INTRODUCTION

Concepts of positive dependence are becoming increasingly important in probability, statistics and their applications. While these concepts are traditionally viewed as focusing on positive and negative dependence for random vectors, they also are related to broader issues in the modeling and the analysis of multivariate data, and, in particular, ordinal data.

Historically, positive dependence for the multivariate normal distribution had been synonymous with positive correlations. Other subsequently developed multivariate distributions were often interpreted with this perspective. It was eventually realized that positive correlations can have substantially different meanings for other multivariate distributions than they have for the multivariate normal. In fact, it has been more recently demonstrated that several different important positive dependence concepts which are equivalent for multivariate normal distributions are not equivalent, in general, for multivariate distributions. In particular the concept of association is stronger than positive orthant dependence which is stronger than positive correlations. Thus, in a certain sense, many of the positive dependence concepts discussed or referenced in this volume, are outgrowths of original attempts to nonparametrically capture and extend certain properties of the multivariate normal. Additionally, other types of dependence came about from extending certain univariate properties, such as the memoryless property of the exponential distribution.

From the point of view expressed above, the theoretical origins of this research area include the fundamental works of Lehmann (1966) concerning orthant dependence, Esary, Proschan and Walkup (1967) dealing with the concept of association, and Marshall and Olkin (1967) modeling multivariate distributions. These three papers drew upon a rich historical stream and, in turn, have spawned numerous applications and inspired other related dependence concepts. The sources of this historical stream range broadly from reliability and mathematical inequalities to nonparametric statistical modeling and measures of association. Some of the researchers involved in these pioneering efforts include Goodman, Hardy-Littlewood-Polya, Hoeffding, Karlin, Kendall, Kruskal, Lancaster, Šidák, Sobel, and Tukey.

Barlow and Proschan (1981) further developed positive dependence concepts in their book (first printed in 1975) particularly in the bivariate case, and Tong (1980) provides additional material and development. A parallel development occurred independently in the mathematical physics literature (see Fortuin, Kastelyn and Ginibre (1971)). For a long time it was felt that negative dependence concepts were the mirror image of positive dependence. That this was not the case was demonstrated by Karlin and Rinott (1980), Block, Savits, and Shaked (1982), and by Joag-dev and Proschan (1983) among others. Related review articles on inequalities and dependence are those by Eaton (1982) and by Block and Sampson

(1983).

Currently, techniques and approaches are being developed specifically for modeling positive dependence for discrete multivariate distributions, with a perspective for applications to multivariate ordinal contingency tables. These techniques draw upon a diverse body of research including log-linear contingency table modeling (Bishop, Fienberg and Holland (1975)) and canonical representations of probability distributions (Lancaster (1969)). Due in part to natural constraints upon the discrete probabilities imposed by positive and negative dependence concepts, techniques from order-restricted inference have been fruitful in dealing with problems related to estimation (e.g., Robertson, Wright and Dykstra (1989)). The book by Agresti (1984) synthesizes and summarizes some of the basic connections between positive dependence and ordinal contingency tables, and the applications of order restricted inference to this area.

This broad field of research concerning positive and negative dependence can be categorized into four areas. These areas are currently in different stages of development.

(a) *Fundamental Concepts of Positive Dependence.* This area consists of the probabilistic modeling of concepts for positive and negative dependence. The primary foci are properties of dependence, orderings, related inequalities, measures and their interrelationships.

(b) *Applications to Probability Theory, Reliability Theory, Mathematical Physics, and Other Areas.* This area applies the theoretical developments in the preceding item, (a), in developing results in other research areas. Examples include (i) probability limit theorems under positive dependence conditions; (ii) system reliability bounds assuming certain dependence structures for component life lengths; (iii) models for causal relationships in the social sciences; and (iv) repeated testing schemes.

(c) *Statistical Considerations Involving Positive and Negative Dependence.* This relatively new area concerns estimation, hypothesis testing and distribution theory based upon data taken from multivariate distributions having certain positive or negative dependence structures, or having certain monotone order relationships. Instances of such research are: (i) testing for component life lengths having certain distributional structures such as quadrant dependence; and (ii) testing whether all odds ratio in an $r \times c$ contingency table are greater than or equal to 1 (which is the dependence concept called TP_2).

(d) *Inter-Relationships of Positive Dependence Concepts With Other Aspects of Statistics.* This area concerns the connections between the ideas developed in the theory of positive and negative dependence, with other areas of statistics. A large body of research is concerned with the interpretation of positive and negative dependence concepts for the multivariate normal distribution and other parametric families of distributions. Examples include: (i) conditions

for certain positive dependence concepts to hold for elliptically symmetric distributions; and (ii) results for when certain parametric families of distributions are well-ordered by the parameter with respect to various positive dependence orderings. An emerging research area is the connection between various probability models for ordinal contingency tables and positive dependence concepts. For instance, certain types of odds-ratios properties have a natural interpretation as positive dependence concepts.

Bibliography of Pertinent Literature

AGRESTI, A. (1984). *Analysis of Ordinal Categorical Data.* John Wiley, New York.

BARLOW, R.E., BARTHOLOMEW, D.J., BREMNER, J.M., and BRUNK, H.D. (1972). *Statistical Inference Under Order Restrictions.* John Wiley, New York.

BARLOW, R. and PROSCHAN, F. (1981). *Statistical Theory of Reliability and Life Testing.* To Begin With, Silver Spring, Md.

BISHOP, Y.M., FIENBERG, S.E., and HOLLAND, P.W. (1975). *Discrete Multivariate Analysis.* MIT Press, Cambridge.

BLOCK, H.W. and SAMPSON, A.R. (1983). Inequalities on distributions: bivariate and multivariate. *Encyclopedia of Statistical Sciences, Volume 4* (S. Kotz and N.L. Johnson, eds.), John Wiley and Sons, Inc.

BLOCK, H.W., SAVITS, T.H., and SHAKED, M. (1982). Some concepts of negative dependence. *Ann. Prob.* **10**, 765–772.

EATON, M.L. (1982). A review of selected topics in multivariate probability inequalities. *Ann. Statist.* **10** 11–43.

ESARY, J.D., PROSCHAN, F., and WALKUP, D.W. (1967). Association of random variables with applications. *Ann. Math. Statist.* **38** 1466–1474.

FORTUIN, C., KASTELYN, P., and GINIBRE, J. (1971). Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.* **22** 89–103.

GOODMAN, L.A. and KRUSKAL, W.H. (1979). *Measures of Association for Cross-classifications.* Springer-Verlag, New York.

JOAG-DEV, K. and PROSCHAN, F. (1983). Negative association of random variables with applications. *Ann. Statist.* **11** 286–295.

KARLIN, S. and RINOTT, Y. (1980). Classes of orderings of measures and related correlation inequalities. II. Multivariate reverse rule distributions. *J. Multi. Anal.* **10** 499–516.

LANCASTER, H.O. (1969). *The Chi-squared Distribution.* John Wiley and Sons, New York.

LEHMANN, E.L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

MARSHALL, A.W. and OLKIN, I. (1967). A multivariate exponential distribution. *J. Amer. Statist. Assoc.* **62** 30–49.

ROBERTSON, T.J., WRIGHT, F.T., and DYKSTRA, R.L. (1988). *Order Restricted Statistical Inference.* John Wiley and Sons, New York.

TONG, Y.L. (1980). *Probability Inequalities in Multivariate Distribution.* Academic Press, New York.