New Developments and Applications in Experimental Design
IMS Lecture Notes – Monograph Series (1998) Volume 34

SOME RECENT ADVANCES IN MINIMUM ABERRATION DESIGNS

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The objective of this article is to review recent advances in the theory of characterizing minimum aberration designs in terms of their complementary designs. The approach is very powerful for identifying minimum aberration designs whose complementary design sizes are small. Using this theory and some designs from Chen, Sun and Wu (1993), we identify all minimum aberration 2^{n-m} designs whose complementary design size is less than 64.

1. Introduction and definitions. Fractional factorial designs have a long history of successful use in many scientific investigations. A 2^{n-m} fractional factorial design is a 2^{-m} fraction of the 2^n design, it has n factors but 2^{n-m} runs. Each factor is represented by one of the numbers $1, 2, ..., n$, which are called *letters.* A product (juxtaposition) of a subset of these letters is called a *word.* The number of letters in a word is called its *length.* Associated with every regular 2^{n-m} fractional factorial design is a set of m words, W_1, W_2,...,W_m, called *generators.* The set of distinct words formed by all possible products involving m generators gives the *defining relation* of the fraction. The *resolution* of such a design is defined as the length of the shortest word in the defining relation [Box and Hunter (1961)]. Resolution is a commonly used criterion for selecting regular fractional factorial designs. In a design of resolution r, no c-factor effect is confounded with effects involving less than $r - c$ factors. Let $D(2^{n-m})$ be a regular 2^{n-m} fractional factorial design, the vector $W(D) = (A_1(D), A_2(D), ..., A_n(D))$ is defined as the *wordlength pattern* of $D(2^{n-m})$, where $A_i(D)$ is the number of words of length i in its defining relation. For related additional information concerning fractional factorial designs, see Raktoe, Hedayat and Federer (1981).

In situations where there is little prior knowledge about the possible greater importance of factorial effects, often experimenters prefer to use a design with the highest

[1] Research sponsored by National Science Foundation Grant No. DMS-9304014 and National Cancer Institute Grant No. P01-CA48112.

Received September, 1997; revised April, 1998.

AMS 1991 subject classifications. Primary 62K15; secondary 62K05.

Key words and phrases. Finite projective geometry, fractional factorial design, resolution, wordlength pattern.

possible resolution. However using resolution alone may mislead the selection process in some situations because all fractional factorial designs with the maximum resolution are not equally desirable. Fries and Hunter (1980) introduced the minimum aberration criterion for discriminating among all 2^{n-m} designs with the same resolution. For two 2^{n-m} designs d_1 and d_2 with wordlength patterns $W(d_1)$ and $W(d_2)$ respectively, d_1 is said to have less aberration than d_2 if $A_s(d_1) < A_s(d_2)$ where s is the smallest integer such that $A_s(d_1) \neq A_s(d_2)$. A 2^{n-m} design has *minimum aberration* if no other 2^{n-m} design has less aberration. Franklin (1984) extended this criterion to p^{n-m} fractional factorial designs, where p is a prime power. Both resolution and minimum aberration are defined under the assumption: (a) lower order interactions are more important than higher order interactions, and (b) interactions of the same order are equally important. The criterion of minimum aberration sequentially minimizes A_1, A_2, A_3,.., and can rank-order almost any two designs. Minimizing the numbers of short-length words generally leads to the estimability of more lower-order interactions. For example, if we assume that three-factor and higher-order interactions are negligible, designs of maximum resolution IV with minimum number of words of length four may provide more estimable two-factor interactions. This can be best illustrated by the following example from Fries and Hunter (1980).

EXAMPLE 1. There are precisely three non-isomorphic 2^{7-2} fractional factorial designs with resolution IV,

$$\begin{aligned} d_1 &: \; I = 1236 = 2347 = 1467, \\ d_2 &: \; I = 1236 = 1457 = 234567, \\ d_3 &: \; I = 4567 = 12346 = 12357. \end{aligned}$$

It can be shown that IV is the maximum attainable resolution for a 2^{7-2} design. Their wordlength patterns are

$$\begin{aligned} W(d_1) &= (0,0,0,3,0,0,0), \\ W(d_2) &= (0,0,0,2,0,1,0), \\ W(d_3) &= (0,0,0,1,2,0,0). \end{aligned}$$

Looking through these wordlength patterns, clearly d_3 has minimum aberration. By comparing with other two designs, we can see that d_3 has the smallest number of two-factor interactions confounded with each other, i.e., it provides the largest number of estimable two-factor interactions when three-factor and higher-order interactions are negligible.

The minimum aberration criterion plays a fundamental role in practical selection of factorial designs (Wu and Chen (1992)), characterization of such designs is an important problem in design theory. For catalogues of some minimum aberration designs, see Franklin (1984), Chen and Wu (1991), Chen (1992) and Chen, Sun and Wu (1993). Meanwhile the criterion may also lead to other good overall properties. Krouse (1994) studied optimal first order 2^{n-m} designs which are locally robust to misspecification of the prior distribution parameter. These designs turn out to have minimum aberration. Cheng, Steinberg and Sun (1998) studied the performances of minimum aberration designs under two different criteria quantifying the notion of model robustness, estimation capacity and the expected number of suspected two-factor interactions. They showed that minimum aberration is a good surrogate for both of these criteria.

For factor screening experiments in many scientific investigations, regular 2^{n-m} designs with resolution III or IV are widely used. The study of minimum aberration designs with low resolution has a direct impact on practical experimentation. Chen (1993), Chen and Hedayat (1996) and Tang and Wu (1996) proposed a method for characterizing minimum aberration 2^{n-m} designs in terms of their complementary designs. Suen, Chen, and Wu (1997) extended the method to identify minimum aberration p^{n-m} designs. This method is very powerful for identifying minimum aberration designs with low resolution. In this paper, we review recent advances in the theory of characterizing minimum aberration designs. For simplicity, we focus entirely on two-level designs, even though some of the results can be generalized to the case where the number of levels is a prime power. In Section 2, we study the representation of a fractional factorial design and its complementary design in terms of finite geometry. Some theorems and rules for characterizing minimum aberration designs are stated in Section 3. Using these rules and some designs from Chen, Sun, and Wu (1993), we identify all minimum aberration 2^{n-m} designs whose complementary design size is less than 64 in Section 4.

2. Preliminaries. First, we discuss a geometric interpretation of regular fractional factorial designs by using techniques of finite projective geometry [see Bose (1947) for a detailed discussion].

Let $D(2^{n-m})$ be a regular 2^{n-m} fractional factorial design. A word in the defining relation of $D(2^{n-m})$ can be represented by a binary row vector. Let G be an $m \times n$ matrix of rank m over the finite field $GF(2)$ whose rows are m generators of D, that is, $W_1,...,W_m$. Each treatment combination in D is viewed as a row vector $\mathbf{x}$ which satisfies the equation $\mathbf{x}G' = \mathbf{0}$. The subspace generated by the rows of G is the defining relation of $D(2^{n-m})$. Let B_n be an $(n-m) \times n$ matrix whose $n-m$ rows form a basis of the solution space of the equation $\mathbf{x}G' = \mathbf{0}$. Following the notation and concepts in Chen and Hedayat (1996), a $D(2^{n-m})$ can be represented as row vectors as follows:

$$D(2^{n-m}) = \{\mathbf{x} : \mathbf{x} = \mathbf{u}' B_n, \mathbf{u} \in EG(n-m, 2)\}, \tag{2.1}$$

where $EG(n-m, 2)$ is the Euclidean geometry of dimension $n-m$ over $GF(2)$ and $\mathbf{u}$ is represented by a column vector. B_n is called the factor representation of the fractional factorial design $D(2^{n-m})$. One such matrix B_n can be obtained by writing down the coordinates of n points of $PG(n-m-1, 2)$ as columns, where $PG(\nu, 2)$ is the projective geometry of dimension ν over $GF(2)$. Then a regular fractional factorial design as in (2.1) is determined by a set of n points of $PG(n-m-1, 2)$. The factor representation can be viewed as a subset of n points of $PG(n-m-1, 2)$.

Let $k = n-m$ and a $2^{n-(n-k)}$ design with resolution III or higher be determined by a subset of n distinct points of $PG(k-1, 2)$. Let $B_n = \{\mathbf{a}_1, ..., \mathbf{a}_n\}$ be a subset of n distinct points of $PG(k-1, 2)$. Such a subset can be obtained by deleting $2^k - 1 - n$ points from $PG(k-1, 2)$. Without loss of generality, we can represent all points of

$PG(k-1,2)$ as

$$(2.2) \qquad \underbrace{\mathbf{a}_1, ..., \mathbf{a}_n}_{B_n}, \underbrace{\mathbf{a}_{n+1}, ..., \mathbf{a}_{2^k-1}}_{\overline{B}_n},$$

where the first n points are all points of B_n, and $\overline{B}_n$ denotes all points of $PG(k-1,2)\backslash B_n = \{\mathbf{a}_{n+1}, ..., \mathbf{a}_{2^k-1}\}$. The set $\overline{B}_n$ contains the remaining $\overline{n}$ points in $PG(k-1,2)$, where $\overline{n} = 2^k - 1 - n$. Let D and $\overline{D}$ be the two fractional factorial designs corresponding to B_n and $\overline{B}_n$ as their factor representations respectively. $\overline{D}$ is called the *complementary design* of D. Since the defining relation of D consists of all those words $\mathbf{w}$, $\mathbf{w} \neq \mathbf{0}$, for which $B_n\mathbf{w}' = \mathbf{0}'$, a word of length l corresponds to l points of B_n, say $\mathbf{a}_{i_1}$, $\mathbf{a}_{i_2}$,..., $\mathbf{a}_{i_l}$, such that $\mathbf{a}_{i_1}+\mathbf{a}_{i_2}+...+\mathbf{a}_{i_l}=\mathbf{0}$. We illustrate these concepts through the following example. For convenience, a point of $PG(k-1,2)$ is denoted by $i_1 i_2 \cdots i_l$ if the i_1th, i_2th,...,i_lth coordinates of this point are 1 and all others are zeros.

EXAMPLE 2. Let d be a 2^{4-1} fractional factorial design with the factor representation B_4, where

$$B_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The factor representation B_4 can be considered as a subset $\{1,2,3,12\}$ of $PG(2,2)$, and it can be obtained by deleting $B_3 = \{13,23,123\}$ from $PG(2,2)$,

$$\underbrace{1,2,3,12}_{B_4}, \underbrace{13,23,123}_{B_3}.$$

The complementary design $\overline{d}$ corresponds to B_3 as its factor representation. The wordlength pattern of d is

$$W(d) = (0,0,1,0).$$

The wordlength pattern of $\overline{d}$ is

$$W(\overline{d}) = (0,0,0).$$

Bose (1947) studied the problem of determining the maximum number of factors that can be accommodated in a regular fractional factorial design when interactions up to a given order are left unconfounded. For $2^{k-1} < n \leq 2^k - 1 (k = n - m)$, the maximum resolution of any 2^{n-m} design is equal to III. Therefore, the complementary design of a minimum aberration 2^{n-m} design with resolution III has relatively small number of factors. It is by far much easier to handle the wordlength pattern of a small complementary design than that of a large 2^{n-m} design. However, the important question is how to link the wordlength pattern of a 2^{n-m} design with that of its complementary design.

3. Characterization of minimum aberration designs in terms of their complementary designs. Suen, Chen, and Wu (1997) studied the relationship between the wordlength patterns of a regular fractional factorial design and its complementary design. By applying the techniques in coding theory, they obtained explicit identities between the wordlength pattern of a 2^{n-m} design and that of its complementary design. These identities provide a powerful tool for characterizing minimum aberration 2^{n-m} designs in terms of their complementary designs.

THEOREM 1 [Suen, Chen and Wu(1997)]. *Let $W(D)$ and $W(\overline{D})$ be the wordlength patterns of a 2^{n-m} design D and its complementary design $\overline{D}$, respectively. Then*

$$(3.1)\qquad A_i(D) = C_i + C_{i0} + \sum_{j=3}^{i-1} C_{ij} A_j(\overline{D}) + (-1)^i A_i(\overline{D}), \quad for \;\; i = 3, ..., n,$$

where, with $k = n-m$, $C_i = 2^{-k}[P_i(0;n) - P_i(2^{k-1};n)]$, and $P_i(j;n) = \sum_{s=0}^{i} (-1)^s \binom{j}{s} \binom{n-}{i-}$

and $C_{ij} = (-1)^{i-\left[\frac{i-j}{2}\right]} \binom{2^{k-1}-1-\overline{n}}{\left[\frac{i-j}{2}\right]}$.

Note that $A_i(\overline{D})$ in (3.1) are 0, for $i > \overline{n}$. We extend the definition of $\binom{n}{s}$ to allow n and s to be any integers:

$$\binom{n}{s} = \begin{cases} \dfrac{n(n-1)\cdots(n-s+1)}{s(s-1)\cdots 1} & for \;\; positive \;\; s \\ 1 & for \;\; s = 0 \\ 0 & for \;\; negative \;\; s. \end{cases}$$

The identities in (3.1) have explicit forms so that the wordlength pattern of a 2^{n-m} fractional factorial design can be readily calculated from that of its complementary design. Using these equations, some rules for identifying minimum aberration 2^{n-m} designs in terms of their complementary designs can be easily established.

RULE 1. A 2^{n-m} design D^* with $\overline{D^*}$ as its complementary design (of size $\overline{n}$) has minimum aberration if
(i) $A_3(\overline{D^*})$ is the maximum among all complementary designs of size $\overline{n}$,
(ii) $\overline{D^*}$ is the unique design satisfying (i).

RULE 2. A 2^{n-m} design D^* has minimum aberration if
(i) $A_3(\overline{D^*})$ is the maximum among all complementary designs of size $\overline{n}$,
(ii) $A_4(\overline{D^*})$ is the minimum among all complementary designs of size $\overline{n}$ whose number of words of length three equals $A_3(\overline{D^*})$,
(iii) $\overline{D^*}$ is the unique design satisfying (ii).

More generally, by noting the relation

$$A_i(D) = (-1)^i A_i(\overline{D}) + lower \;\; order \;\; terms,$$

we can develop similar rules for identifying minimum aberration designs if it is necessary to minimize the numbers of words of length $i(\geq 5)$.

The following example illustrates the application of these rules for identifying minimum aberration designs.

EXAMPLE 3. All points of $PG(2,2)$ can be represented as

$$PG(2,2) = \{1, 2, 3, 12, 13, 23, 123\}.$$

Any 2^{4-1} design of resolution III or higher can be determined by a subset of four distinct points of $PG(2,2)$. The factor representation of its complementary design consists of the remaining three points of $PG(2,2)$. Up to isomorphism (Pu (1989)), there are two different complementary designs $\overline{d}_1$ and $\overline{d}_2$ corresponding to factor representations $B_3^1 = \{13, 23, 123\}$, and $B_3^2 = \{12, 13, 23\}$ respectively. The wordlength patterns of these complementary designs are $W(\overline{d}_1) = (0,0,0)$ and $W(\overline{d}_2) = (0,0,1)$. Let d_1 and d_2 be two 2^{4-1} designs corresponding to factor representations which result in deleting B_3^1 and B_3^2 from $PG(2,2)$ respectively. Since $\overline{d}_2$ has the maximum number of words of length three between $\overline{d}_1$ and $\overline{d}_2$, the design d_2 has minimum aberration. By (3.1), we have the identities,

$$\begin{aligned} A_3(D) &= 1 - A_3(\overline{D}), \\ A_4(D) &= A_3(\overline{D}). \end{aligned}$$

The wordlength patterns of d_1 and d_2 can be calculated from those of their complementary designs,

$$\begin{aligned} W(d_1) &= (0,0,1,0), \\ W(d_2) &= (0,0,0,1). \end{aligned}$$

Whenever a 2^{n-m} design of resolution three or higher has minimum aberration, its number of words of length three must be minimized, i.e., its complementary design has the maximum number of words of length three among all complementary designs of size $\overline{n}$. For $\overline{n} = 2^k - 1 - n = 2^r + q$ $(0 \le q < 2^r)$, the factor representation of the complementary design is an $\overline{n}$-subset of $PG(k-1,2)$. Let rank of an $\overline{n}$-subset of $PG(k-1,2)$ be the maximum number of independent points in the subset. The rank of any $\overline{n}$-subset of $PG(k-1,2)$ is at least $r+1$. In Example 3, the 3-subset B_3^2 containing the maximum number of words of length three has the minimum rank two, while the rank of B_3^1 with no word of length three is three. Chen and Hedayat (1996) discovered that the factor representation of the complementary design $\overline{D}$ of size $2^r + q$ $(0 \le q < 2^r)$ containing the maximum number of words of length three must have the minimum rank $r+1$. Furthermore, Chen and Hedayat (1996) obtained one factor representation of the complementary design containing the maximum number of words of length three.

THEOREM 2 [Chen and Hedayat(1996)]. *Let $\overline{n} = 2^k - 1 - n = 2^r + q$ and $0 \le q < 2^r$ $(r < k)$. The maximum number of words of length three in the complementary design of size $\overline{n}$ is*

$$\frac{(2^r-1)(2^r-2)}{(2^2-1)(2^2-2)} + \binom{q+1}{2}. \tag{3.2}$$

One factor representation of the complementary design containing the maximum number of words of length three is

$$PG(r-1,2) \cup \{\mathbf{a}, \mathbf{a}+\mathbf{a}_1, ..., \mathbf{a}+\mathbf{a}_q\}, \tag{3.3}$$

where $PG(r-1,2)$ is an $(r-1)$-flat of $PG(k-1,2)$, $\mathbf{a}_i \in PG(r-1,2)$ for $i=1,...,q$, and $\mathbf{a} \in PG(k-1,2) \setminus PG(r-1,2)$.

REMARK. The set (3.3) is one factor representation of the complementary design containing the maximum number of words of length three, the structure is not unique [see Chen and Hedayat (1996) for a detailed discussion].

Since the factor representation of the complementary design $\overline{D}$ of size 2^r+q $(0 \le q < 2^r)$ containing the maximum number of words of length three must have the minimum rank $r+1$, the factor representation of the complementary design of minimum aberration 2^{n-m} design should be an $\overline{n}$-subset of an r-flat of $PG(k-1,2)$ which can be viewed as a copy of $PG(r,2)$ embedded in $PG(k-1,2)$. For simplicity, it is stated as an $\overline{n}$-subset of $PG(r,2)$. To search for minimum aberration 2^{n-m} design with its complementary design of size $\overline{n}=2^r+q$ $(0 \le q < 2^r)$, we only need to consider all $\overline{n}$-subsets of $PG(r,2)$. The maximum number of words of length three in the complementary design can be used to further narrow down possible candidates for the complementary design of the minimum aberration design. This provides an efficient way to identify minimum aberration designs in terms of their complementary designs.

4. Some families of minimum aberration 2^{n-m} designs. The classification rules discussed in Section 3 are very powerful for constructing minimum aberration designs whose complementary design sizes are relatively small. Applying Rules 1 and 2, Chen and Hedayat (1996) constructed all minimum aberration 2^{n-m} designs whose complementary design size is less than 16. Using the complete catalogue of two-level fractional factorial designs for 32 and 64 runs provided by Don X. Sun, a coauthor of Chen, Sun, and Wu (1993) and these classification rules, we obtain the following factor representations of the complementary designs of size $\overline{n}$ with the minimum rank $(16 \le \overline{n} \le 63)$. Deleting these factor representations from any $PG(k-1,2)$ yield subsets corresponding to minimum aberration designs. Hence we have constructed all minimum aberration 2^{n-m} designs whose complementary design size is less than 64.

$\overline{n}=16$

$B_{16}=\{1,2,3,4,5,12,13,23,123,14,24,124,34,134,234,1234\}$

$\overline{n}=17$

$B_{17}=\{1,2,3,4,5,12,13,23,123,14,24,124,34,134,234,1234,15\}$

$\overline{n}=18$

$B_{18}=\{1,2,3,4,5,12,13,23,123,14,24,124,34,134,234,1234,$
$15,25\}$

$\overline{n}=19$

$B_{19}=\{1,2,3,4,5,12,13,23,123,14,24,124,34,134,234,1234,$
$15,25,35\}$

$\overline{n} = 20$

$B_{20} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234,$
$15, 25, 35, 45\}$

$\overline{n} = 21$

$B_{21} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 24, 124, 34, 134, 15, 25, 125, 35,$
$135, 45, 145\}$

$\overline{n} = 22$

$B_{22} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 15, 25, 125,$
$35, 135, 45, 145\}$

$\overline{n} = 23$

$B_{23} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 15, 25, 125,$
$35, 135, 45, 235, 145\}$

$\overline{n} = 24$

$B_{24} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 15,$
$25, 125, 35, 135, 45, 235, 145, 245\}$

$\overline{n} = 25$

$B_{25} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 1234, 15,$
$25, 125, 35, 135, 45, 245, 345, 12345\}$

$\overline{n} = 26$

$B_{26} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 1234,$
$15, 25, 125, 35, 135, 45, 235, 145, 245, 345\}$

$\overline{n} = 27$

$B_{27} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 1234, 15,$
$25, 125, 35, 135, 45, 235, 145, 245, 345, 12345\}$

$\overline{n} = 28$

$B_{28} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 1234, 15,$
$25, 125, 35, 135, 45, 235, 145, 245, 345, 2345, 12345\}$

$\overline{n} = 29$

$B_{29} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 1234, 15,$
$25, 125, 35, 135, 45, 235, 145, 245, 345, 2345, 1345, 12345\}$

$\overline{n} = 30$

$B_{30} = \{1, 2, 3, 4, 5, 12, 13, 23, 123, 14, 234, 24, 124, 34, 134, 1234, 15,$
$25, 125, 35, 135, 45, 235, 145, 245, 345, 2345, 1245, 1345, 12345\}$

$\overline{n} = 31$

$B_{31} = PG(4, 2)$

$\overline{n} = 32$

$B_{32} = \{1, 12, 13, 23, 14, 24, 34, 1234, 15, 25, 35, 126, 136, 1235, 45,$
$1245, 1345, 2345, 146, 246, 346, 12346, 156, 256, 356, 12356, 456, 236,$
$12456, 13456, 23456, 123456\}$

$\overline{n} = 33$

$B_{33} = \{1, 2, 12, 13, 23, 14, 24, 34, 1234, 15, 25, 35, 126, 136, 1235, 45,$
$1245, 1345, 2345, 146, 246, 346, 12346, 156, 256, 356, 12356, 456, 236,$
$12456, 13456, 23456, 123456\}$

$\overline{n} = 34$

$$B_{34} = \{1, 2, 3, 12, 13, 23, 14, 24, 34, 1234, 15, 25, 35, 1235, 45, 1245, 1345, 2345, 126, 136, 236, 146, 246, 346, 12346, 156, 256, 356, 12356, 456, 12456, 13456, 23456, 123456\}$$

$\overline{n} = 35$

$$B_{35} = \{1, 2, 3, 4, 12, 13, 23, 14, 24, 34, 1234, 15, 25, 35, 1235, 45, 1245, 1345, 2345, 126, 136, 236, 146, 246, 346, 12346, 156, 256, 356, 12356, 456, 12456, 13456, 23456, 123456\}$$

$\overline{n} = 36$

$$B_{36} = \{1, 2, 3, 4, 5, 12, 13, 23, 14, 24, 34, 1234, 15, 25, 35, 1235, 45, 1345, 2345, 126, 136, 236, 146, 246, 346, 12346, 156, 256, 356, 12356, 456, 1245, 12456, 13456, 23456, 123456\}$$

$\overline{n} = 37$

$$B_{37} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 14, 24, 34, 1234, 15, 25, 35, 126, 1235, 1245, 1345, 2345, 146, 246, 346, 12346, 156, 256, 356, 12356, 456, 136, 236, 45, 12456, 13456, 23456, 123456\}$$

$\overline{n} = 38$

$$B_{38} = \{2, 3, 4, 5, 6, 12, 13, 23, 14, 24, 34, 134, 1234, 15, 25, 125, 35, 1245, 1345, 2345, 12345, 126, 136, 236, 146, 246, 346, 12346, 156, 1235, 45, 256, 356, 12356, 456, 12456, 13456, 123456\}$$

$\overline{n} = 39$

$$B_{39} = \{2, 3, 4, 5, 6, 12, 13, 23, 14, 24, 34, 134, 1234, 15, 25, 125, 35, 45, 1245, 1345, 2345, 126, 146, 246, 346, 12346, 156, 256, 356, 12356, 1235, 456, 136, 236, 12456, 3456, 13456, 23456, 123456\}$$

$\overline{n} = 40$

$$B_{40} = \{2, 3, 4, 5, 6, 12, 13, 23, 14, 24, 34, 134, 1234, 15, 25, 35, 135, 1235, 45, 145, 1245, 1345, 2345, 146, 246, 346, 12346, 156, 256, 356, 126, 12356, 456, 136, 236, 12456, 3456, 13456, 23456, 123456\}$$

$\overline{n} = 41$

$$B_{41} = \{2, 3, 4, 5, 6, 12, 13, 23, 14, 24, 34, 134, 1234, 15, 25, 125, 35, 45, 1235, 1245, 1345, 2345, 126, 136, 236, 146, 246, 346, 12346, 156, 145, 256, 356, 12356, 456, 2456, 12456, 3456, 13456, 23456, 123456\}$$

$\overline{n} = 42$

$$B_{42} = \{2, 3, 4, 5, 6, 12, 13, 23, 14, 24, 34, 134, 1234, 15, 25, 125, 35, 45, 145, 245, 1245, 1345, 2345, 126, 136, 236, 146, 246, 346, 12346, 156, 1235, 256, 356, 12356, 456, 2456, 12456, 3456, 13456, 23456, 123456\}$$

$\overline{n} = 43$

$$B_{43} = \{1, 2, 3, 4, 6, 12, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 15, 25, 135, 1235, 45, 145, 245, 1245, 345, 1345, 2345, 12345, 126, 136, 236, 256, 356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 35, 156, 123456\}$$

$\overline{n} = 44$

$B_{44} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 25, 125, 35, 135, 235, 1235, 45, 1245, 1345, 2345, 146, 246, 346, 256, 356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 15, 12346, 156, 123456\}$

$\overline{n} = 45$

$B_{45} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 125, 35, 135, 235, 1235, 45, 145, 1245, 1345, 2345, 146, 246, 346, 256, 356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 15, 25, 12346, 156, 123456\}$

$\overline{n} = 46$

$B_{46} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 25, 125, 35, 135, 235, 1235, 45, 145, 1245, 1345, 2345, 146, 246, 346, 256, 356, 2356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 1234, 15, 12346, 156, 123456\}$

$\overline{n} = 47$

$B_{47} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 25, 125, 35, 135, 235, 1235, 45, 145, 1245, 1345, 2345, 146, 246, 346, 12346, 156, 256, 356, 2356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 15, 2346, 23456, 123456\}$

$\overline{n} = 48$

$B_{48} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 124, 34, 134, 234, 1234, 15, 25, 125, 35, 1235, 45, 145, 245, 1245, 345, 1345, 236, 146, 346, 1346, 2346, 156, 12345, 136, 256, 356, 1356, 2356, 12356, 456, 1456, 24, 2456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 49$

$B_{49} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 124, 34, 134, 1234, 15, 25, 125, 35, 135, 1235, 45, 145, 1245, 345, 1345, 136, 236, 24, 146, 246, 346, 1346, 2346, 12346, 156, 256, 356, 1356, 2356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 50$

$B_{50} = \{1, 2, 3, 4, 5, 6, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 15, 25, 125, 35, 135, 235, 1235, 45, 245, 1245, 345, 1345, 2345, 126, 136, 236, 146, 246, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 2356, 12356, 456, 2456, 12456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 51$

$B_{51} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 34, 134, 234, 1234, 15, 25, 125, 35, 24, 235, 1235, 45, 145, 245, 1245, 345, 2345, 12345, 126, 136, 146, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 1356, 2356, 12356, 1456, 246, 2456, 12456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 52$

$B_{52} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 34, 134, 234, 1234, 15, 25, 125, 35, 235, 1235, 45, 145, 245, 1245, 345, 2345, 12345, 126, 136, 236, 24, 146, 246, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 1356, 2356, 12356, 1456, 2456, 12456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 53$

$B_{53} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 34, 134, 234, 1234, 15, 25, 35, 135, 24, 235, 1235, 45, 145, 245, 345, 1345, 2345, 12345, 126, 136, 236, 146, 246, 1246, 346, 1346, 2346, 12346, 256, 1256, 356, 1356, 2356, 156, 12356, 1456, 2456, 12456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 54$

$B_{54} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 14, 24, 124, 34, 134, 234, 1234, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 345, 1345, 2345, 12345, 126, 136, 236, 146, 1246, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 1356, 2356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 55$

$B_{55} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 15, 25, 125, 35, 135, 235, 1235, 45, 245, 1245, 345, 1345, 2345, 12345, 126, 136, 236, 1236, 146, 1246, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 1356, 2356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 123456\}$

$\overline{n} = 56$

$B_{56} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 15, 125, 35, 235, 1235, 45, 145, 245, 1245, 345, 1345, 2345, 12345, 126, 136, 1236, 1456, 146, 1246, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 2356, 12356, 456, 2456, 12456, 3456, 13456, 23456, 25, 236, 1356, 123456\}$

$\overline{n} = 57$

$B_{57} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 12345, 126, 136, 2345, 1456, 146, 1246, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 12356, 456, 2456, 12456, 3456, 13456, 23456, 15, 25, 236, 1236, 1356, 2356, 123456\}$

$\overline{n} = 58$

$B_{58} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 12345, 126, 136, 2345, 1456, 146, 246, 1246, 346, 1346, 2346, 12346, 156, 256, 1256, 356, 2356, 12356, 456, 2456, 12456, 3456, 13456, 23456, 15, 25, 236, 1236, 1356, 123456\}$

$\overline{n} = 59$

$B_{59} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 12345, 126, 136, 2345, 156, 1456, 146, 246, 1246, 346, 1346, 2346, 12346, 56, 256, 1256, 2356, 12356, 456, 2456, 12456, 3456, 13456, 23456, 15, 25, 236, 1236, 356, 1356, 123456\}$

$\overline{n} = 60$

$B_{60} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 12345, 126, 136, 1236, 46, 2345, 1456, 146, 246, 1246, 346, 1346, 2346, 12346, 156, 256, 1356, 2356, 12356, 456, 56, 2456, 12456, 3456, 13456, 23456, 15, 25, 236, 1256, 356, 123456\}$

$\overline{n} = 61$

$B_{61} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 46, 1345, 12345, 126, 236, 1236, 2345, 156, 146, 246, 1246, 346, 1346, 2346, 12346, 56, 256, 1356, 2356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 15, 25, 36, 136, 1256, 356, 123456\}$

$\overline{n} = 62$

$B_{62} = \{1, 2, 3, 4, 5, 6, 12, 13, 23, 123, 14, 24, 124, 34, 134, 234, 1234, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 46, 12345, 26, 236, 1236, 2345, 156, 146, 246, 1246, 346, 1346, 2346, 12346, 56, 256, 1356, 2356, 12356, 456, 1456, 2456, 12456, 3456, 13456, 23456, 15, 25, 126, 36, 136, 1256, 356, 123456\}$

$\overline{n} = 63$

$B_{63} = PG(5, 2).$

Acknowledgment. The authors thank the editors and two referees for their valuable comments and suggestions.

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