SHOULD MINIMAL REPAIR DEPEND ON INFORMATION?

BY ELJA ARJAS AND ILKKA NORROS

University of Oulu and Technical Research Centre of Finland

The notion of minimal repair with respect to a history is defined in terms of a general filtration and a completely unpredictable stopping time. An inequality relating compensator transformations with respect to the minimal history of a one-point process and a richer history is proven. Applied to minimal repair, this result shows that the modeling of minimal repair in a "black box" sense always gives a stochastically longer life length than a more realistic model.

1. Introduction. The notion of minimal repair was introduced in reliability theory by Barlow and Hunter (1960). Its intuitive meaning is putting the system back to operation when it fails in such a way that the situation immediately preceding the failure is restored. The traditional probabilistic model is the following. Consider a nonnegative random variable S (the life length of the system) with a continuous distribution function F. When the system fails, say at time $S = s$, it is given an additional lifetime S' with conditional distribution

$$P(S' > t \mid S = s) = P(S > s + t \mid S > s) = (1 - F(s + t))/(1 - F(s)).$$

Equivalently, the minimal repair model can be defined in terms of the cumulative hazard function

$$R(t) = -\ln(1 - F(t)) = \int_0^t \frac{dF(s)}{1 - F(s)}$$

as follows. The original failure point $S(\omega)$ is "erased" and the hazard of the additional life time S' at age $t - S(\omega)$ is given the same value as the original hazard would have had at time t had there been no failure, that is, $dR(t)$. If minimal repairs are made repeatedly, the sequence of repair times is a nonhomogeneous Poisson process with integrated intensity $R(t)$.

This simple notion of minimal repair has obvious intuitive appeal. However, it may not be a realistic description of any actual repair done on a failed system.

AMS 1980 subject classifications. 90B25, 60G55.

Key words and phrases. Minimal repair, imperfect repair, compensator transformation, stochastic order.

In a sense, it treats the system as a black box, without any reference to what caused the failure and what repair was needed to put the system back to operation. Bergman makes this point in a review paper (Bergman, 1985), distinguishing between "statistical" and "physical" minimal repair. The comments on minimal repair on page 51 in Ascher and Feingold (1984) should also be mentioned.

As a very simple example illustrating this problem in the definition of minimal repair, consider a system consisting of two components in parallel, with independent $Exp(1)$ distributed life lengths. The system fails when the component with the longer life fails. A natural concept of minimal repair of the system would be the restoration of the working condition of this component when it fails, leaving the component which failed earlier in the down state. The additional lifetime obtained by this kind of repair is clearly independent of the failure time and $Exp(1)$ distributed. The black box model would consider a system life length S with distribution $P(S \leq t) = F(t) = (1-e^{-t})^2$ and an additional lifetime S' with conditional distribution

$$P(S' > t \mid S = s) = \frac{1 - F(s+t)}{1 - F(s)} = e^{-t}\frac{2 - e^{-s-t}}{2 - e^{-s}},$$

which is stochastically larger than $Exp(1)$ for every s.

In this example, the more realistic model gives a less optimistic estimate of the total life length than the black box model. We shall show below in Section 3 that this is always the case. Section 2 is devoted to the definition of minimal repair with respect to a general history. A more detailed exposition of Sections 2 and 3 can be found in Norros (1987) and Arjas and Norros (1989), respectively.

2. Minimal Repair with Respect to a General History. Consider a probability space $(\Omega, \mathcal{F}, P)$, a history (filtration) $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$ and an $\mathbf{F}$-stopping time S, satisfying the following conditions:

(i) Ω is a Polish space, that is, a complete separable metric space;

(ii) $\mathcal{F}$ is the completion w.r.t. P of $\mathcal{B}(\Omega)$, the Borel σ-field of Ω;

(iii) $\mathbf{F}$ satisfies Dellacherie's "usual conditions", that is, it is right continuous and $\mathcal{F}_0$ contains all P-null sets;

(iv) S is completely unpredictable and a.s. finite.

We denote by N the simple point process $N_t = 1_{\{t \geq S\}}$ and by A the $\mathbf{F}$-compensator of N. By (iv), A is a continuous process.

In this section we show how the stopping time S can be "minimally repaired". In other words, we define probability measures Q_n, $n = 1, 2, \ldots$, such that under Q_n, things behave as if S had been n times minimally repaired before its final failure. In our construction we need the following well-known result and some facts about the prediction process.

LEMMA 2.1. *For any finite stopping time T, the conditional distribution of $A_S - A_T$ given $\mathcal{F}_T$ is the $Exp(1)$ distribution on the set $\{S > T\}$.*

The notion of a *prediction process* was introduced by Knight (1975). Aldous

(1981) developed a somewhat different approach, which was applied in Norros (1985). The definition given below differs slightly from the above-mentioned ones since we are considering an abstract history instead of a process.

We denote by $\mathcal{P}(\Omega)$ the space of all probability measures on $\mathcal{B}(\Omega)$, endowed with the topology of weak convergence. $\mathcal{P}(\Omega)$ is in turn a Polish space, and $\mathcal{P}(\mathcal{P}(\Omega))$ can be defined in a similar manner.

THEOREM 2.2. *There exists a $\mathcal{P}(\Omega)$-valued cadlag process μ such that for any stopping time T, μ_T is a regular version of the conditional probability $P(\,\cdot\mid\mathcal{F}_T)$.*

The proof can be found in Aldous (1981), and it is reproduced in Norros (1985). In this paper, we call the process μ of Theorem 2.2 simply *the prediction process.*

PROPOSITION 2.3. *Let Y be a bounded $\mathcal{B}(\Omega)$-measurable random variable defined on $(\Omega, \mathcal{F}, P)$. Denote by M^Y a cadlag version of the martingale $E[Y\mid\mathcal{F}_t]$. Then the process $\int Y\,d\mu_{t-}$ is indistinguishable from M^Y_{t-}. In particular, $\int Y\,d\mu_{T-} = M^Y_{T-}$ a.s. for every stopping time T.*

For a proof, see, for example, Norros (1985). Note that, although $\mu_T = P[\,\cdot\mid\mathcal{F}_T]$, μ_{T-} is *not* $P[\,\cdot\mid\mathcal{F}_{T-}]$. For example, T is $\mathcal{F}_{T-}$-measurable, but if T is completely unpredictable, then the random measure μ_{T-} gives T a continuous distribution a.s. The random measure μ_{T-} is not a conditional distribution with respect to any σ-field, but it tells what the prediction was immediately before T occurred.

Intuitively, the difference between the σ-fields $\mathcal{F}_{S-}$ and $\mathcal{F}_S$ is that in $\mathcal{F}_{S-}$, it is known when S occurs, but it is not known what else happens at time S. For example, if S is a point in a marked point process, then S is known in $\mathcal{F}_{S-}$, but the mark is $\mathcal{F}_S$-measurable and may not be $\mathcal{F}_{S-}$-measurable.

Now we proceed to the construction of the "minimal repair" of S. Suppose that we choose an ω with distribution P and start proceeding at time 0. Suddenly S occurs. In order to make a minimal repair, we have to change our ω to another, say ω', which is indistinguishable from ω strictly before the time $S(\omega)$ and satisfies $S(\omega') > S(\omega)$. Moreover, ω' should be chosen according to an appropriate distribution among the candidates satisfying these conditions. This reasoning can be formalized by means of the prediction process.

Indeed, $\mu_{S-}(\omega)$ gives the conditional distribution with respect to the history strictly before $S(\omega)$ when it was not yet known that S would appear at time $S(\omega)$. Thus, if we choose ω' according to the distribution $\mu_{S-}(\omega)$, we may proceed further "as if nothing had happened". Intuitively, this means that at any time prior to the ultimate system failure it is not even "known" whether there has been a minimal repair or not.

In order to be more rigorous, let κ be a $\mathcal{B}(\Omega)$-measurable $\mathcal{P}(\Omega)$-valued random variable such that $\kappa = \mu_{S-}$ a.s. (κ can be constructed by means of the regular conditional distribution of μ_{S-} w.r.t. $\mathcal{B}(\Omega)$). Now define successively the probability measures $Q_0,\ Q_1,\ Q_2,\ldots$ on $\mathcal{B}(\Omega)$ by

$$Q_0 = P,\ Q_{n+1}(A) = \int_\Omega \kappa(\omega)(A)Q_n(d\omega).$$

Q_n is the measure which is obtained when S is deferred n times in the sense of minimal repair. The following theorem shows that the measures Q_n have a density (w.r.t. P) which has a very simple expression.

THEOREM 2.4. *For any n, the probability measure Q_n is absolutely continuous w.r.t. P, with the Radon-Nikodym derivative*

$$\frac{dQ_n}{dP} = \frac{1}{n!}A_S^n.$$

Moreover, for any stopping time T,

$$\left(\frac{dQ_n}{dP}\right)_{\mathcal{F}_T} = e^{A_T}1_{\{S>T\}}\left(1-\frac{1}{n!}\int_0^{A_T} x^n e^{-x}dx\right)+\frac{1}{n!}A_S^n 1_{\{S\le T\}}.$$

PROOF. We prove the first assertion by induction. It holds trivially for $n=0$. Suppose that it holds for some fixed value of n. Let Y be any bounded $\mathcal{B}(\Omega)$-measurable random variable. We have to show that

$$\int Y dQ_{n+1} = EY\frac{1}{(n+1)!}A_S^{n+1}.$$

Denote by M^Y a cadlag version of the martingale $E[Y \mid \mathcal{F}_t]$. Now, by the definition of κ, Proposition 2.3, the rules of Stieltjes stochastic calculus and Dellacherie's integration formula (Dellacherie (1972), IV T 47),

$$\int_\Omega Y dQ_{n+1} = \int_\Omega Q_n(d\omega)\int_\Omega \kappa(\omega)(d\omega')Y(\omega') = \int_\Omega M_{S-}^Y dQ_n = E\frac{1}{n!}A_S^n M_{S-}^Y$$

$$= E\frac{1}{n!}\int_0^\infty A_t^n M_{t-}^Y dN_t = E\int_0^\infty M_{t-}^Y\frac{1}{n!}A_t^n dA_t = E\int_0^\infty M_t^Y\frac{1}{(n+1)!}dA_t^{n+1}$$

$$= EM_\infty^Y\frac{1}{(n+1)!}A_\infty^{n+1} = EY\frac{1}{(n+1)!}A_S^{n+1}.$$

The proof of the second equation is based on Lemma 2.1.:

$$\left(\frac{dQ_n}{dP}\right)_{\mathcal{F}_T}1_{\{S>T\}} = 1_{\{S>T\}}\frac{1}{n!}E[A_S^n \mid \mathcal{F}_T]$$

$$= 1_{\{S>T\}}\frac{1}{n!}\int_0^\infty (A_T+x)^n e^{-x}dx = 1_{\{S>T\}}\frac{1}{n!}\int_{A_T}^\infty x^n e^{-(x-A_T)}dx$$

$$= e^{A_T}1_{\{S>T\}}\left(1-\frac{1}{n!}\int_0^{A_T} x^n e^{-x}dx\right). \quad \|$$

Theorem 2.4 gives the justification for the following definition.

DEFINITION 2.5. With notation as above, the probability measure corresponding to an *n-fold $\mathcal{F}$-minimal repair of the stopping time S* is the measure Q_n defined by

$$\frac{dQ_n}{dP} = \frac{1}{n!}A_S^n.$$

We close this section with a remark concerning the situation where the stopping time S is eliminated completely by repeating "minimal repairs" indefinitely. Since S is a.s. finite, a measure describing this operation in the sense of Theorem 2.4 can not in general be defined on $\mathcal{F}_\infty$, and if it can, it will be concentrated on the P-null set $\{S = \infty\}$. However, such a measure can be defined on each sub-σ-field $\mathcal{F}_T$, where T is a stopping time such that A_T is bounded. Letting n go to infinity in the second assertion of Theorem 2.4, it is seen that this measure, say, Q_∞^T, is absolutely continuous w.r.t. P on $\mathcal{F}_T$ and

$$\frac{dQ_\infty^T}{dP|_{\mathcal{F}_T}} = e^{A_T}1_{\{S>T\}}.$$

3. Stochastic Comparison of Transformed Distributions. Let $(\Omega, \mathcal{F}, P)$, $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$, S and N be as in the previous section. Let $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0}$ be the history generated by the one point counting process N. Let $A^{\mathbf{F}}$ and $A^{\mathbf{G}}$ be the $\mathbf{F}$- and $\mathbf{G}$-compensators of N, respectively. Since S is assumed to be completely unpredictable w.r.t. $\mathbf{F}$, both compensators are continuous.

We consider the following kind of transformations of the compensators. Let $g : [0,\infty] \to [0,\infty]$ be an increasing differentiable function such that $g(0) = 0$ and $g(\infty) = \infty$. Consider the continuous increasing process $B^{\mathbf{F}}$ defined by

$$B_t^{\mathbf{F}} = g(A_t^{\mathbf{F}}),\ t \geq 0.$$

By the next well-known Girsanov type theorem, we can modify the probability P in an absolutely continuous way so that $B^{\mathbf{F}}$ is the compensator of N with respect to the new measure. For a proof, see Jacod (1975), Proposition 4.3 and Theorem 4.5.

THEOREM 3.1. *Denote by L the process*

$$L_t = g'(A_t^{\mathbf{F}})^{N_t}e^{A_t^{\mathbf{F}} - B_t^{\mathbf{F}}}.$$

L is a uniformly integrable martingale with expectation 1. *Define the probability measure Q on $\mathcal{F}$ by*

$$\frac{dQ^{\mathbf{F}}}{dP} = L_\infty = g'(A_S^{\mathbf{F}})e^{A_S^{\mathbf{F}} - B_S^{\mathbf{F}}}.$$

Then the $(Q^{\mathbf{F}}, \mathbf{F})$-compensator of S is $B^{\mathbf{F}}$.

We consider two special cases of this transformation. The first is minimal repair. Indeed, choosing $g(x) = x - \ln(1+x)$, we have

$$L_\infty = \frac{A_S^{\mathbf{F}}}{1 + A_S^{\mathbf{F}}} \exp(A_S^{\mathbf{F}} - (A_S^{\mathbf{F}} - \ln(1 + A_S^{\mathbf{F}}))) = A_S^{\mathbf{F}},$$

which is the density corresponding to an $\mathcal{F}$-minimal repair of S (Definition 2.5).

The other case is the linear transformation

$$g(x) = \alpha x, \ \alpha \in (0,1),$$

which could be called *proportional improvement* since the hazard is reduced by a fixed percentage. One also quickly concludes that this is equivalent to the *imperfect repair* of Brown and Proschan (1983), where the device is repeatedly minimally repaired with probability $(1-\alpha)$ up to the first unsuccessful repair attempt. See also Shaked and Shanthikumar (1986).

We can now prove the main result of this paper.

THEOREM 3.2. *With the notation as above, suppose that the function g is such that $e^{-g(-\ln x)}$ is concave for $x \in (0,1)$. Then S is stochastically smaller under $Q^{\mathbf{F}}$ than under $Q^{\mathbf{G}}$.*

PROOF. Let T be an $\mathbf{F}$-stopping time such that $A_T^{\mathbf{F}}$ is bounded. We first observe that (cf. Norros (1986), Proposition 5.1)

$$E 1_{\{S > t\wedge T\}} e^{A_{t\wedge T}^{\mathbf{F}}} = 1.$$

Thus, $\exp(A_{t\wedge T}^{\mathbf{F}})$ can be viewed as a density function on the set $\{S > t \wedge T\}$. Denote

$$f(x) = e^{-g(-\ln x)}, \ x \in (0,1),$$

which by assumption is a concave function. Now, by Jensen's inequality,

$$\begin{aligned} Q^{\mathbf{F}}(S > t\wedge T) &= E 1_{\{S>t\wedge T\}} e^{A_{t\wedge T}^{\mathbf{F}} - g(A_{t\wedge T}^{\mathbf{F}})} = E 1_{\{S>t\wedge T\}} e^{A_{t\wedge T}^{\mathbf{F}}} f(e^{-A_{t\wedge T}^{\mathbf{F}}}) \\ &\le f(E 1_{\{S>t\wedge T\}} e^{A_{t\wedge T}^{\mathbf{F}}} e^{-A_{t\wedge T}^{\mathbf{F}}}) = f(P(S > t\wedge T)). \end{aligned}$$

Applying this for $T = T_n = \inf\{t : A_t^{\mathbf{F}} \ge n\}$ and letting $n \to \infty$ we obtain, since $S \le \sup T_n$ a.s.,

$$Q^{\mathbf{F}}(S > t) \le f(P(S > t)).$$

But

$$f(P(S > t)) = e^{-g(-\ln P(S>t))} = e^{-g(A_t^{\mathbf{G}})} = e^{-B_t^{\mathbf{G}}} = Q^{\mathbf{G}}(S > t). \quad \|$$

It is easy to see that the transformations corresponding to minimal repair and imperfect repair both satisfy the conditions of Theorem 3.2. Thus we have the following corollaries, which are of certain practical interest.

COROLLARY 3.3. *Consider the change of distributions which corresponds to exactly one successful minimal repair on a failed device. Then the* **F***-hazard transformation leads to a stochastically shorter life length than the corresponding* **G***-hazard transformation.*

COROLLARY 3.4. *Consider the change of distributions which corresponds to a fixed proportional improvement, or, equivalently, imperfect repair with a constant probability for successful minimal repair. Then the* **F***-hazard transformation leads to a stochastically shorter life length than the corresponding* **G***-hazard transformation.*

We conclude with some remarks. First, we return to the distinction between "physical" and "statistical" minimal repair. Recall that the history **F** can be completely general, as long as S remains completely unpredictable. On the other hand, one could argue that if the minimal repair is an actual physical operation performed on a failed device, which returns it to the state immediately preceding the failure, the history should be one "giving a full description of the internal state".

It is also interesting that our inequalities hold irrespective of all dependencies that a conditioning on a history might reveal, whether positive or negative. This is not obvious since usually stochastic comparison results involving conditioning require some form of stochastic monotonicity.

Finally, we mention two open problems. First, Proposition 5.4 in Norros (1986) shows that in the case of proportional improvement, the stochastic comparison in Corollary 3.4 is reversed if **F** is the internal history of a set of component life lengths and if the transformation is made on the compensators of *all* components. Would a similar result hold for the minimal repair also, or for a more general class of compensator transformations? Second, does Theorem 3.2 hold when **G** is larger than the minimal history but smaller than **F**?

REFERENCES

ALDOUS, D. (1981). Weak convergence and the general theory of processes. Incomplete draft of a monograph.

ARJAS, E. and NORROS, I. (1989). Change of life distribution via a hazard transformation: An inequality with application to minimal repair. *Math. Oper. Res.* **14** 355–361.

ASCHER, H. and FEINGOLD, H. (1984). *Repairable Systems Reliability.* Marcel Dekker, New York and Basel.

BARLOW, R.E. and HUNTER, L. (1960). Optimum preventive maintenance policies. *Operations Research* **1** 90–100.

BROWN, M. and PROSCHAN, F. (1983). Imperfect repair. *J. Appl. Prob.* **20** 851–859.

BERGMAN, B. (1985). On reliability theory and its applications (with discussion). *Scand. J. Statist.* **12** 1–41.

DELLACHERIE, C. (1972). Capacités et processus stochastiques. Springer Verlag, Berlin.

JACOD, J. (1975). Multivariate point processes: Predictable projection, Radon-Nikodym derivatives, representation of martingales. *Z. Wahrsch. Verw. Geb.* **34** 235–253.

KNIGHT, F. (1975). A predictive view of continuous time processes. *Ann. Prob.* **3** 573–596.

NORROS, I. (1985). Systems weakened by failures. *Stoch. Proc. Appl.* **20** 181–196.

NORROS, I. (1986). A compensator representation of multivariate life length distributions, with applications. *Scand. J. Statist.* **13** 99–112.

NORROS, I. (1987). The "minimal repair", elimination and externalization of a totally inaccessible stopping time. Reports of Dep. of Math., Univ. of Helsinki.

SHAKED, M. and SHANTHIKUMAR, J.G. (1986). Multivariate imperfect repair. *Operations Research* **34** 437–448.

DEPARTMENT OF APPLIED MATHEMATICS
AND STATISTICS
UNIVERSITY OF OULU
LINNANMAA
SF–90570 OULU, FINLAND

TELECOMMUNICATIONS LABORATORY
TECHNICAL RESEARCH CENTRE OF
FINLAND
OTAKAARI 7 B
SF–02150 ESPOO, FINLAND