c}{>} G$ if and only if $\bar{G} \overset{c}{>} \bar{F}$.*

PROOF. Since $F^{-1}(1-t) = \bar{F}^{-1}(t)$ on (0,1), we have $\bar{G}\bar{F}^{-1}(t) = 1 - GF^{-1}(1-t)$. Thus FG^{-1} is convex if and only if $\bar{G}\bar{F}^{-1}$ is convex. $\|$

REMARK. An immediate consequence of Theorems 2.7 and 2.8 is that $F \overset{c}{>} G$ implies both $\frac{F}{G}$ and $\frac{\bar{F}}{\bar{G}}$ are nondecreasing.

We end this section with comparisons of our convex-ordering and other partial orderings. We note that the Weibull family is not convex-ordered with respect to the shape parameter λ, but it is convex-ordered in the sense of Van Zwet. For an example showing that convex-ordering of distribution does not imply convex-ordering in the sense of Van Zwet, consider $F_1(t) = t^2$ and $F_2(t) = 1 - \sqrt{1 - t^2}$

for t in $[0,1]$. Then F_2 is more convex than F_1, i.e., $F_2 \overset{c}{>} F_1$. Since $F_1^{-1}F_2(t) = (1-\sqrt{1-t^2})^{\frac{1}{2}}$ is not convex, F_2 is not convex-ordered with respect to F_1 in Van Zwet's sense.

When F is absolutely continuous with respect to the Lebesgue measure λ, the failure rate function of F is defined to be $\tau_F = \frac{f}{\bar{F}}$, where $f = \frac{dF}{d\lambda}$. F has a larger failure rate function than G if $\tau_F(t) \geq \tau_G(t)$ for all $t \geq 0$. The next theorem compares convex-ordering with failure rate ordering.

THEOREM 2.9. *If F and G are absolutely continuous distributions with respective densities f and g, then $F \overset{c}{>} G$ implies $\tau_F(t) \leq \tau_G(t)$ for all t.*

PROOF. By Theorem 2.3, $f(t_1)g(t) \leq f(t)g(t_1)$ for all $t_1 \leq t$. Integrating this over $[t_1,\infty)$, we have $f(t_1)\bar{G}(t_1) \leq \bar{F}(t_1)g(t_1)$ for all t_1. ||

Comparing this result with the convex-hazard order of Lee (1981), which requires that $\frac{\tau_F}{\tau_G}$ be a nondecreasing function of t, we see that convex-ordering neither implies nor is implied by the convex hazard function ordering of Lee.

3. Preservation of Convex-Ordering Under Operations. In this section, we show that our notion of convex-ordering is preserved under various standard statistical operations.

First, we show that convex-ordering is preserved under mixture of distributions.

THEOREM 3.1. *If $F_\alpha \overset{c}{>} G_\beta$ for each pair (α,β), then $\int F_\alpha d\mu(\alpha) \overset{c}{>} \int G_\beta d\nu(\beta)$ for any mixing distribution μ and ν.*

PROOF. The proof can be split into two parts. Suppose that $F_\alpha \overset{c}{>} G$ for each α. We will show that $\int F_\alpha d\mu(\alpha) \overset{c}{>} G$ for any mixing distribution μ. Let $f_\alpha = \frac{dF_\alpha}{dG}$, then by Theorem 2.3, f_α is nondecreasing for each α. Thus $\int f_\alpha d\mu(\alpha)$ is nondecreasing, and this implies that $\int F_\alpha d\mu(\alpha) \overset{c}{>} G$.

A similar proof shows that if $F \overset{c}{>} G_\beta$ for each β, then $F \overset{c}{>} \int G_\beta d\nu(\beta)$ for any mixing distribution ν.

These two results establish Theorem 3.1. ||

It should be noted that the condition in Theorem 3.1 cannot be weakened to $F_\alpha \overset{c}{>} G_\alpha$ for each α, as shown in the following example.

EXAMPLE 3.2. Let

$$\bar{F}_1(t) = e^{-t/1.1},\ \bar{F}_2(t) = e^{-t/5.1},\ \bar{G}_1(t) = e^{-t},\ \bar{G}_2(t) = e^{-t/5}.$$

Then $F_1 \overset{c}{>} G_1$ and $F_2 \overset{c}{>} G_2$. However, $G_2 \overset{c}{>} F_1$. Let $F = \frac{1}{2}(F_1+F_2)$, $G = \frac{1}{2}(G_1+G_2)$. To check whether $F \overset{c}{>} G$, consider the ratio of the derivatives of F and G,

$$h(t) = \frac{dF}{dG}(t) = \frac{\frac{1}{1.1}e^{-t/1.1} + \frac{1}{5.1}e^{-t/5.1}}{e^{-t} + \frac{1}{5}e^{-t/5}}$$

We note that h is continuous, but h is not increasing because $h(4.6) = 1.04 > 1.026 = h(6)$. Thus the ordering $F \overset{c}{>} G$ is shown to be false in view of Theorem 2.3.

Next, we show that convex-ordering is preserved under formation of certain coherent structures. We begin with some basic definitions and notations from reliability.

Consider n independent components, each of which is either functioning or not. We use the binary variable x_i to indicate the state of the i-th component:

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{otherwise.} \end{cases}$$

The state of a system composed of these components is determined by the states of the components. The function $\phi(x_1, \ldots, x_n)$ is called the *structure function* of the system and is defined by

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if system is functioning} \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE. A k-out-of-n system functions if and only if at least k out of the n components function. The structure function is given by

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

The i-th component is *irrelevant* to the structure ϕ if ϕ is constant in x_i. We consider *monotone* systems, that is, systems for which $\phi(x_1, \ldots, x_n) \geq \phi(y_1, \ldots, y_n)$ whenever $x_i \geq y_i$ for all $i = 1, \ldots, n$. If a monotone system has no irrelevant components, it is said to be a *coherent* system.

Let $P(X_i = 1) = p_i$ denote the reliability of the i-th component; the system reliability is given by

$$h_\phi(p_1, \ldots, p_n) = P(\phi(x_1, \ldots, x_n) = 1).$$

Denote the life distribution of the i-th component by F_i; then the life distribution F_ϕ of the system is given by

$$F_\phi(t) = 1 - h_\phi(\bar{F}_1(t), \ldots, \bar{F}_n(t)).$$

As a special case, we will consider a parallel structure of n components, i.e., a 1-*out-of-n* system. The life distribution of the system is given by the product $\Pi_{i=1}^{n} F_i(t)$. The following theorem shows that convex-ordering is preserved under formation of parallel systems.

THEOREM 3.3. *Suppose $F_i \overset{c}{>} G_j$ for each pair (i,j). Then $\Pi_{i=1}^{n} F_i \overset{c}{>} \Pi_{i=1}^{n} G_i$.*

PROOF. It suffices to prove the theorem for $n = 2$. We first establish that $F_1 \cdot F_2$ is absolutely continuous ($<<$) with respect to $G_1 \cdot G_2$. From our assumption it follows that $F_i << G_j$, $i = 1, 2$, $j = 1, 2$. Note that $(F_1 \cdot F_2)(x) = F_1(x)F_2(x)$ is a distribution function. The distribution function $G_1 \cdot G_2$ is similarly defined. It is easy to establish that

$$G_1(E)G_2(E) \leq (G_1 \cdot G_2)(E) \leq G_1(E) + G_2(E)$$

and

$$F_1(E)F_2(E) \leq (F_1 \cdot F_2)(E) \leq F_1(E) + F_2(E),$$

for intervals E and unions of intervals E and extending it for Borel sets E. Now, if $(G_1 \cdot G_2)(E) = 0$, then either $G_1(E)$ or $G_2(E) = 0$. Either case implies that both $F_1(E) = 0$ and $F_2(E) = 0$ and thus $(F_1 \cdot F_2)(E) = 0$. Thus $F_1 \cdot F_2 << G_1 \cdot G_2$. Let λ be a measure dominating all the G_i's. Let $f_i = \frac{dF_i}{d\lambda}$ and $g_i = \frac{dG_i}{d\lambda}$. Then

$$\frac{d(F_1 \cdot F_2)}{d(G_1 \cdot G_2)} = \frac{f_1 \cdot F_2 + f_2 \cdot F_1}{g_1 \cdot G_2 + g_2 \cdot G_1} = \left[\frac{g_1 \cdot G_2}{f_1 \cdot F_2} + \frac{g_2 \cdot G_1}{f_1 \cdot F_2}\right]^{-1} + \left[\frac{g_1 \cdot G_2}{f_2 \cdot F_1} + \frac{g_2 \cdot G_1}{f_2 \cdot F_1}\right]^{-1},$$

is a nondecreasing function, since $\frac{f_i}{g_i}$, $\frac{F_i}{G_j}$ are nondecreasing. Hence $F_1 \cdot F_2 \overset{c}{>} G_1 \cdot G_2$ by Theorem 2.3. ||

It is natural to ask whether we can relax the assumption of Theorem 3.3 to $F_i \overset{c}{>} G_i$, all i? A counterexample paralleling Example 3.2 is given below.

EXAMPLE 3.4. Let F_1, F_2, G_1, G_2 be defined as in Example 3.2. Then

$$k(t) = \frac{d(F_1 \cdot F_2)}{d(G_1 \cdot G_2)}(t) = \frac{f_1(t)F_2(t) + f_2(t) \cdot F_1(t)}{g_1(t)G_2(t) + g_2(t) \cdot G_1(t)}$$

is continuous and $k(4.5) = 1.01944 > 1.01935 = k(4.7)$. Theorem 2.3 shows that the ordering $F_1 \cdot F_2 \overset{c}{>} G_1 \cdot G_2$ does not hold.

Another important coherent structure is the k-out-of-n system. We now show that convex-ordering is preserved under formation of such systems with independent components.

THEOREM 3.5. *Let $F_{n,k}$ be the life distribution of a k-out-of-n system with independent components with life distributions $F_1, \ldots, F_n$. Similarly let $G_{n,k}$ be the life distribution of a k-out-of-n system with independent components with life distributions $G_1, \ldots, G_n$. Suppose that $F_i \overset{c}{>} G_j$ for each pair (i, j). Then $F_{n,k} \overset{c}{>} G_{n,k}$.*

PROOF. Suppose $G_{n,k}(E) = 0$. Then $G_j(E) = 0$ for some j. Hence $F_i(E) = 0$ for all i. Thus $F_{n,k}(E) = 0$, so that $F_{n,k} << G_{n,k}$.

Let λ be a measure which dominates all the G_i's. Let $f_i = \frac{dF_i}{d\lambda}$ and $g_i = \frac{dG_i}{d\lambda}$, $i = 1, \ldots, n$. Then for a k-out-of-n system, the densities of $F_{n,k}$ and $G_{n,k}$ with respect to λ are

$$\begin{aligned} f_{n,k} &= \frac{n!}{(k-1)!(n-k)!} \Sigma f_{\alpha_1} \bar{F}_{\alpha_2} \ldots \bar{F}_{\alpha_k} F_{\alpha_{k+1}} \ldots F_{\alpha_n}, \\ g_{n,k} &= \frac{n!}{(k-1)!(n-k)!} \Sigma g_{\alpha_1} \bar{G}_{\alpha_2} \ldots \bar{G}_{\alpha_k} G_{\alpha_{k+1}} \ldots G_{\alpha_n}, \end{aligned}$$

where the summation is taken over all permutations $(\alpha_1, \ldots, \alpha_n)$ of the integers $1, 2, \ldots n$. The ratio of derivatives of $F_{n,k}$ with respect to $G_{n,k}$ is

$$\frac{dF_{n,k}}{dG_{n,k}} = \frac{\Sigma f_{\alpha_1} \cdot \bar{F}_{\alpha_2} \cdot \ldots \bar{F}_{\alpha_k} \cdot F_{\alpha_{k+1}} \cdot \ldots F_{\alpha_n}}{\Sigma g_{\alpha_1} \cdot \bar{G}_{\alpha_2} \cdot \ldots \bar{G}_{\alpha_k} \cdot G_{\alpha_{k+1}} \cdot \ldots G_{\alpha_n}}.$$

To complete the proof that this is a nondecreasing function, we show that the ratio of every term in the numerator to any term in the denominator is non-decreasing. A typical term is of the form

$$\frac{f_1 \cdot \bar{F}_2 \cdot \ldots \bar{F}_k \cdot F_{k+1} \cdot \ldots F_n}{g_{\alpha_1} \cdot \bar{G}_{\alpha_2} \cdot \ldots \bar{G}_{\alpha_k} \cdot G_{\alpha_{k+1}} \cdot \ldots G_{\alpha_n}},$$

which is nondecreasing since $\frac{f_i}{g_j}$, $\frac{\bar{F}_i}{\bar{G}_j}$, $\frac{F_i}{G_j}$ are nondecreasing. By Theorem 2.3, we conclude that $F_{n,k} \overset{c}{>} G_{n,k}$. ||

When all components are identical, the following theorem shows among other results, that the reliability of a k-out-of-n system is more convex than the reliability of a $(k-1)$-out-of-n system.

THEOREM 3.6. *Let $h_{n,k}(p)$ be the reliability function of a k-out-of-n system with identical components. Then $h_{n,k+1} \overset{c}{>} h_{n+1,k+1} \overset{c}{>} h_{n,k} \overset{c}{>} h_{n+1,k}$.*

PROOF.

$$\begin{aligned} h_{n,k}(p) &= \Sigma_{i=1}^{n} \binom{n}{i} p^i (1-p)^{n-i} \\ &= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} \int_0^p t^{k-1}(1-t)^{n-k} dt, \end{aligned}$$

the incomplete Beta function. Taking the derivative of $h_{n,k}$, we have

$$h'_{n,k}(p) = \frac{n!}{(n-k)(k-1)!} p^{k-1}(1-p)^{n-k}.$$

Thus,

$$\frac{h'_{n,k}(p)}{h'_{n+1,k}(p)} = \frac{n-k+1}{n+1} \cdot \frac{1}{1-p}$$

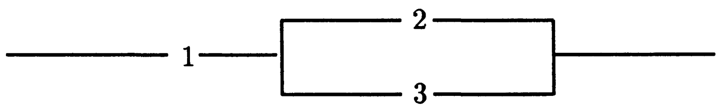
is increasing in p, establishing that $h_{n,k} \overset{c}{>} h_{n+1,k}$. The remaining inequalities can be proved similarly. ||

Since the distribution of the $(n-k+1)$-th order statistic corresponds to the life distribution of a k-out-of-n system of identical components, the following corollary is essentially a restatement of Theorem 3.6.

COROLLARY 3.7. *Let $F_{n,k}$ be the distribution of the $(n-k+1)$-th order statistic in a sample of size n from F. Then $F_{n,k+1} \overset{c}{>} F_{n+1,k+1} \overset{c}{>} F_{n,k} \overset{c}{>} F_{n+1,k}$.*

In view of Theorem 3.5, one might ask whether convex-ordering is preserved under formation of coherent systems. The example below shows that this is not true in general.

EXAMPLE. Consider the coherent structure of identical components presented in the following diagram.



The system reliability is $h_\phi(p) = p^2(2-p)$. For $\bar{F}(t) = e^{-t}$ and $\bar{G}(t) = e^{-2t}$, we have $F \overset{c}{>} G$; but

$$\frac{f_\phi(t)}{g_\phi(t)} = \frac{4-3e^{-t}}{2e^{-2t}(4-3e^{-2t})}$$

which is continuous and not monotone nondecreasing in t.

In order to show that convex-ordering is preserved under convolution, we need to consider the class of Pólya frequence densities of order 2 (PF_2).

DEFINITION 3.8. f is a *Pólya frequency function of order 2* (PF_2) if for all $\Delta > 0$, $f(x+\Delta)/f(x)$ is decreasing in x, $-\alpha < x < \alpha$.

An equivalent definition is that $\log f(x)$ is concave. Note that each PF_2 function $f(x)$ defines a TP_2 function, $h(x,y) = f(x-y)$.

The following theorem, due to Ghurye and Wallace (1959), gives a sufficient condition on convex families for preservation of convexity under convolution.

THEOREM 3.9. *Let $\{F_\alpha\}$ and $\{G_\alpha\}$ be convex families with PF_2 densities. Then $(F_\alpha * G_\alpha)$ is a convex family.*

Denote the n-fold convolution of F by $F^{(n)}$. Then for life distribution F with log concave density, $F^{(n+1)}$ is more convex than $F^{(n)}$. This is a special case of Theorem 1 in Karlin and Proschan (1960).

THEOREM 3.10. *Let F be a life distribution with PF_2 density, then $\{F^{(n)}\}$ is a convex family with respect to n.*

4. Application of Convex-Ordering. Very often in life testing we do not know the exact form of the distribution, but based on physical evidence, we know something about the properties of the distribution. For example, in situations where a normal distribution is assumed, we might suspect that the tail of the underlying distribution is, in fact, heavier than that of the normal distribution. Therefore, we want to test the normal assumption against convex-ordered alternatives. In this section, we will present an inequality for convex families and apply this inequality to develop tests of such a hypothesis.

THEOREM 4.1. *(Hardy, Littlewood, and Pólya (1952), p. 75.) $F \overset{c}{>} G$ if and only if*

$$F^{-1}\left(\Sigma_{i=1}^{n}\lambda_i F(x_i)\right) \geq G^{-1}\left(\Sigma_{i=1}^{n}\lambda_i G(x_i)\right)$$

for all x_i and $\lambda_i \geq 0$, $i = 1, \ldots, n$, such that $\Sigma_{i=1}^{n}\lambda_i = 1$.

Note that $\Sigma_{i=1}^{n}\lambda_i F(x_i)$ is a weighted average. We now apply this result to hypothesis testing.

APPLICATION 4.2. Let $X_1, \ldots, X_n$ be a random sample. Suppose we wish to test:

$$H_0 : X_1, \ldots, X_n \sim G \ \text{(known)}$$

against the alternative

$$H_1 : X_1, \ldots, X_n \sim F,\ \ F \overset{c}{>} G \ \text{ but otherwise unknown.}$$

Notice that when H_1 holds, $G^{-1}\left(\frac{1}{n}\Sigma_{i=1}^{n}G(X_i)\right) \leq F^{-1}\left(\frac{1}{n}\Sigma_{i=1}^{n}F(X_i)\right)$.

The right hand side can be estimated by $F_n^{-1}\left(\frac{1}{n}\Sigma_{i=1}^{n}F_n^{-1}(X_i)\right) = F_n^{-1}((n+1)/2n) \approx$ median of the sample where F_n is the empirical distribution function. Therefore, our test procedure is to reject H_0 if $G^{-1}\left(\frac{1}{n}\Sigma_{i=1}^{n}G(X_i)\right)$ is sufficiently smaller than $F_n^{-1}(\frac{1}{2})$. Recall that G is known.

5. Convex-Ordering for Symmetric Distribution Functions. In this section, we consider convex orderings between continuous symmetric distribution functions: $\bar{F}(x) = F(-x)$ for all x.

DEFINITION 5.1. $F \overset{sc}{>} G$ if F and G are continuous symmetric distribution functions and $F \overset{c}{>} G$ on $[0, \infty)$; i.e., $F << G$ and FG^{-1} is concave on $(-\infty, 0]$ and convex on $[0, \infty)$.

Examples of such ordered distributions are:

1) Normal.
Let F_α be the distribution function on $N(0,\alpha^2)$, $\alpha > 0$. Then $\alpha_2 > \alpha_1 \Rightarrow F_{\alpha_2} \overset{sc}{>} F_{\alpha_1}$

2) Double Exponential.
Let F_α be the density function given by

$$f_\alpha(x) = \frac{1}{2\alpha} e^{-|x|/\alpha}, \ \alpha > 0, \ -\infty < x < \infty.$$

Then $\alpha_2 > \alpha_1 \Rightarrow F_{\alpha_2} \overset{sc}{>} F_{\alpha_1}$.

A characterization of this ordering is given in the next theorem.

THEOREM 5.2. *$F \overset{sc}{>} G$ if and only if $f(t) = \frac{dF}{dG}(t)$ is nondecreasing in $|t|$ for almost every t.*

PROOF. By Theorem 2.3, f is increasing on $[0,\infty)$ and decreasing on $(-\infty, 0]$. ||

Thus, if we wish to show $F \overset{sc}{>} G$, we need only to consider f on the positive axis. As an immediate consequence of this and Theorem 3.1, we have:

THEOREM 5.3. *Let $F_\alpha \overset{sc}{>} G_\beta$ for each pair of (α, β). Then $\int F_\alpha d\mu(\alpha) \overset{sc}{>} \int G_\beta d\nu(\beta)$ for any mixing distributions μ and ν.*

When F is more convex than G, then G is more peaked about the origin than F, as shown in Theorem 5.5 below. We now compare this notion of relative peakedness to the following definition given by Birnbaum (1948).

DEFINITION 5.4. Y is more peaked than X if

$$P(|Y| \geq t) \leq P(|X| \geq t) \ \text{ for all } t \geq 0.$$

If X and Y have symmetric distribution functions F and G respectively, then this is equivalent to $G(t) \geq F(t)$ for all $t \geq 0$.

THEOREM 5.5. *If $F \overset{sc}{>} G$, then G is more peaked than F.*

PROOF. Since $F \overset{c}{\geq} G$ on $[0,\infty)$, it follows that $FG^{-1}(u) - \frac{1}{2}$ is convex on $[0, \frac{1}{2})$, and hence $(FG^{-1}(u) - \frac{1}{2})/(u - \frac{1}{2})$ is increasing on (0,1). This implies that $(F(t) - \frac{1}{2})/(G(t) - \frac{1}{2})$ is increasing on $(0,\infty)$ and is less than or equal to 1. This shows that $F(t) \leq G(t)$ for $t > 0$. Since F and G are symmetric, it follows that G is more peaked than F. ||

Since the product of symmetric distribution functions need not be symmetric, we do not have a result analogous to Theorem 3.3. It can also be shown that this ordering is not necessarily preserved under convolution. If $F \overset{sc}{>} G$, then we can show that the even central moments of F are greater than those of G. To prove this we need the following result.

LEMMA 5.6. *(Barlow and Proschan, 1975, p. 120.)*
Let $W(x)$ be a Lebesgue-Stieltjes measure, not necessarily positive for which $\int_t^\infty dW(x) \geq 0$ for all t, and let $h \geq 0$ be increasing. Then $\int_{-\infty}^\infty h(x)dW(x) \geq 0$.

THEOREM 5.7. *$F \overset{sc}{>} G \Rightarrow \mu_{2n}(F) \geq \mu_{2n}(G)$ for all n.*

PROOF. $F \overset{sc}{>} G \Rightarrow F(t) \leq G(t)\ \forall\, t \geq 0$. Let

$$W(x) = \begin{cases} F(x) - G(x) & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h(x) = \begin{cases} x^{2n} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Lemma 5.6, $\mu_{2n}(F) \geq \mu_{2n}(G)$. ||

COROLLARY 5.8. *If $F \overset{sc}{>} G$, then $Var\, F \geq Var\, G$.*

We conclude this section with the following theorem.

THEOREM 5.9. *Let the distribution functions F and G be absolutely continuous with densities f and g. If $F \overset{sc}{>} G$ and F is unimodal, then G is unimodal.*

PROOF. Both g/f and f are nonnegative and decreasing on $[0, \infty)$; thus g is decreasing on $[0, \infty)$. ||

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