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Gaussianity and nonlinearity of foreign exchange rates

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Abstract: In this paper, we test on the Gaussianity and nonlinearity of the foreign exchange rate return series by the Gaussianity test due to Kariya, Tsay, Terui and Li (1994) and by the five well-known nonlinearity tests for stationary time series. The daily returns of the foreign exchange rate we consider exhibit the strong non-Gaussianity or nonlinearity, but a central limit effect is observed with observational frequency longer even under stringent test.

Key words: KTTL test for Gaussianity, exchange rate, nonlinearity tests.

AMS subject classification: Primary 62M15; secondary 62M10.

1 Introduction

In economic time series analysis, Gaussianity is often assumed for modeling and for constructing asymptotic tests such as unit root test, cointegration test, e.t.c. or tabulating their critical points. In particular, in finance it is quite common to develop models and theories under the assumption of Brownianity or Gaussianity and apply them to real data. For example, the Black-Scholes stock option theory assumes that log prices follow a Brownian process or equivalently that returns of a stock follow a Gaussian process and the so-called CAPM (capital assets pricing market) model assumes normality for returns at least in their original forms. Although these theories have been developed in a less restrictive way for normality, most of empirical modelling and many time series tests frequently used for financial series are constructed under Gaussianity. Therefore it is quite important to check whether our financial data is consistent with the assumption.

In this paper, applying the Gaussianity test proposed by Kariya, Tsay, Terui and Li (1994), shortened as the KTTL test[1] below, we test on the Gaussianity of the five foreign exchange rate returns and by considering the close relationship between the Gaussianity and linearity, we apply some nonlinearity tests to these data sets. The KTTL test of Gaussianity is a test that checks the consistency of the moment structure of a series with that of a Gaussian process up to an arbitrary order of the moments. In fact, the Gaussianity or equivalently normality is completely characterized by the moment structure (see Billingsley, 1986). It is also noted that the KTTL test is consistent with any stationary time series structure, and that such tests as the skewness test and the kurtosis test e.t.c. are not only partial tests, but also they assume *i.i.d.*-ness (independently-and-identically-distributed-ness). On the other hand, most of nonlinearity tests for stationary time series developed so far employ a linear process as a null hypothesis and set up, as alternatives, specific nonlinear models with additive noise. In fact, although there is a gap between non-Gaussianity and nonlinearity in stationary time series structure, induced time series from specific nonlinear model is almost always non-Gaussian as far as we assume the Gaussianity on the additive noise of nonlinear model, of which assumption is common in practical testing procedures. In this sense, nonlinearity tests constitute some kinds of the Gaussianity test.

We describe the KTTL test in Section 2. In Section 3, some nonlinearity testing methods for stationary time series are explained. Our empirical observations on the Gaussianity and nonlinearity of the return of foreign exchange rate are provided in Section 4. We observe the followings:

The daily series of the foreign exchange rate return show strong non-Gaussianity by the KTTL tests and nonlinearity by the five nonlinearity tests. The analysis of different observational intervals show the operation of a central limit effect in the sense that the p-values get larger as the observational period become longer for most of series. The KTTL omnibus test shows that the 6th order moment structures of monthly series are not consistent with those of a Gaussian process, although the moment structures up to the 4th order are not always incompatible with those of Gaussianity, where most of previous tests, including Jack-Bera test, etc., take only up to the 4th order moments into considerations.

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[1]FORTRAN code for the KTTL test is available on request via e-mail : terui@econ.tohoku.ac.jp.

2 The KTTL test for Gaussianity

In this section, we describe the KTTL test of Gaussianity. To do so, we first summarize the KTTL test of multinormality, from which the KTTL test of Gaussianity follows.

Let $\boldsymbol{Y} = (y_1, y_2, \ldots, y_n)'$ follow multivariate normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)'$ and covariance matrix $\Sigma = (\sigma_{ij})$, and denote the standardized variate of y_i as $z_i = (y_i - \mu_i)/\sqrt{\sigma_{ii}}$. Define $w_j^{(p)} = h^{(p)}(z_j)$ as the p-th order Hermite polynomial of z_j for $p = 1, \ldots, P$ and $j = 1, \ldots, n$. For example, $w_j^{(1)} = z_j, w^{(2)} = (z_j^2 - 1)/\sqrt{2}, w_j^{(3)} = (z_j^3 - 3z_j)/\sqrt{6}, w_j^{(4)} = (z_j^4 - 6z_j^2 + 3)/\sqrt{24}$ and so on.

Let

$$\psi_{ij}^{(p,q)} = \mathrm{Cov}(w_i^{(p)}, w_j^{(q)}). \tag{1}$$

Then, under the null hypothesis of multinormality of $\boldsymbol{Y}$, it holds that for every $i, j = 1, \ldots, n$ and $p, q = 1, \ldots, P$, $E(w_i^{(p)}) = 0$ and $\psi_{ij}^{(pq)} = \phi_{ij}^{(pq)}$ with

$$\phi_{ij}^{(p,q)} = \begin{cases} \phi_{ij}^p & p = q \\ 0 & p \neq q, \end{cases} \tag{2}$$

where $\phi_{ij} = \mathrm{Cov}(w_i^{(1)}, w_j^{(1)})$. This characterization is shown in Kendall and Stuart (1964, p.600). Granger and Newbold (1986, p.308) discussed its application to time series as a technique of instantaneous transformation.

Let $\boldsymbol{Y}_t, t = 1, \ldots, T$, be the vectors of observation of $\boldsymbol{Y}$ and define the sample version of z_i as $u_{it} = (y_{it} - \overline{y}_{i\cdot})/\sqrt{s_{ii}}$, where $\overline{y}_{i\cdot} = \frac{1}{T}\sum_{t=1}^T y_{it}$ and $s_{ij} = \frac{1}{T}\sum_{t=1}^T (y_{it} - \overline{y}_{i\cdot})(y_{jt} - \overline{y}_{j\cdot})$. And denote the corresponding transformed variates of u_{it} via Hermite polynomials as $v_{it}^{(p)} = h^{(p)}(u_{it})$ for $i = 1, \ldots, n$ and $p = 1, \ldots, P$. Under the assumption of the existence of the $2P$th order moments of $y_1, y_2, \ldots, y_n$, where P is the maximum order of the Hermite transformation under consideration and fixed in advance, two kinds of estimator for the covariance between $w_i^{(p)}$ and $w_j^{(q)}$ are used to construct an asymptotic test of multinormality. One is a consistent estimator only under the null hypothesis of multinormality, and the other is a consistent estimator under any arbitrary distribution. The former is given by

$$\hat{\phi}_{ij}^{(p,q)} = \begin{cases} (\hat{\phi}_{ij})^p & p = q \\ 0 & p \neq q, \end{cases} \tag{3}$$

where $\hat{\phi}_{ij}$ is the sample correlation coefficient between $z_i = w_i^{(1)}$ and $z_j = w_j^{(1)}$, and the the latter is the conventional sample covariance estimate of $\psi_{ij}^{(p,q)}$;

$$r_{ij}^{(pq)} = \frac{1}{T}\sum_{t=1}^{T} v_{it}^{(p)} v_{jt}^{(q)}. \tag{4}$$

The proposed test employs the difference between (3) and (4) as a test statistic.

Define a $Pn \times Pn$ symmetric matrix $\Re$ by

$$\Re = \begin{bmatrix} \boldsymbol{R}^{(1,1)} & \boldsymbol{R}^{(1,2)} & \dots & \boldsymbol{R}^{(1,P)} \\ \boldsymbol{R}^{(2,1)} & \boldsymbol{R}^{(2,2)} & \dots & \boldsymbol{R}^{(2,P)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \boldsymbol{R}^{(P,1)} & \boldsymbol{R}^{(P,2)} & \dots & \boldsymbol{R}^{(P,P)} \end{bmatrix}, \tag{5}$$

where the $\boldsymbol{R}^{(p,q)}$ is an $n \times n$ matrix whose (i,j) element is $r_{ij}^{(p,q)}$. Now compose $Pn(Pn+1)/2$ dimensional vector $\boldsymbol{r}_P$ as follows:

$$\boldsymbol{r}^{(p,p)} = Vech(\boldsymbol{R}^{(p,p)}): \quad n(n+1)/2 \times 1 \tag{6}$$

$$\boldsymbol{r}^{(p,q)} = Vec(\boldsymbol{R}^{(p,q)}): \quad n^2 \times 1 \tag{7}$$

$$\boldsymbol{r}_{P_1} = (\boldsymbol{r}^{(1,1)'}, \boldsymbol{r}^{(2,2)'}, \cdots, \boldsymbol{r}^{(P,P)'})': \quad f_1 = Pn(n+1)/2 \times 1 \tag{8}$$

$$\boldsymbol{r}_{P_2} = (\boldsymbol{r}^{(1,2)'}, \boldsymbol{r}^{(1,3)'}, \cdots; \boldsymbol{r}^{(2,3)'}, \cdots, \boldsymbol{r}^{(P-1,P)'})': \quad f_2 = P(P-1)n^2/2 \times 1 \tag{9}$$

$$\boldsymbol{r}_P = (\boldsymbol{r}'_{P_1}, \boldsymbol{r}'_{P_2})': \quad f = f_1 + f_2 = Pn(Pn+1)/2 \times 1 \tag{10}$$

Here for any $n \times n$ symmetric matrix $\boldsymbol{A} = (a_{ij})$,

$$Vech(\boldsymbol{A}) = (a_{11}, a_{12}, \cdots, a_{1n}; a_{22}, a_{23}, \cdots, a_{2n}; \cdots; a_{nn})$$

and for any $n \times n$ matrix $\boldsymbol{B} = (b_{ij})$,

$$Vec(\boldsymbol{B}) = (b_{11}, b_{12}, \cdots, b_{1n}; b_{21}, b_{22}, \cdots, b_{2n}; \cdots; b_{n1}, b_{n2}, \cdots, b_{nn}).$$

Correspondently, by replacing $r_{ij}^{(p,q)}$ by $\psi_{ij}^{(p,q)}$, we can derive an $f \times 1$ parameter vector $\boldsymbol{\psi}_P$ under the alternative hypothesis. Also under the null hypothesis, in the same way we define, by replacing $r_{ij}^{(p,q)}$ by $\hat{\phi}_{ij}^{(p,q)}$ and $\phi_{ij}^{(p,q)}$, we can define

$$\hat{\boldsymbol{\phi}}_P = (\hat{\boldsymbol{\phi}}'_{P1}, \boldsymbol{O}')' \quad \text{and} \quad \boldsymbol{\phi}_P = (\boldsymbol{\phi}'_{P1}, \boldsymbol{O}')', \tag{11}$$

which correspond to $\boldsymbol{r}_P$ and $\boldsymbol{\psi}_P$ respectively. Then, assuming the $2P$th order moment of y_{it}, $\boldsymbol{r}_P$ follows the asymptotic multivariate normal distribution,

$$\sqrt{T}(\boldsymbol{r}_P - \boldsymbol{\psi}_P) \sim N_f(\boldsymbol{O}, \boldsymbol{\Lambda}) \quad (\boldsymbol{\Lambda} \text{ is the function of } \boldsymbol{\psi}_P). \tag{12}$$

Similarly it follows from (3) under the null hypothesis that

$$\sqrt{T}(\hat{\boldsymbol{\phi}}_P - \boldsymbol{\phi}_P) \sim N_f(\boldsymbol{O}, \boldsymbol{\Lambda}^\star) \quad (\boldsymbol{\Lambda}^\star \text{ is a function of } \phi_{ij}^{(1,1)}). \tag{13}$$

The KTTL test detects the equivalence between $\boldsymbol{\psi}_P$ and $\boldsymbol{\phi}_P$ under the null hypothesis based on the difference of $\boldsymbol{r}_P$ and $\hat{\boldsymbol{\phi}}_P = (\hat{\boldsymbol{\phi}}'_{P_1}, \boldsymbol{O}')$. As we have $\{n(n+1)/2\}$ equivalent relationships; $\hat{\phi}_{ij}^{(1,1)} \equiv r_{ij}^{(1,1)}$, the test is $g = \{f - n(n+1)/2\}$ dimensional vector

$$\boldsymbol{C}_{Po} = \boldsymbol{C}_{Po}(\boldsymbol{r}_P) = (\boldsymbol{C}'_{P1}, \boldsymbol{C}'_{P2})', \tag{14}$$

where

$$\boldsymbol{C}_{P1} = (\boldsymbol{c}^{(2,2)\prime}, \cdots, \boldsymbol{c}^{(P,P)\prime})' \tag{15}$$

$$\boldsymbol{C}_{P2} = (\boldsymbol{c}^{(1,2)\prime}, \cdots, \boldsymbol{c}^{(1,P)\prime}; \cdots; \boldsymbol{c}^{(P-1,P)\prime})' \quad \text{and} \tag{16}$$

$$\boldsymbol{c}^{(p,q)} = \begin{cases} \boldsymbol{r}^{(p,p)} - \hat{\boldsymbol{\phi}}^{(p,p)} & p = q \\ \boldsymbol{r}^{(p,q)}, & p \neq q. \end{cases} \tag{17}$$

In fact, $\boldsymbol{C}_{Po}$ is expressed as

$$\boldsymbol{C}_{Po} = [\boldsymbol{O}\ \boldsymbol{I}](\boldsymbol{r}_P - \hat{\boldsymbol{\phi}}_P), \tag{18}$$

where $\boldsymbol{O}$ is the $(f - n(n+1)n/2) \times 1$ zero matrix, and $\boldsymbol{I}$ is the $(f - n(n+1)n/2)$ dimensional identity matrix. Under the null hypothesis, where $\boldsymbol{\psi}_P = \boldsymbol{\phi}_P$, it holds from (12) and (13) that

$$\sqrt{T}\boldsymbol{C}_{Po} \sim N_g(\boldsymbol{O}, \boldsymbol{J}(\boldsymbol{\phi}_P)\boldsymbol{\Lambda}\boldsymbol{J}(\boldsymbol{\phi}_P)'), \tag{19}$$

where the $(f - n(n+1)/2) \times f$ matrix $\boldsymbol{J}(\boldsymbol{\phi}_P)$ is

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{\phi}_P) &= \left.\frac{\partial \boldsymbol{C}_{Po}}{\partial \boldsymbol{r}_P}\right|_{\boldsymbol{r}_P = \boldsymbol{\phi}_P} \\ &= \left.\left(\frac{\partial \boldsymbol{c}^{(2,2)}}{\partial \boldsymbol{r}_P}, \ldots, \frac{\partial \boldsymbol{c}^{(P,P)}}{\partial \boldsymbol{r}_P}; \frac{\partial \boldsymbol{c}^{(1,2)}}{\partial \boldsymbol{r}_P}, \ldots, \frac{\partial \boldsymbol{c}^{(P-1,P)}}{\partial \boldsymbol{r}_P}\right)\right|_{\boldsymbol{r}_P = \boldsymbol{\phi}_P} \end{aligned} \tag{20}$$

and the matrix differential is given by

$$\frac{\partial \boldsymbol{c}^{(p,q)}}{\partial \boldsymbol{r}_P} = \left(\frac{\partial c_{ij}^{(p,q)}}{\partial r_{kl}^{(a,b)}} \right). \tag{21}$$

The KTTL test statistic for multinormality is a Wald type chi- square test defined by

$$W_o = T\boldsymbol{C}'_{Po} \left[\hat{\boldsymbol{J}} \hat{\boldsymbol{\Lambda}} \hat{\boldsymbol{J}}' \right]^{-1} \boldsymbol{C}_{Po}, \tag{22}$$

where $\hat{\boldsymbol{J}} = J(\hat{\boldsymbol{\phi}}_P)$ and $\hat{\boldsymbol{\Lambda}} = (\hat{\lambda}_{ij,kl}^{(pq,ab)})$ with

$$\hat{\lambda}_{ij,kl}^{(pq,ab)} = \frac{1}{T} \sum_{i=1}^{T} (v_{it}^{(p)} v_{jt}^{(q)} - r_{ij}^{(pq)})(v_{kt}^{(a)} v_{lt}^{(b)} - r_{kl}^{(ab)}). \tag{23}$$

The KTTL test of Gaussianity for univariate series is a modification of the above test. Let $\{x_t\}$ be a univariate stationary process with $E(x_t) = \mu$ and $\mathrm{Cov}(x_t, x_{t-k}) = \gamma_k$ and assume the mixing condition $\sum_{k=-\infty}^{\infty} |k||\gamma_k| < \infty$. Then the following two methods for constructing a test are proposed.

[I] Overlapping method

Set, for $i = 1, \ldots, n$,

$$y_{it} = x_{t-i+1}, \tag{24}$$

and corresponding to *i.i.d.* case, we define $z_{it} = (y_t - \mu)/\sqrt{\gamma_0} = (x_{t-i+1} - \mu)/\sqrt{\gamma_0}$.

[II] Non-overlapping method

For a given positive integer n and the realizations $\{x_1, \cdots, x_N\}$, set n-dimensional non-overlapping random vectors $\{y_t = (y_{1t}, \cdots, y_{nt})'\}$ with

$$y_{it} = x_{n(t-1)+i}, i = 1, \cdots, n; t = 1, \cdots, T, \tag{25}$$

where $T = [N/n]$ is the integer part of N/n. Then we define $z_{it} = (x_{n(t-1)+1} - \mu)/\sqrt{\gamma_0}$.

For both [I] and [II], it follows that $\mathrm{Cov}(z_{it}, z_{jt}) = \gamma_{i-j}/\gamma_0 = \rho_{i-j} = \phi_{ij}$. Defining $w_{it}^{(p)} = h^{(p)}(z_{it})$, just like *i.i.d.* case, under the Gaussianity of x_t, it holds that $\psi_{ij}^{(p,q)} = \phi_{ij}^{(p,q)}$ with

$$\phi_{ij}^{(pq)} = \begin{cases} (\phi_{ij})^p & p = q \\ 0 & p \neq q. \end{cases} \tag{26}$$

The sample variates of z_{it} and $w_{it}^{(p)}$ are defined as $u_{it} = (y_{it} - \overline{y}_i)/\sqrt{s_{ii}}$, $v_{it}^{(p)} = h^{(p)}(u_{it})$ for $i = 1, \ldots, n$ and $p = 1, \ldots, P$, where $\overline{y}_i = \frac{1}{T} \sum_{t=1}^{T} y_{it}$ and

$s_{ij} = \frac{1}{T}\sum_{t=max(i,j)}^{T}(y_{it} - \overline{y}_i)(y_{jt} - \overline{y}_j)$. For time series data, the estimates corresponding to (3) and (4) should be respectively modified as

$$r_{ij}^{(pq)} = \frac{1}{T}\sum_{t=max(i,j)}^{T} v_{it}^{(p)} v_{jt}^{(q)} \tag{27}$$

and

$$\hat{\phi}_{ij}^{(pq)} = \begin{cases} \hat{\rho}_{i-j}^{p} & p = q \\ 0 & p \neq q \end{cases} \tag{28}$$

for $i, j = 1, \ldots, n : p, q = 1, \ldots, P$. Under the Gaussianity of $\{x_t\}$ with the mixing condition above, Keenan (1983) proved the asymptotic normality that $\sqrt{T}(\boldsymbol{r}_P - \boldsymbol{\phi}_P) \to N(\boldsymbol{O}, \boldsymbol{J\Lambda J'})$. Therefore, exactly following the argument of the *i.i.d.* case, we obtain the asymptotically χ^2 test statistic (22) with d.f. $\{f - n(n+1)/2\}$ for testing the Gaussianity of a stationary time series.

The test in (22) test is an omnibus test which detects departures from Gaussianity, and it can be decomposed into two parts. The first part tests departures from the even moment structure when $p = q$ and the second part tests departures from the odd moment structure when $p \neq q$. It is useful to have a separate test for each part when we are interested in the symmetry and the tail behavior of the underlying distribution separately. Corresponding to the dimensions of $\boldsymbol{C}_{P1}$ and $\boldsymbol{C}_{P2}$, the appropriate decomposition of $\boldsymbol{\Lambda}$ and $\boldsymbol{J}$ produces the following test statistics:

$$W_1 = T\boldsymbol{C}'_{P1}\left[\hat{\boldsymbol{J}}_{11}\hat{\boldsymbol{\Lambda}}_{11}\hat{\boldsymbol{J}}'_{11}\right]^{-1}\boldsymbol{C}_{P1}, \text{ and} \tag{29}$$

$$W_2 = T\boldsymbol{C}'_{P2}\left[\hat{\boldsymbol{\Lambda}}_{22}\right]^{-1}\boldsymbol{C}_{P2}. \tag{30}$$

Under the null hypothesis of Gaussianity, the asymptotic distributions of these two test statistics are χ^2 with degrees of freedom $f_1 - n(n+1)/2$ and f_2 respectively. In fact, setting $\boldsymbol{A} \equiv [\hat{\boldsymbol{J}}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{J}}']^{-1}$, and decomposing $\boldsymbol{A}$ appropriately according to the dimension of $\boldsymbol{C}_{P1}$ and $\boldsymbol{C}_{P2}$, then we have the following relationship among W_o, W_1 and W_2;

$$\begin{aligned} W_o &= T\boldsymbol{C}'_{Po}\left[\hat{\boldsymbol{J}}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{J}}'\right]^{-1}\boldsymbol{C}_{Po} \\ &\equiv T(\boldsymbol{C}_{P1},' \boldsymbol{C}'_{P2})\begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12} & \boldsymbol{A}_{22} \end{pmatrix}(\boldsymbol{C}_{P1}, \boldsymbol{C}_{P2}) \\ &= T\boldsymbol{C}'_{P1}\boldsymbol{A}_{11}\boldsymbol{C}_{P1} + T\boldsymbol{C}'_{P2}\boldsymbol{A}_{22}\boldsymbol{C}_{P2} + 2T\boldsymbol{C}'_{P1}\boldsymbol{A}_{12}\boldsymbol{C}_{P2} \\ &= W_1 + W_2 + 2\boldsymbol{C}_{12}, \end{aligned} \tag{31}$$

where $\boldsymbol{C}_{12} = T\boldsymbol{C}'_{P1}\boldsymbol{A}_{12}\boldsymbol{C}_{P2}$. Now considering here $\frac{\boldsymbol{C}'_{P1}\boldsymbol{A}_{12}\boldsymbol{C}_{P2}}{\sqrt{W_1}\sqrt{W_2}}$ may be regarded as a kind of correlation between $\boldsymbol{C}_{P1}$ and $\boldsymbol{C}_{P2}$. In fact, in the case where both $\boldsymbol{C}_{P1}$ and $\boldsymbol{C}_{P2}$ are one dimensional vector, $C_{12}/(\sqrt{W_1}\sqrt{W_2})$ is shown to be equivalent to negative correlation between C_1 and C_2.

3 Some nonlinearity tests

Let

$$X_t = h(X_{t-1}, X_{t-2}, \cdots, X_{t-p}) + e_t \tag{32}$$

be an autoregressive nonlinear time series model, where $\{e_t\}$ is *i.i.d.* with mean zero. If we assume the innovation e_t as Gaussian, nonlinearity test is equivalent to Gaussianity test. Here we use the five well known nonlinearity tests:

(i) Ori-F test by Tsay (1986)
(ii) Aug-F test by Luukkonenn, Saikkonen and Teräsvirta (1988)
(iii) CUSUM test by Petruccelli and Davis (1986)
(iv) TAR-F test by Tsay (1989)
(v) New-F test by Tsay (1988)

All of these tests set up, as a null hypothesis, a linear process. Based on the Volterra expansion of (32) around $\boldsymbol{O} = (0, 0, \cdots)'$

$$x_t = \mu + \sum_{u=1}^{\infty} \psi_u x_{t-u} + \sum_{u,v=1}^{\infty} \psi_{uv} x_{t-u} x_{t-v} + \sum_{u,v,w=1}^{\infty} \psi_{uvw} x_{t-u} x_{t-v} x_{t-w} + \cdots + e_t, \tag{33}$$

where

$$\begin{aligned} \mu &= h(\boldsymbol{O}), \phi_u = \left.\frac{\partial h}{\partial x_{t-u}}\right|_O, \phi_{uv} = \left.\frac{\partial^2 h}{\partial x_{t-u}\partial x_{t-v}}\right|_O, \phi_{uvw} \\ &= \left.\frac{\partial^3 h}{\partial x_{t-u}\partial x_{t-v}\partial x_{t-w}}\right|_O, e.t.c., \end{aligned}$$

the Ori-F and Aug-F tests detect against the nonlinearity of the second and third order polynomials respectively. The CUSUM, TAR-F and New-F tests assume the threshold type nonlinear alternatives;

$$x_t = \beta_0^{(j)} + \sum_{i=1}^{p} \beta_i^{(j)} x_{t-i} + a_t^{(j)} \quad (j = 1, 2), \tag{34}$$

where $\{a_t^{(j)}\}$ is the innovation of mean zero and variance σ_j^2. The New-F test covers the most extensive alternatives of nonlinearity, including ExpAR proposed by Haggan and Ozaki (1981)

$$\begin{aligned} X_t &= \{\phi_1 + \pi_1 \exp(-\gamma X_{t-1}^2)\} X_{t-1} + \{\phi_2 + \pi_2 \exp(-\gamma X_{t-1}^2)\} X_{t-2} + \\ &\qquad \cdots + \{\phi_p + \pi_p \exp(-\gamma X_{t-1}^2)\} X_{t-p} + e_t \end{aligned} \tag{35}$$

and bilinear models by Granger and Anderson (1978), Subba Rao and Gabr (1984) and others

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = \sum_{j=0}^{r} c_j e_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{k} b_{ij} X_{t-i} e_{t-j}. \tag{36}$$

The detailed procedures and distributional properties regarding these tests are found in Granger and Teräsvirta (1993).

In order to implement these tests, the order p of autoregression for all the tests and the value of delay parameter d for the tests (iii), (iv), and (v) need to be specified. We set the maximum of p as 10 and let d run from 1 to 10. Each of nonlinearity tests with different set of (p, d) brings out different results. We employ the most significant result of the test among all the combinations of (p, d).

4 Foreign exchange rate

Since the work of Westerfield (1977), a battery of research about the foreign exchange rate has been done. For a survey, see Levich (1985), Isard (1988), Mills (1993) and Campbell, Lo and McKilay (1997). Many of those works, for example, So (1987), Wolff (1987), Enders (1988) and etc., employ the Gaussianity of the process as a basic assumption. The careful investigations of this assumption should be needed in advance.

In this section, we investigate the Gaussianity and nonlinearity of five foreign exchange rate return series against US dollar: FRF (French Franc), JPY (Japanese Yen), CHF (Swiss Franc), GBR (English Starling), DEM (German Mark). We deal with three kinds of data; daily (1992.1.1.- 1993.12. 31. (552 samples)), weekly (1984.1.2. - 1993.12.27. (521 samples: Mondays)) and monthly (1983.10.31. - 1994.10.31. (132 samples)). Over the observational period, we assume a homogeneity between these data.

We define a exchange rate return x_t at time t as $x_t = log\ R_t - log\ R_{t-1}$ where R_t is a exchange rate at t.

4.1 Gaussianity

In this article, we use the overlapping method of the KTTL test because macroeconomic time series usually do not have enough numbers of the sample. Denote the KTTL test with $P = 2$ and $n = 1$ by O-21 and so on.

Applying the KTTL test to the data, we move $P = 2, 3$ and $n = 1, \cdots, 4$ and the p-values of the tests O-21 through O-32 are tabulated in Table 1. The other tests with greater than $P = 3$ and $n = 2$ (that is, O-33 and O-34) reject the Gaussianity very strongly. In the following, we enumerate the empirical findings from the results.

1) **Daily series**

a) All the tests applied to other than CHF series reject the Gaussianity. The marginal moment test W_1 rejects the Gaussianity with 5% significant level for all the series. In fact, the maximum value of the p-values is 0.03955 for CHF.

b) In O-21 and O-22 tests, the results of the W_2 test are not significant for many cases.

c) The W_0 test can not reject the Gaussianity of CHF even when P and n are large.

2) **Weekly series**

a) The Gaussianity of JPY is rejected more strongly than other series.

b) Compared with daily series, the number of significant series increases.

3) **Monthly series**

a) In case of $P = 2$, not only the omnibus test W_0, but also the marginal tests W_1 and W_2 can not reject the Gaussianity for all the series.

b) Compared with daily and weekly series, many p-values of all the tests, W_o, W_1 and W_2, are larger.

c) The W_1 and W_2 tests reject the Gaussianity of GBR strongly, however the W_o test can not necessarily be inconsistent with the Gaussianity (O-22, O-23 and O-31).

d) The W_o test rejects the Gaussianity more strongly when we set $P = 3$. This means that the 6th order moment structures of data sets are not consistent with those of a Gaussian process, although the moment structures up to the 4th order are not always incompatible with those of Gaussianity.

e) In case of $P = 3$, we have some cases where the omnibus test W_0 reject the null hypothesis more strongly than the marginal tests W_1 and W_2 (JPY, CHF, DEM of O-31 and JPY, CHF of O- 32). This means that $2C_{12}$ of (21) takes the large positive values for those cases.

As a whole, we observe that, given P, the hypothesis of Gaussianity tends to be rejected more strongly, as n gets larger and in case of $P = 3$, and the Gaussianity is completely rejected for almost returns, especially in case of $n \geq 2$. Further we observe that a central limit effect is working on weekly series in the sense that the p-values get larger as the observational period become longer.

Table 1: The KTTL Tests: Foreign Exchange Rate Returns.

Variable	Daily(552 samples)			Weekly(521 samples)			Monthly(132 samples)		
	$P(W_o)$	$P(W_1)$	$P(W_2)$	$P(W_o)$	$P(W_1)$	$P(W_2)$	$P(W_o)$	$P(W_1)$	$P(W_2)$
O-21(D.F.: $W_o:2$, $W_1:1$, $W_2:1$)									
FRF	0.01311	0.00622	0.57667	0.11414	0.02135	0.35430	0.47818	0.28592	0.09538
JPY	0.01405	0.00244	0.35201	0.00674	0.00112	0.05900	0.66199	0.69024	0.49823
CHF	0.14530	0.03956	0.84395	0.42304	0.23533	0.72896	0.72221	0.79985	0.37821
GBR	0.00355	0.00006	0.15180	0.26114	0.00262	0.03956	0.26506	0.11417	0.77253
DEM	0.01549	0.00567	0.73782	0.22792	0.07984	0.74623	0.55148	0.53422	0.16216
O-22(D.F.:$W_o:7$, $W_1:3$, $W_2:4$)									
FRF	0.00076	0.00020	0.11010	0.05483	0.00034	0.45788	0.75493	0.41573	0.06835
JPY	0.01096	0.00000	0.01128	0.00305	0.00000	0.06881	0.87798	0.88747	0.81214
CHF	0.20441	0.01291	0.36325	0.63944	0.16946	0.43741	0.94473	0.75097	0.73328
GBR	0.00083	0.00000	0.00000	0.35538	0.00002	0.02632	0.53963	0.00000	0.00006
DEM	0.02160	0.00128	0.81563	0.07231	0.00989	0.96215	0.78226	0.78435	0.17848
O-23(D.F.:$W_o:15$, $W_1:6$, $W_2:9$)									
FRF	0.00037	0.00000	0.00090	0.02444	0.00000	0.29071	0.84104	0.22524	0.00350
JPY	0.01999	0.00000	0.00439	0.00126	0.00000	0.08059	0.85650	0.25579	0.83568
CHF	0.28879	0.00219	0.09646	0.71675	0.16591	0.52572	0.98131	0.78192	0.81381
GBR	0.00019	0.00000	0.00000	0.47406	0.00000	0.00168	0.73477	0.00000	0.00000
DEM	0.03287	0.00014	0.69558	0.04213	0.00118	0.99706	0.81120	0.89435	0.08016
O-24(D.F.:$W_o:26$, $W_1:10$, $W_2:16$)									
FRF	0.00026	0.00000	0.00000	0.00261	0.00000	0.22997	0.72710	0.07933	0.00000
JPY	0.00819	0.00000	0.00005	0.00016	0.00000	0.01655	0.87774	0.09242	0.75604
CHF	0.43501	0.00029	0.03210	0.48250	0.01646	0.60762	0.87173	0.83366	0.48292
GBR	0.00002	0.00000	0.00000	0.48250	0.01646	0.60762	0.17422	0.00000	0.00000
DEM	0.02310	0.00000	0.20145	0.01875	0.00006	0.99967	0.67201	0.87080	0.01459
O-31(D.F.:$W_o:5$, $W_1:2$, $W_2:3$)									
FRF	0.00001	0.00000	0.00000	0.30700	0.00001	0.00636	0.0 0000	0.33776	0.00000
JPY	0.00000	0.00000	0.00000	0.00002	0.00000	0.00000	0.00 012	0.07334	0.03814
CHF	0.09266	0.00000	0.00000	0.86862	0.04989	0.34839	0.0 6818	0.18915	0.38700
GBR	0.00004	0.00000	0.00000	0.00573	0.00000	0.00000	0.2 5814	0.00000	0.00001
DEM	0.00973	0.00000	0.00023	0.35201	0.00266	0.31204	0.0 0000	0.61632	0.00000
O-32(D.F.:$W_o:18$, $W_1:6$, $W_2:12$)									
FRF	0.00000	0.00000	0.00000	0.22459	0.00000	0.00239	0.00000	0.01589	0.00000
JPY	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00266	0.00000
CHF	0.13266	0.00000	0.00000	0.45164	0.00005	0.03728	0.00217	0.15251	0.13535
GBR	0.00000	0.00000	0.00000	0.00009	0.00000	0.00000	0.00000	0.00000	0.00000
DEM	0.00429	0.00000	0.00000	0.51822	0.00008	0.52567	0.00000	0.18092	0.00000

$P(W_i)$ means the p-value of the test $W_i, i = 0, 1, 2$.

4.2 Nonlinearity

Table 2 shows the results of applying the five nonlinearity tests explained in Section 3 to the foreign exchange returns. The CUSUM test produces very different results from other tests. In Terui and Kariya (1996), it is shown that, applying these tests to 214 Japanese stock returns, the empirical distributions of the p-values of the CUSUM test have a very different shape of distribution function from those of four other tests. Further the simulation studies by Tsay (1988, 1989) indicate the low power of the CUSUM test and we can see this observation as an empirical evidence of it. Therefore we leave out the results of the CUSUM test.

From the table, we observe;

1) **Daily series**

a) FRF and GBR are non-Gaussian judging from all the nonlinearity tests.

b) JPY is incompatible with a Gaussian process for all the tests. It is possible for JPY to have a nonlinearity other than what the five tests assume as alternative, because the KTTL test also supports the non Gaussianity.

c) For CHF, the Ori-F and Aug-F tests support the Gaussianity and the TAR-F and New-F tests suggests the non-Gaussianity. Therefore CHF can have a threshold type nonlinearity.

d) DEM is compatible with a Gaussian process by the Aug-F test and inconsistent by the TAR-F and New-F tests. The threshold type nonlinearity might exit for DEM.

2) **Weekly series**

a) FRF, CHF and DEM could have threshold type nonlinearity because the TAR-F and New-F tests reject the null hypothesis of linearity.

b) The non-Gaussianity is suggested for JPY by all the tests.

c) GBR is compatible with a Gaussian process for all the tests.

3) **Monthly series**

a) There are no returns with the p-values greater than 1%, except for the TAR-F test applied to JPY.

Table 2: Nonlinearity Tests: Foreign Exchange Rate Returns.

Variable	Ori-F	Aug-F	TAR-F	CUSUM	New-F
Daily					
FRF	0.00000	0.00000	0.00110	0.10151	0.00053
JPY	0.14047	0.24706	0.25270	0.74493	0.05492
CHF	0.46322	0.68311	0.02595	0.04235	0.02791
GBR	0.00012	0.00003	0.00262	0.00920	0.00241
DEM	0.03721	0.08136	0.00511	0.02198	0.02273
Weekly					
FRF	0.08246	0.06449	0.00341	0.18164	0.05208
JPY	0.01152	0.01331	0.00090	0.11705	0.00000
CHF	0.46697	0.68680	0.02721	0.04223	0.01986
GBR	0.06608	0.14518	0.11549	0.33748	0.07851
DEM	0.06415	0.05298	0.00125	0.11915	0.00035
Monthly					
FRF	0.07120	0.04798	0.11767	0.04712	0.10081
JPY	0.04101	0.01300	0.00159	0.12351	0.03721
CHF	0.03099	0.07967	0.10196	0.15927	0.04794
GBR	0.17833	0.22842	0.04982	0.13293	0.02420
DEM	0.10053	0.12918	0.15184	0.06969	0.11015

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