Statistics, Probability and Game Theory
IMS Lecture Notes – Monograph Series (1996) Volume 30

BLACKWELL GAMES

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Abstract

Blackwell games are infinite games of imperfect information. The two players simultaneously make their moves and are then informed of each other's moves. Payoff is determined by a Borel measurable function f on the set of possible resulting sequences of moves. A standard result in Game Theory is that finite games of this type are determined. Blackwell proved that infinite games are determined, but only for the cases where the payoff function is the indicator function of an open or G_δ set [2, 3]. For games of perfect information, determinacy has been proven for games of arbitrary Borel complexity [6, 7, 8]. In this paper I prove the determinacy of Blackwell games over a $G_{\delta\sigma}$ set, in a manner similar to Davis' proof of determinacy of games of $G_{\delta\sigma}$ complexity of perfect information [5].

There is also extensive literature about the consequences of assuming AD, the axiom that *all* such games of perfect information are determined [9, 11]. In the final section of this paper I formulate an analogous axiom for games of imperfect information, and explore some of the consequences of this axiom.

1 Introduction

Imagine two players playing a game of Blind Chess. The only board they have is in their minds, and they make their moves merely by announcing them. Someone who doesn't know the rules would find a game like this difficult to follow. If that someone was of a literal bend, he might describe it like this:

> "There were two players, playing against each other. The first player said something, and I was told it was her move, and that she had made the move by saying it. The other player thought for a while, and then announced his own move. Then the first player made a move again, then the second player, and so forth. The moves always sounded similar, something like 'pawn from ee-four to ee-five'. So I think they couldn't just say anything,

[1]My thanks go to Michiel van Lambalgen and Tonny Hurkens, for their guidance, ideas and meticulous proofreading. This research was partially funded by the Netherlands Organization for Scientific Research (NWO), under grant PGS 22-262.

> but had to select their moves from only a few possible options. And suddenly they stopped, and shook hands, and I was told that the first player had won, apparently because of the moves she and her opponent had played."

If no one gave the poor fellow a copy of the rules of Chess, the way a sequence of moves determines which player wins would probably seem quite arbitrary. And our hypothetical observer might be quite impressed that apparently chess-players are able to memorize this long list of what the result is of each possible sequence of moves.

Of course, the game of Chess is not really that arbitrary, and those of us who play chess only need to know a few simple rules to figure out which player has won. But we can use this concept of a game to construct a quite general mathematical game $\Gamma_{p.i.}(f)$.

> Let there be given two finite sets X and Y, an integer n, and a function f assigning to each sequence w of length n of pairs $(x_i, y_i) \in X \times Y$, a *payoff* $f(w) \in \mathbb{R}$. Two players are playing against each other. Each player, in turn, makes a move by selecting an element $x_1 \in X$ or $y_1 \in Y$, respectively, and announcing his or her selection. Then they each in turn make a second move, and a third move, and continue making moves until n rounds have been played. This generates a sequence w of length n of pairs $(x_i, y_i) \in X \times Y$. Then they stop, and player II pays player I the amount $f(w)$.

With the right choices for X, Y, n and f, the game $\Gamma_{p.i.}(f)$ can 'emulate' the game of Chess. For if we let X and Y be the set of all possible chess moves, and $n = 6350^2$, then a sequence w corresponds to a finished game of chess. We now set $f(w) = 1$, $f(w) = 0$, or $f(w) = \frac{1}{2}$, depending on whether the corresponding game is a win for White, a win for Black, or a draw.[3] And voilà, we have our Chess emulator.

But Chess is not the only game that can be 'emulated' in this manner. The same can be done with Noughts-and-Crosses, Connect-Four, Go and Checkers. In general, the games $\Gamma_{p.i.}(f)$ can emulate any game G that has the following properties:

- There are two players.
- There is no element of chance

[2]The fifty-move rule is a rule in chess stating that if no piece has been captured and no pawn has been moved for fifty turns, the game is a draw. Under the fifty-move rule, a game of chess can last a maximum of 6350 moves.

[3]If w does not correspond to a *legal* chess game, we count it as a win for White if the first illegal move is made by Black, and vice versa.

- Moves are essentially made by selecting them and announcing them.
- There is no hidden information: a player knows all the moves made so far when making her current move, and there is nothing going on simultaneously either (*Perfect Information*).
- If one player loses (a certain amount) the other player wins (that same amount) (*Zero-Sum*).
- The game can last no more than a certain number of rounds (*Finite Duration*).
- There is a maximum number of alternatives each player can select from (*Finite Choice-of-Moves*).

Any results for the games $\Gamma_{p.i.}(f)$ apply to all the games with these properties.

David Blackwell described the concept of a *strategy* as [4]:

> Imagine that you are to play the White pieces in a single game of chess, and that you discover you are unable to be present for the occasion. There is available a deputy, who will represent you on the occasion, and who will carry out your instructions exactly, but who is absolutely unable to make any decisions of his own volition. Thus, in order to guarantee that your deputy will be able to conduct the White pieces throughout the game, your instructions to him must envisage every possible circumstance in which he may be required to move, and must specify, for each such circumstance, what his choice is to be. Any such complete set of instructions constitutes what we shall call a *strategy*.

Thus, a strategy for a given player in a given game consists of a specification, for each position in which he or she is required to make a move, of the particular move to make in that position. In turn, a position can be specified by the moves made to get to that position. If we apply this to the game $\Gamma_{p.i.}(f)$, a strategy becomes a function from the set of sequences of length $< n$ of pairs $(x_i, y_i) \in X \times Y$, to the set of possible selections X, Y respectively.

Given strategies for each of the players, the outcome of the game is determined: each move follows from the current position and the strategy of the player whose turn it is to move, and determines the next position. So, the totality of all the decisions to be made can be described by a single decision - the choice of a strategy. This is the *normal form* of a game: the two players independently make a single move, which consists of selecting a strategy, and then payoff is calculated and made.

Of course, there are good strategies and bad strategies. The *value* of a strategy for a given player is the result of that strategy against the best counterstrategy. The *value* of a game for a given player is the best result that that player can guarantee, i.e. the value of that player's best strategy. A game is called *determined* if its value is the same for both players. That value is the result that will occur if both players are playing perfectly.[4]

Victor Allis [1] recently demonstrated that in a game of Connect-Four, the first player can win, i.e. has a strategy that wins against any counterstrategy. And countless persons throughout the ages have independently discovered that in the game of Noughts-and-Crosses, both players can force a draw. These are both examples of determinacy. It can be shown (using induction) that any game $\Gamma(f)$ as defined above is determined, and hence any game with all of the properties mentioned above is determined. In the case of Go, Chess, and Checkers, this means that either one of the players has a winning strategy, or both players have a drawing strategy.

Now consider the game of Scissors-Paper-Stone. In this game, the two players simultaneously 'throw' one of three symbols: 'Stone' (hand balled in a fist), 'Paper' (hand flat with the palm down) or 'Scissors' (middle and forefinger spread, pointing forwards). If both players throw the same symbols, the result is a draw; otherwise, Paper beats Stone, Stone beats Scissors, and Scissors beats Paper (the reason being that "Paper wraps Stone, Stone blunts Scissors, and Scissors cut Paper"). In this game, the players do not make moves in turn, but simultaneously. In other words, both players make moves, and neither player knows what move the other is making. This is an example of a game of *Imperfect Information.*

The strategy 'Throw Stone' is a losing strategy, because it loses against the counterstrategy 'throw Paper'. The same can be said for *any* strategy of the type 'throw *this*', for both players. So in terms of the concept of strategy described above, this game is not determined. On the other hand, consider the 'strategy' 'throw Scissors 1/3 of the time, throw Paper 1/3 of the time, and throw Stone the remaining 1/3 of the time'. Against any other strategy, this strategy loses, draws and wins 1/3 of the time each, for an 'average result' of a draw. This strategy does not fit in the concept of strategy given above, but it is clearly worth considering.

Strategies of this new type are called *mixed* strategies, as opposed to the old type of strategies, the *pure* strategies. A mixed strategy for a given player in a given game consists of a specification, for each position in which he or she is required to make a move, of the *probability distribution* to be used

[4] In more general cases, we allow ϵ-approximation, i.e. a game is *determined* iff there exists a value v such that for any $\epsilon > 0$, the two players have strategies guaranteeing them a payoff of at least $v - \epsilon$ or at most $v + \epsilon$, respectively.

to determine what move to make in that position.[5] Given mixed strategies for each of the players, the outcome of the game is not determined, but we can calculate the *probability* of each outcome. If we assign values to winning and losing ('the loser pays the winner one dollar'), then we can calculate the average profit/loss one player can expect to make from the other, playing those strategies.

The *value* of a mixed strategy is therefore the *expected average result* against the best counterstrategy. And a game is called determined if, for some value v, one of the players has a strategy with which she can always expect to make (on average) at least v, no matter what the other plays, while the other player has a strategy with which he can always expect to lose (on average) at most v, no matter what the other plays. As before, it can be shown (using induction and a theorem of Von Neumann) that all finite two-person zero-sum games with Imperfect Information (i.e. the games with the properties mentioned above, except that players make moves at the same time instead of one after the other) are determined.

All the games mentioned so far are of *finite* duration. Let, as before, X and Y be two finite sets, and let f be a function assigning to each countably infinite sequence w of pairs $(x_i, y_i) \in X \times Y$, a payoff $f(w) \in \mathbb{R}$.[6] We first consider games of infinite duration and *perfect* information:

> Two players are playing against each other. Each player, in turn, makes a move by selecting an element $x_1 \in X$ or $y_1 \in Y$, respectively, and announcing his or her selection. Then they each in turn make a second move, and a third move, and continue making moves for a *countably infinite* number of rounds. This generates an infinite sequence w of pairs $(x_i, y_i) \in X \times Y$. 'Then' they stop, and player II pays player I the amount $f(w)$.

The problem with infinite games, of course, is that the outcome is only known after an infinite number of moves, and thus it is impractical to play the game as it is. But our concept of a strategy as a specification of which move to make in each position, is still valid in the case of games of infinite duration. And given strategies for both players we can construct the infinite sequence of moves that will be played (or the probability distribution thereof), and apply the payoff function to obtain our (expectation of the) outcome. Hence we can still play the game in a fashion, by using its *normal form.*

The concepts of values and determinateness carry over as well. But it is no longer provable that all such games are determined. For some payoff functions f, such as bounded Borel-measurable functions f, it has been

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[5] Standard game theory defines a mixed strategy as a probability distribution on pure strategies, but the above definition can be shown to be equivalent to that one.

[6] We tacitly assume f to be bounded, as otherwise things get ugly.

proven that the infinite game of perfect information $\Gamma_{p.i.}(f)$ is determined. But using the Axiom of Choice, a nonmeasurable payoff function f can be constructed such that $\Gamma(f)$ is not determined [10]. The axiom AD, the axiom that all games $\Gamma(f)$ are determined, is widely used as an alternative to AC [9, 11].

A game of infinite duration and *imperfect* information is similar, except that both players make their n^{th} move at the same time. These games are called Blackwell games, named after David Blackwell, the first one to describe and study these games [2]. For Blackwell games, it has been proven that $\Gamma(f)$ is determined for the case that f is the indicator function of an open or G_δ set. In this article I prove determinacy of $\Gamma(f)$ for the case that f is the indicator function of a $G_{\delta\sigma}$ set. But the general case of Borel-measurable functions is still open.

2 Definitions, Lemmas and Terminology

2.1 Games, Strategies and Values

The definitions in this subsection are fairly standard, and merely formalize the intuitive concepts from the introduction. The lemmas are all basic properties of game-values. For reasons of conciseness, no proofs are given in this section.

In order to define what a Blackwell game is, we first need some sets. Let X and Y be two finite, nonempty sets, and put $Z = X \times Y$. An *(infinite) play* is a countably infinite sequence w of pairs $(x, y) \in Z$. We write W for the set of all plays, i.e. $W = Z^{\mathbb{N}}$.

Definition 2.1 Let $f : W \to \mathbb{R}$ be a bounded Borel (measurable) function (i.e. a bounded function such that $f^{-1}[O]$ is a Borel set for every open set $O \subseteq \mathbb{R}$). The *Blackwell game* $\Gamma(f)$ with *payoff function* f is the two-person zero-sum infinite game of imperfect information played as follows: Player I selects an element $x_1 \in X$ (*makes the move* x_1) and, simultaneously, player II selects an element $y_1 \in Y$. Then both players are told $z_1 = (x_1, y_1)$, and the game *is at* or *has reached* position (z_1). Then player I selects $x_2 \in X$ and, simultaneously, player II selects $y_2 \in Y$. Then both players are told $z_2 = (x_2, y_2)$, and the game is at position (z_1, z_2). Then both players simultaneously selects $x_3 \in X$ and $y_3 \in Y$, etc. Thus they produce a play $w = (z_1, z_2, \ldots)$. Then player II pays player I the amount $f(w)$, ending the game.

A *position* or *finite play* (*of length* k) is a finite sequence p (of length k) of pairs $(x, y) \in Z$. We write P for the set of all positions, i.e. $P = Z^{<\omega}$.

Some notation and terminology that we are going to use:
w usually denotes an infinite play, p denotes a finite play or position.
$p_{|n}$, $w_{|n}$ denote the sequences consisting of the first n moves made in p, w respectively (counting a pair (x, y) as one move).
$p^\frown p'$, $p^\frown w$ denote the sequences consisting of the finite sequence p followed by the finite sequence p' or the infinite sequence w, respectively.
$\text{len}(p)$ denotes the length of a finite sequence p.
e denotes the position of length 0, i.e. the empty sequence.
W_n denotes the set of all finite plays of length n, i.e. $W_n = Z^n$, for $n \in \mathbb{N}$.
$p \subset w$ denotes that $w_{|\text{len}(p)} = p$, and we say that w *hits* or *passes through* p.
$p \subset p'$ denotes that $p'_{|\text{len}(p)} = p$ and $p' \neq p$, and we say that p' *follows* p, and p *precedes* p'.
$p \subseteq p'$ denotes that $p'_{|\text{len}(p)} = p$, and we say that p' *follows or is equal to* p.
$[p]$ denotes the set $\{w \in W \mid w \supset p\}$ of all plays hitting the position p.
$[H]$ denotes the set $\{w \in W \mid \exists p \in H : w \supset p\}$ of all plays hitting any position in a set of positions H.
We sometimes write a sequence $((x_1, y_1), (x_2, y_2), \ldots)$ as $(x_1, y_1, x_2, y_2, \ldots)$.
$\Gamma(S)$ denotes the game $\Gamma(I_S)$, where I_S is the indicator function of $S \subseteq W$.

Remark 2.2 We give W the usual topology by letting the basic open sets be the sets of the form $[H]$ for some some set $H \subseteq W_n$ of positions of fixed length n. Then the open subsets of W are exactly those of the form $[H]$ for some set H of positions. The G_δ subsets of W are exactly those of the form $\{w \in W \mid \#\{p \in H \mid w \text{ hits } p\} = \infty\}$ for some set H of positions. Note that under this topology, W is a compact space.

Definition 2.3 A *strategy* for player I in a Blackwell game $\Gamma(f)$ is a function σ assigning to each position p a probability distribution on X. More formally, σ is a function $P \to [0, 1]^X$ satisfying $\forall p \in P : \sum_{x \in X} \sigma(p)(x) = 1$.
Analogously, a *strategy* for player II is a function τ assigning to each position p a probability distribution on Y.

Definition 2.4 Let σ and τ be strategies for players I, II in a Blackwell game $\Gamma(f)$. σ and τ determine a probability measure $\mu_{\sigma,\tau}$ on W, induced by

$$\mu_{\sigma,\tau}[p] = P\{w \mid w \text{ hits } p\} = \prod_{i=1}^{n} \Big(\sigma(p_{|(i-1)})(x_i) \bullet \tau(p_{|(i-1)})(y_i) \Big) \qquad (2.1)$$

for any position $p = (x_1, y_1, \ldots, x_n, y_n) \in P$.
The *expected income* of player I in $\Gamma(f)$, if she plays according to σ and player II plays according to τ, is the expectation of $f(w)$ under this probability measure:

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \int f(w) d\mu_{\sigma,\tau}(w) \qquad (2.2)$$

Definition 2.5 Let $\Gamma(f)$ be a Blackwell game. The *value* of a strategy σ for player I in $\Gamma(f)$ is the expected income player I can guarantee if she plays according to σ. Similarly, the *value* of a strategy τ for player II in $\Gamma(f)$ is the amount to which player II can restrict player I's income if he plays according to τ. I.e.

$$\begin{aligned} \mathrm{val}(\sigma \text{ in } \Gamma(f)) &= \inf_{\tau} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) && (2.3) \\ \mathrm{val}(\tau \text{ in } \Gamma(f)) &= \sup_{\sigma} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) && (2.4) \end{aligned}$$

Definition 2.6 Let $\Gamma(f)$ be a Blackwell game. The *lower value* of $\Gamma(f)$ is the smallest upper bound on the income that player I can guarantee. Similarly, the *upper value* of $\Gamma(f)$ is the largest lower bound on the restrictions player II can put on player I's income. I.e.

$$\begin{aligned} \mathrm{val}^{\downarrow}(\Gamma(f)) = \sup_{\sigma} \mathrm{val}(\sigma \text{ in } \Gamma(f)) &= \sup_{\sigma} \inf_{\tau} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) && (2.5) \\ \mathrm{val}^{\uparrow}(\Gamma(f)) = \inf_{\tau} \mathrm{val}(\tau \text{ in } \Gamma(f)) &= \inf_{\tau} \sup_{\sigma} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) && (2.6) \end{aligned}$$

Clearly, for all games $\Gamma(f)$, $\mathrm{val}^{\downarrow}(\Gamma(f)) \leq \mathrm{val}^{\uparrow}(\Gamma(f))$. If $\mathrm{val}^{\uparrow}(\Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma(f))$, then $\Gamma(f)$ is called *determined*, and we may write $\mathrm{val}(\Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma(f))$.

Definition 2.7 Let $\Gamma(f)$ be a Blackwell game, and let $\epsilon > 0$. A strategy σ for player I in $\Gamma(f)$ is *optimal* if $\mathrm{val}(\sigma \text{ in } \Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma(f))$. A strategy σ for player I in $\Gamma(f)$ is *ϵ-optimal* if $\mathrm{val}(\sigma \text{ in } \Gamma(f)) > \mathrm{val}^{\downarrow}(\Gamma(f)) - \epsilon$. Similarly, a strategy τ for player II in $\Gamma(f)$ is *optimal* if $\mathrm{val}(\tau \text{ in } \Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma(f))$, and *$\epsilon$-optimal* if $\mathrm{val}(\tau \text{ in } \Gamma(f)) < \mathrm{val}^{\uparrow}(\Gamma(f)) + \epsilon$.

Some basic properties of these values are:

Lemma 2.8 *Let f, g be two payoff functions such that for all $w \in W$, $f(w) \leq g(w)$. Then* $\mathrm{val}^{\downarrow}(\Gamma(f)) \leq \mathrm{val}^{\downarrow}(\Gamma(g))$ *and* $\mathrm{val}^{\uparrow}(\Gamma(f)) \leq \mathrm{val}^{\uparrow}(\Gamma(g))$.

Lemma 2.9 *Let f be a payoff function, and let $a, c \in I\!\!R, a \geq 0$. Then* $\mathrm{val}^{\downarrow}(\Gamma(af+c)) = a\,\mathrm{val}^{\downarrow}(\Gamma(f)) + c$ *and* $\mathrm{val}^{\uparrow}(\Gamma(af+c)) = a\,\mathrm{val}^{\uparrow}(\Gamma(f)) + c$.

Lemma 2.10 *Let f be a payoff function, and let $f_{sw} : (Y \times X)^{I\!N} \to I\!\!R$ be defined by*

$$f_{sw}((y_1, x_1), (y_2, x_2), \ldots) = f((x_1, y_1), (x_2, y_2), \ldots) \qquad (2.7)$$

Then

$$\begin{aligned} \mathrm{val}^{\downarrow}(\Gamma(-f)) &= -\mathrm{val}^{\uparrow}(\Gamma_{sw}(f_{sw})) && (2.8) \\ \mathrm{val}^{\uparrow}(\Gamma(-f)) &= -\mathrm{val}^{\downarrow}(\Gamma_{sw}(f_{sw})) && (2.9) \end{aligned}$$

where $\Gamma_{sw}(f_{sw})$ is the Blackwell game with payoff function f_{sw} in which player I selects moves from Y and player II selects moves from X.

Lemma 2.11 *Let $(f_i)_i$ be a sequence of functions $f_i : W \to [a, b]$ such that $(f_i)_i$ converges pointwise to a function $f : W \to [a, b]$. Then for any two strategies σ, τ, $\lim_{i\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f_i)) = E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))$*

2.2 Starting and Stopping

Definition 2.12 Let $f : W \to I\!R$ be a bounded Borel function, and $p = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$ a position. The *subgame* $\Gamma(f, p)$ *starting from position* p is the game played like $\Gamma(f)$, except that the players start at round $n+1$, and the first n moves are supposed to have been $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. The game $\Gamma(f, p)$ is played exactly the same as the game $\Gamma(g)$, where g is the payoff function defined by $g(w) = f(p^\frown w)$.

As before, strategies σ and τ determine a probability measure $\mu_{\sigma,\tau \text{ in } \Gamma(f,p)}$ on W. This measure is equal to the conditional probability measure obtained from $\mu_{\sigma,\tau}$ given $[p]$, i.e.

$$\mu_{\sigma,\tau \text{ in } \Gamma(f,p)}(S) = \frac{\mu_{\sigma,\tau}(S \cap [p])}{\mu_{\sigma,\tau}[p]} \quad \text{if } \mu_{\sigma,\tau}[p] = 0 \tag{2.10}$$

The expected income of player I, the value of a strategy σ, etc. are defined for the games $\Gamma(f, p)$ in the same manner as for the games $\Gamma(f)$.

Definition 2.13 A *stopping position* in a Blackwell game $\Gamma(f)$ is a position p, such that for all plays $w, w' \in [p]$, $f(w) = f(w')$. We will denote this value by $f(p)$. A *stopset* in a Blackwell game $\Gamma(f)$ is a set H of stopping positions, such that no stopping position $p \in H$ precedes another stopping position $p' \in H$.

We will often define a payoff function f using the following format:

$$\begin{aligned} f(p) &= \text{formula1 for } p \in H \\ f(w) &= \text{formula2 if } w \notin [H] \end{aligned}$$

where H is a set of positions such that no position $p \in H$ precedes another position $p' \in H$. Then H is a stopset in the game $\Gamma_H(f)$.

Remark 2.14 If p is a stopping position, any moves made at or after p will not affect the outcome of the game. It is often convenient to assume that both players will stop playing if a stopping position is reached. If $\Gamma(f)$ is a Blackwell game, and H is a stopset, we write $\Gamma_H(f)$ to explicitly denote that players stop playing at the positions in H. In this case, we only require strategies to be defined on nonstopping positions. Similarly, with respect to a subgame $\Gamma(f, p)$, we only require strategies to be defined on positions that are following or equal to p. In fact it is occasionally necessary to assume that a strategy is *not* defined on positions outside the subgame proper, for instance to combine strategies for different subgames into one big strategy.

Using stopsets, a finite game can be treated as a special type of infinite game. For finite games, we have determinacy, as well as a kind of continuity of the value function.

Definition 2.15 Let $\Gamma(f)$ be a Blackwell game. If, for some n, all positions in W_n are stopping positions, then $\Gamma(f)$ is called *finite (of length n)*. If $\Gamma(f)$ is finite, we can stop after playing n rounds, and we will denote this by writing $\Gamma_n(f)$.

Theorem 2.16 (Von Neumann's Minimax Theorem[12]) *Let $\Gamma_1(f)$ be a finite one-round Blackwell game (i.e. of length 1). Then $\Gamma_1(f)$ is determined, and both players have optimal strategies.*

Theorem 2.17 *Let $\Gamma_n(f)$ be a finite Blackwell game of length n. Then $\Gamma_n(f)$ is determined, and both players have optimal strategies.*

Lemma 2.18 *Let $n \in \mathbb{N}$. Let $(f_i)_i$ be a sequence of payoff functions $f_i : W_n \to [a,b]$ such that $(f_i)_i$ converges to a payoff function $f : W_n \to [a,b]$. Then* $\mathrm{val}(\Gamma_n(f)) = \lim_{i\to\infty} \mathrm{val}(\Gamma_n(f_i))$.

2.3 Equivalent Truncated Subgames

In games like Chess, Go, or even Risk or Monopoly, a player is usually allowed to give up if he has no hope of winning. He doesn't have to play it out in the hope that the other player will make a mistake. Two players can agree beforehand to stop in certain positions, and pay out the value of the game at that position rather than continue playing. Provided their assessment of that value is accurate, this does not change the value of the total game. We will call a game resulting from such an alteration a *truncated subgame.*

Definition 2.19 Let f, g be two payoff functions, and H a stopset in $\Gamma(g)$. $\Gamma_H(g)$ is an *equivalent truncated subgame* of $\Gamma(f)$ (*truncated at H*), if for any play $w \notin [H]$, $f(w) = g(w)$, and for any $p \in H$, $g(p) = \mathrm{val}(\Gamma(f,p))$.
$\Gamma_H(g)$ is a *truncated subgame, equivalent for player I [player II]*, if for any play $w \notin H$, $f(w) = g(w)$, and for any $p \in H$, $g(p) = \mathrm{val}^{\downarrow}(\Gamma(f,p))$ [$g(p) = \mathrm{val}^{\uparrow}(\Gamma(f,p))$]. In all three cases, $\Gamma(f)$ is called an *extension* of $\Gamma_H(g)$.

Note that $\Gamma_H(g)$ is an equivalent truncated subgame of $\Gamma(f)$ iff it is a truncated subgame equivalent for both player I and player II.

Lemma 2.20 *Let $\Gamma(f)$ be a Blackwell game, and let $\Gamma_H(g)$ be a truncated subgame of $\Gamma(f)$, truncated at a set of positions H, equivalent for player I [for player II]. Then* $\mathrm{val}^{\downarrow}(\Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma_H(g))$ *[*$\mathrm{val}^{\uparrow}(\Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma_H(g))$*]. Furthermore, for any $\epsilon > 0$, any ϵ-optimal strategy for player I [player II] in $\Gamma_H(g)$ (if it is undefined on all positions at or after positions in H) can be extended to an ϵ-optimal strategy for player I [player II] in $\Gamma(f)$.*

Sketch of proof: We find an ϵ-optimal strategy for the truncated subgame $\Gamma_H(g)$, and for the appropriate δ, δ-optimal strategies for the games $\Gamma(f,p)$ starting at positions $p \in H$, i.e. the positions where $\Gamma_H(g)$ stops. Then we tie them together, and calculate how well the combination strategy performs against opposing strategies.

Corollary 2.21 *Let $\Gamma(f)$ be a Blackwell game, and let $\Gamma_H(g)$ be an equivalent truncated subgame of $\Gamma(f)$ (truncated at H). If $\Gamma_H(g)$ is determined, then $\Gamma(f)$ is determined, and* $\text{val}(\Gamma(f)) = \text{val}(\Gamma_H(g))$. *Furthermore, for any $\epsilon > 0$, any ϵ-optimal strategy for player I or player II in $\Gamma_H(g)$ can be extended to an ϵ-optimal strategy for player I or player II in $\Gamma(f)$.*

Corollary 2.22 *Let $\Gamma(f), \Gamma_H(g)$ be Blackwell games. If for any $p \in H$, $g(p) \leq \text{val}^{\downarrow}(\Gamma(f,p))$, and for any $w \notin [H]$, $g(w) \leq f(w)$, then* $\text{val}^{\downarrow}(\Gamma_H(g)) \leq \text{val}^{\downarrow}(\Gamma(f))$. *Similarly for the value and the upper value, and for $\geq$ instead of $\leq$.*

Truncated subgames may be nested. If we have a nested series of truncated subgames, then we may extend a strategy for the smallest subgame to a strategy for *all* subgames. This allows us to approximate complicated games with a series of simpler, truncated subgames, obtain a strategy that is (ϵ-)optimal in all the subgames. The final lemma in this section allows us to prove results for that strategy in the original game.

Definition 2.23 Let, for $n \in I\!N$, f_n be a payoff function, and H_n a set of stopping positions in $\Gamma(f_n)$. If for all $n \in I\!N$, $\Gamma_{H_n}(f_n)$ is a truncated subgame of $\Gamma_{H_{n+1}}(f_{n+1})$, and equivalent to $\Gamma_{H_{n+1}}(f_{n+1})$ [for player I, II], then the series of games $(\Gamma_{H_n}(f_n))_{n \in I\!N}$ is called a *nested series of equivalent truncated subgames* [equivalent for player I, II].

Lemma 2.24 *Let $(\Gamma_{H_i}(g_i))_{i \in I\!N}$ be a nested series of truncated games equivalent for player I [player II]. Then all the games have the same lower value [upper value]. Furthermore, we can find a strategy for player I [player II] that is ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$.*

Sketch of proof: Basically, we apply Lemma 2.20 a number of times and use induction. The proof is straightforward, except for a slight complication involving the domain on which the strategies are defined. This complication is solved using the observations that if we truncate a game, any stopping position remains a stopping position, and that strategies can be assumed to be undefined on stopping positions.

Corollary 2.25 *Let $(\Gamma_{H_i}(g_i))_{i \in I\!N}$ be a nested series of equivalent truncated subgames. If $\Gamma_{H_0}(g_0)$ is determined, then all the games are determined, and all the games have the same value. Furthermore, we can find strategies for player I and player II that are ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$.*

Remark 2.26 If the component games involved all have optimal strategies, then we can extend optimal strategies with optimal strategies to optimal strategies, i.e. drop the ϵ in the above lemmas and corollaries.

3 Determinateness Results

3.1 Generalized Open Games

In this subsection we prove determinacy of a class of 'generalized open games', where payoff for a play is calculated as the supremum of values associated with the positions hit in the play. In addition we derive a result for these and open games comparable to the compactness of W.

Theorem 3.1 *Let $u : P \to \mathbb{R}$ be a bounded function, and let $f : W \to \mathbb{R}$ be the payoff function defined by $f(w) = \sup_{j \in \mathbb{N}} u(w_{|j})$. Then $\Gamma(f)$ is determined, and*

$$\mathrm{val}(\Gamma(f)) = \lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n)) \tag{3.11}$$

where $f_n(w) = \sup_{j \le n} u(w_{|j})$.

Sketch of proof: Showing that $\lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n))$ exists and is not greater than the lower value of $\Gamma(f)$ is not difficult. To show that it is not less than the upper value, we approximate $\Gamma(f)$ with a collection of finite auxiliary games $\Gamma_n(g_n)$ such that the payoff at the stopping positions is an *estimate* of the value of the game at that point. We then show that these auxiliary games form a nested series of equivalent finite truncated subgames. This allows us to find a strategy that is optimal in each of the truncated subgames. This strategy is also a strategy in the game $\Gamma(f)$, and has a value in $\Gamma(f)$ equal to $\lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n))$.

Proof: Without loss of generality we may assume that the function u has range $[0,1]$. For any $p \in P$, and any $n \in \mathbb{N}$, the game $\Gamma_n(f_n, p)$ is finite (of length $\le n$), and thus determined. It is easily seen that $f_0 \le f_1 \le f_2 \le \ldots \le f \le 1$. Consequently, for any $p \in P$,

$$\mathrm{val}(\Gamma_0(f_0,p)) \le \mathrm{val}(\Gamma_1(f_1,p)) \le \mathrm{val}(\Gamma_2(f_2,p)) \le \ldots \le \mathrm{val}^{\downarrow}(\Gamma(f,p)) \le 1 \tag{3.12}$$

For all $p \in P$, $\lim_{k\to\infty} \mathrm{val}(\Gamma_k(f_k,p))$ exists, since all monotone non-decreasing bounded sequences converge. Furthermore, for all $p \in P$,

$$\lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n,p)) \le \mathrm{val}^{\downarrow}(\Gamma(f,p)) \tag{3.13}$$

Define for any $n \in \mathbb{N}$ the payoff function $g_n : W_n \to [0,1]$ by

$$g_n(p) = \lim_{k\to\infty} \mathrm{val}(\Gamma_k(f_k,p)) \text{ for } p \in W_n \tag{3.14}$$

Then for all $p \in W_n$, $g_n(p) \geq \text{val}(\Gamma_n(f_n, p)) = f_n(p)$.
Furthermore, the games $\Gamma_n(g_n)$ form a nested series of equivalent truncated subgames. For fix $n \in \mathbb{N}$, $p \in W_n$. Define for $k \in \mathbb{N}$, $h_{n+1,k} : W_{n+1} \to \mathbb{R}$ by $h_{n+1,k}(p') = \text{val}(\Gamma_k(f_k, p'))$ for $p' \in W_{n+1}$. Then

$$\begin{aligned} g_n(p) &= \lim_{k\to\infty} \text{val}(\Gamma_k(f_k, p)) && (3.15)\\ &= \lim_{k\to\infty} \text{val}(\Gamma_{n+1}(h_{n+1,k}, p)) && (3.16)\\ &= \text{val}(\Gamma_{n+1}(\lim_{k\to\infty} h_{n+1,k}, p)) && (3.17)\\ &= \text{val}(\Gamma_{n+1}(g_{n+1}, p)) && (3.18) \end{aligned}$$

(equation (3.16) follows from Corollary 2.21, and equation (3.17) follows from Lemma 2.18 as W_{n+1} is finite).
Since $(\Gamma_n(g_n))_{n\in\mathbb{N}}$ is a nested series of equivalent truncated subgames, by Corollary 2.25 the games $\Gamma_n(g_n)$ all have the same value, say v. Also, we can find a strategy for player II that is ϵ-optimal in all the games $\Gamma_n(g_n)$, and since all the games $\Gamma_n(g_n)$ are finite and hence have optimal strategies, by Remark 2.26 we can even find a strategy that is optimal in all the games $\Gamma_n(g_n)$. So let τ be such a strategy. Then for any strategy σ, and any $n \in \mathbb{N}$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(g_n)) \leq \text{val}(\Gamma_n(g_n)) = v \tag{3.19}$$

Now let σ be any strategy for player I in $\Gamma(f)$. Then

$$\begin{aligned} &E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))\\ &= \lim_{n\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(f_n)) && (3.20)\\ &\leq \lim_{n\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(g_n)) && (3.21)\\ &\leq v && (3.22) \end{aligned}$$

So

$$\text{val}^{\uparrow}(\Gamma(f)) \leq \text{val}(\tau \text{ for player II in } \Gamma(f)) \leq v \tag{3.23}$$

But also

$$v = \text{val}(\Gamma_0(g_0)) = g_o(e) = \lim_{k\to\infty} \text{val}(\Gamma_k(f_k)) \leq \text{val}^{\downarrow}(\Gamma(f)) \tag{3.24}$$

Therefore,

$$\text{val}^{\uparrow}(\Gamma(f)) = \text{val}^{\downarrow}(\Gamma(f)) = \lim_{k\to\infty} \text{val}(\Gamma_k(f_k)) \tag{3.25}$$

□

Corollary 3.2 *Let O be an open set. Then $\Gamma(O)$ is determined.*

Proof: There exists a set of positions H such that $O = [H]$. Then for all $w \in W$, $I_O(w) = \sup_{n\in I\!N} I_H(w_{|n})$. Applying Theorem 3.1 yields the corollary.

□

Corollary 3.3 *Let $O = \bigcup_i O_i$ be the union of open sets. Then* $\mathrm{val}(\Gamma(O)) = \lim_{n\to\infty} \mathrm{val}(\Gamma(\bigcup_{i\leq n} O_i))$.

Proof: As the union of open sets, O is open, and hence there is a set of positions H such that $O = [H]$, i.e.

$$O = \{w \in W \mid \exists p \in H : p \subset w\} \tag{3.26}$$

Define the basic open sets $B_j \subseteq O$ by

$$B_n = \{w \in W \mid \exists p \in H : p \subset w \wedge \mathrm{len}(p) \leq n\} \tag{3.27}$$

then for all $w \in W$,

$$I_O(w) = \sup_{j\in I\!N} I_H(w_{|j}) \tag{3.28}$$

$$I_{B_n}(w) = \sup_{j\leq n} I_H(w_{|j}) \tag{3.29}$$

so applying Theorem 3.1, we find that

$$\mathrm{val}(\Gamma(O)) = \lim_{n\to\infty} \mathrm{val}(\Gamma_n(B_n)) \tag{3.30}$$

For each $m \in I\!N$, B_m is a closed set covered by the open sets $(O_i)_{i\in I\!N}$. So by the compactness of W there is for each $m \in I\!N$ a $n_m \in I\!N$ such that $B_m \subseteq \bigcup_{i=1}^{n_m} O_i$. Then for all $n \geq n_m$,

$$\mathrm{val}(\Gamma_m(B_m)) \leq \mathrm{val}(\Gamma(\bigcup_{i=1}^{n} O_i)) \leq \mathrm{val}(\Gamma(O)) \tag{3.31}$$

The corollary follows immediately.

□

Corollary 3.4 *Let f be a continuous function. Then $\Gamma(f)$ is determined.*

Proof: As W is compact, and f is continuous, $f[W]$ is compact, and hence bounded. Define $u : P \to I\!R$ by $u(p) := \inf_{w\in[p]} f(w)$. Then u is well-defined and bounded, and by the continuity of f, $f(w) = \sup_{n\in I\!N} u(w_{|n})$ for all $w \in W$. Applying Theorem 3.1 yields the corollary.

□

Remark 3.5 In the case of open games (or generalized open games, described later) there is an optimal strategy for player II. This strategy can be described as 'at every position player II plays the optimal one-round strategy, looking at the values the game has *for player I* from all positions directly following that one'. However, for player I there does not always exist an optimal strategy, as the following example shows.

Example 3.6 Consider the following Blackwell game. Each round, both players say either 'Stop' or 'Continue'. If both players say 'Continue', then play continues. Otherwise, the game halts: player II wins (payoff 0) if both players said 'Stop', while player I wins (payoff 1) if only one of the players said 'Stop'. If play continues indefinitely, and neither player ever says 'Stop', then payoff is 0, i.e. player II wins.

This is clearly an open game. An interpretation of this game is, that player II tries to guess on which round player I will say 'Stop', and tries to match her. If player II guesses wrong, i.e. says 'Stop' too soon or not soon enough, then player I wins, if player II guesses right, then he wins.

A strategy of value $1 - \frac{1}{n}$ for player I is, to select at random a number i between 1 and n, and say 'Stop' on round i. Translated to the standard format for strategies, this becomes:

on round 1, say 'Stop' $\frac{1}{n}$ of the time,
on round 2, if not yet stopped, say 'Stop' $\frac{1}{n-1}$ of the time,
on round 3, if not yet stopped, say 'Stop' $\frac{1}{n-2}$ of the time,

$\vdots$

on round n, if not yet stopped, say 'Stop' $\frac{1}{1}$ of the time.

Hence, the value of this game is 1. In fact, the value of this game at any position in which game has not yet ended is 1. But there exists no optimal strategy of value 1. For suppose there exists such a strategy, of value 1. Then on any round (in which play has not yet ended), the chance that player I will say 'Stop' in that round is 0%. For otherwise, the strategy would not score 100% against the counterstrategy that player II says 'Stop' on that round. But then, player I will never say 'Stop', and this strategy will lose against the counterstrategy that player II never says 'Stop'. So any strategy for player I has value strictly less than 1, although there are strategies with values arbitrarily close to 1. This game is an example of a game in which one of the players has no optimal strategy.

3.2 G_δ-sets

Davis' proof of determinacy for $G_{\delta\sigma}$ games of perfect information [5] is based upon the idea of 'imposing restrictions' on the range of moves player II can make. I.e. certain moves are declared 'forbidden', or a loss for player II, in such a way that (a) if player I did not have a win before, she does not get a win now, and (b) a particular G_δ set is now certain to be avoided. By applying this to all the G_δ subsets of a $G_{\delta\sigma}$ set, and using compactness, he shows that if player I cannot force the resulting sequence to be in one of the G_δ sets, player II can force the resulting sequence to be outside all of them.

The union of all the sequences in which one of the 'forbidden' moves is played, is an open set that contains the G_δ set in question. One way of looking at Davis' proof is, that he enlarges each of the G_δ sets to an open set without increasing the (lower) value of the game, in order to be able to apply determinacy of open games.

In this subsection, we show that this holds (in a fashion) for Blackwell games, i.e. that a G_δ set can be 'enlarged' to an open set without increasing the lower value of the game by more than an arbitrarily small amount, even in the presence of a 'background function', a payoff function for those sequences that are not in the G_δ set.

Theorem 3.7 *Let $f : W \to [0,1]$ be a measurable function and let D be a G_δ set. Then*

$$\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) = \inf_{O \supseteq D, O \text{ open}} \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O))) \tag{3.32}$$

Sketch of proof: We define a collection of auxiliary games $\Gamma_{H_i}(g_i)$ of the game $\Gamma(\max(f, I_D))$, in which the amount player I gets at a stopping position p is an *estimate* for the value of $\Gamma(\max(f, I_D))$ at position p, namely $\inf_{O \supseteq D, O \text{ open}} \mathrm{val}(\Gamma(\max(f, I_O), p))$. We then show that these auxiliary games form a nested series of finite truncated subgames, equivalent for player I. This allows us to find a strategy that is ϵ-optimal in each of the truncated subgames. This strategy is also a strategy in the game $\Gamma(\max(f, I_D))$, and has the required value, proving one side of the equation. The other side is trivial.

Proof: Put $v = \inf_{O \supseteq D, O \text{ open}} \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O)))$. For any G_δ set D we can find a set of positions H, such that $D = \{w \in W \mid \#\{p \in H \mid p \subset w\} = \infty\}$. We may assume that $e \in H$.
Define for any $i \in I\!N$,

$$H_i := \{p \in H \mid \text{there are exactly } i \text{ positions } p' \text{ in } H \text{ strictly preceding } p\} \tag{3.33}$$

Define for any $i \in I\!N$ the payoff functions g_i, h_i by

$$\begin{aligned} g_i(p) &= \inf_{O \supseteq D, O \text{ open}} \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O), p)) \text{ for } p \in H_i && (3.34)\\ g_i(w) &= f(w) \text{ if } w \notin [H_i] && (3.35)\\ h_i(p) &= 1 \text{ for } p \in H_i && (3.36)\\ h_i(w) &= f(w) \text{ if } w \notin [H_i] && (3.37) \end{aligned}$$

First, the games $\Gamma_{H_i}(g_i)$ form a nested series of truncated subgames equivalent for player I.

For let $i \in \mathbb{N}$, and fix $p \in H_i$. Let $O \supseteq D$, then for any $p' \in H_{i+1}$ such that $p' \supseteq p$, $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O), p')) \geq g_{i+1}(p')$, and for any $w \supset p$ such that $w \notin [H_{i+1}]$, $\max(f, I_O)(w) \geq f(w) = g_{i+1}(w)$. Hence by Corollary 2.22, for any $O \supseteq D$, $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O), p)) \geq \mathrm{val}^{\downarrow}(\Gamma_{H_{i+1}}(g_{i+1}, p))$. Therefore,

$$g_i(p) \geq \mathrm{val}^{\downarrow}(\Gamma_{H_{i+1}}(g_{i+1}, p)) \tag{3.38}$$

On the other hand, for any $\epsilon > 0$ we can find, for each $p' \in H_{i+1}$, an open set $O_{p'} \supseteq D$ such that

$$\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_{O_{p'}}), p')) \leq g_{i+1}(p') + \epsilon \tag{3.39}$$

Set $O = \bigcup_{p' \in H_{i+1}}([p'] \cap O_{p'})$. Then for all $p' \in H_{i+1}$, $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O), p')) = \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_{O_{p'}}), p')) \leq g_{i+1}(p') + \epsilon$, and for any $w \notin [H_{i+1}]$, $\max(f, I_O)(w) = f(w) = g_{i+1}(w)$. Hence by Corollary 2.22,

$$g_i(p) \leq \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O), p)) \leq \mathrm{val}^{\downarrow}(\Gamma_{H_{i+1}}(g_{i+1}, p)) + \epsilon \tag{3.40}$$

This holds for any $\epsilon > 0$, therefore

$$g_i(p) = \mathrm{val}^{\downarrow}(\Gamma_{H_{i+1}}(g_{i+1}, p)) \tag{3.41}$$

Finally, for any $i \in \mathbb{N}$, and any play $w \notin [H_i]$, we have that $w \notin [H_{i+1}]$, and hence $g_i(w) = f(w) = g_{i+1}(w)$. So $\Gamma_{H_i}(g_i)$ is a truncated subgame of $\Gamma_{H_{i+1}}(g_{i+1})$ equivalent for player I.

Let $\epsilon > 0$.

Since $(\Gamma_{H_i}(g_i))_{i \in \mathbb{N}}$ is a nested series of truncated subgames equivalent for player I, by Lemma 2.24 all the games have the same lower value, namely $\mathrm{val}^{\downarrow}(\Gamma_{H_0}(g_0)) = g_0(e) = v$, and there exists a strategy σ for player I that is ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$, i.e. for any strategy τ, and any $i \in \mathbb{N}$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(g_i)) \geq \mathrm{val}^{\downarrow}(\Gamma_{H_i}(g_i)) - \epsilon = v - \epsilon \tag{3.42}$$

Now let τ be any strategy for player II in $\Gamma(\max(f, I_D))$. Then

$$\begin{aligned} & E(\sigma \text{ vs } \tau \text{ in } \Gamma(\max(f, I_D))) \\ = \; & \lim_{i \to \infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(h_i)) && (3.43) \\ \geq \; & \lim_{i \to \infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(g_i)) && (3.44) \\ \geq \; & v - \epsilon && (3.45) \end{aligned}$$

So σ is a strategy for player I of value at least $v - \epsilon$. This implies that $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \geq v - \epsilon$. This construction can be done for any $\epsilon > 0$, hence

$$\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \geq v \tag{3.46}$$

For any $O \supseteq D$, $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \leq \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O)))$, hence

$$\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \leq \inf_{O \supseteq D, O \text{ open}} \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O))) = v \tag{3.47}$$

Hence $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) = v$.

□

Corollary 3.8 *Let S be a measurable set, and let D be a G_δ set. Suppose that $\Gamma(S \cup D)$ has lower value v. Then for any $\epsilon > 0$, there exist an open set O, $D \subseteq O$, such that $\Gamma(S \cup O)$ has lower value at most $v + \epsilon$.*

Proof: Take $f \equiv I_S$ and apply the non-trivial part of Theorem 3.7.

□

Corollary 3.9 *Let D be a G_δ set. Then $\Gamma(D)$ is determined, and*

$$\mathrm{val}(\Gamma(D)) = \inf_{O \supseteq D, O \text{ open}} \mathrm{val}(\Gamma(O)) \tag{3.48}$$

Proof: For any open set $O \supseteq D$, $\Gamma(O)$ is determined and $\mathrm{val}^{\downarrow}(\Gamma(D)) \leq \mathrm{val}^{\uparrow}(\Gamma(D)) \leq \mathrm{val}(\Gamma(O))$. Applying Theorem 3.7 with $f \equiv 0$ yields the Corollary.

□

3.3 $G_{\delta\sigma}$-sets

In this subsection, we prove the determinacy of $\Gamma(f)$ in the case that f is the indicator function of a $G_{\delta\sigma}$ set. Structurally, this proof is similar to the aforementioned proof by Davis for $G_{\delta\sigma}$ games of perfect information [5]. We apply the results of the previous subsection to the G_δ subsets of a $G_{\delta\sigma}$ set. Corollary 3.3 takes the place of the compactness used in Davis' proof.

Theorem 3.10 *Let $S = \bigcup_i D_i$ be a $G_{\delta\sigma}$ set. Then $\Gamma(S)$ is determined.*

Sketch of proof: Each of the G_δ sets composing the $G_{\delta\sigma}$ set is enlarged to an open set using Corollary 3.8, in such a way that at all times the lower value is not increased by more than ϵ (compared to the original game), where ϵ is arbitrarily small. The resulting union of open sets is itself open, and hence determined, and furthermore Corollary 3.3 allows us to conclude that the total increase of the lower value is still not more than ϵ. This means that the upper value of the original game is also only at most ϵ more than the lower value.
Note that, unlike the previous proofs, this proof does not produce an optimal or ϵ-optimal strategy.

Proof: Put $v = \mathrm{val}^{\downarrow}(\Gamma(S))$. Let $\epsilon > 0$. Using Corollary 3.8, we can find inductively open sets $O_i \supseteq D_i$ such that for all $j \in I\!N$,

$$\mathrm{val}^{\downarrow}(\Gamma(S \cup \bigcup_{i \leq j+1} O_i)) \leq \mathrm{val}^{\downarrow}(\Gamma(S \cup \bigcup_{i \leq j} O_i)) + \epsilon/2^j \tag{3.49}$$

Then for all $j \in I\!N$,

$$\mathrm{val}^{\downarrow}(\Gamma(S \cup \bigcup_{i \leq j} O_i)) \leq v + \epsilon \tag{3.50}$$

and hence, for all $j \in I\!N$,

$$\mathrm{val}(\Gamma(\bigcup_{i \leq j} O_i)) \leq v + \epsilon \tag{3.51}$$

Then by Corollary 3.3,

$$\mathrm{val}(\Gamma(\bigcup_{i \in I\!N} O_i)) \leq v + \epsilon \tag{3.52}$$

Since $S = \bigcup_{i \in I\!N} D_i \subseteq \bigcup_{i \in I\!N} O_i$,

$$\mathrm{val}^{\uparrow}(\Gamma(S)) \leq \mathrm{val}^{\uparrow}(\Gamma(\bigcup_{i \in I\!N} O_i)) = \mathrm{val}(\Gamma(\bigcup_{i \in I\!N} O_i)) \leq v + \epsilon \tag{3.53}$$

This is true for any ϵ, hence $\mathrm{val}^{\uparrow}(\Gamma(S)) = v = \mathrm{val}^{\downarrow}(\Gamma(S))$.

□

Remark 3.11 The proof of Theorem 3.10 shows that any $G_{\delta\sigma}$ set (and a fortiori any set of lesser complexity) can be enlarged to an open set such that the value of the Blackwell game on that set is not increased by more than an arbitrarily small amount. A plausible conjecture is, that this holds for any Borel-measurable set.

This conjecture holds in the case of games of Perfect Information. Such a game, on a Borel-set S, is determined and has value 0 or 1. If it has value 0 then player II has a winning strategy. The set of plays that cannot occur if player II uses that strategy, is an open set, and the game on that set has value 0 as well.

4 The Axiom of Determinacy for Blackwell Games

For Games of Perfect Information, there exists the Axiom of Determinacy, which states that any Game of Perfect Information with finite choice of moves is determined[7]. AD has many interesting consequences, such as the existence of an ultrafilter on $\aleph_1$, the existence of a complete measure on $\mathbb{R}$, the non-existence of a sequence of $\aleph_1$ reals, and the negation of the Axiom of Choice. We can formulate an analogue of AD with respect to Blackwell Games, and look at the consequences of that axiom. But AD is an axiom about games on all subsets of W, not just on the Borel measurable subsets[8]. An analogous axiom for Blackwell Games should therefore not be limited to games with measurable payoff functions. Hence we need to extend the concepts of expectation and value for Blackwell games.

Definition 4.1 Let $\Gamma(f)$ be a Blackwell Game, where f is bounded but not necessarily Borel measurable. Let σ and τ be strategies for players I, II. σ and τ determine a probability measure $\mu_{\sigma,\tau}$ on W, induced by setting

$$\mu_{\sigma,\tau}[p] = P\{w \mid w \text{ hits } p\} = \prod_{i=1}^{n} \Big(\sigma(p_{|(i-1)})(x_i) \bullet \tau(p_{|(i-1)})(y_i)\Big) \qquad (4.1)$$

for any position $p = (x_1, y_1, \dots, x_n, y_n) \in P$.
Instead of the expected income of player I, if she plays according to σ and player II plays according to τ, we now have the *lower* and *upper expected income* :

$$E^-(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \sup_{g \leq f, g \text{ measurable}} \int g(w)\,d\mu_{\sigma,\tau}(w) \qquad (4.2)$$

$$E^+(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \inf_{g \geq f, g \text{ measurable}} \int g(w)\,d\mu_{\sigma,\tau}(w) \qquad (4.3)$$

Lower value and upper value are redefined in the obvious way:

$$\text{val}^{\downarrow}(\Gamma(f)) = \sup_{\sigma} \inf_{\tau} E^-(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \qquad (4.4)$$

$$\text{val}^{\uparrow}(\Gamma(f)) = \inf_{\tau} \sup_{\sigma} E^+(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \qquad (4.5)$$

Note that in the case that f is measurable, these definitions reduce to the old definitions.

[7]Formally, AD is an axiom about games with countable choice of moves, whose payoff function is the indicator function of a set $S \subseteq W$. But in the case of Games of Perfect Information, determinacy for games with countable choice of moves is equivalent to determinacy for games with finite choice of moves, and determinacy for games with 0-1-valued payoff functions is equivalent to determinacy for games with arbitrary bounded payoff functions.

[8]AD with respect only to games on Borel measurable subsets (and finite sets X and Y) is in fact provable from CAC [6, 7].

Definition 4.2 The *Axiom of Determinacy for Blackwell Games* (AD-Bl) is the statement that for every pair of non-empty finite sets X, Y, and every bounded function f on $W = (X \times Y)^{\boldsymbol{N}}$, the Blackwell Game $\Gamma(f)$ is determined, i.e.

$$\mathrm{val}^{\downarrow}(\Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma(f)) \tag{4.6}$$

Theorem 4.3 *Assuming AD-Bl, it follows that all sets of reals are Lebesgue measurable.*

Sketch of proof: Let X and Y be the set $\{0, 1\}$. Then we can construct a mapping $\phi : W \to [0, 1]$ such that if either σ or τ is the strategy that assigns the $\frac{1}{2}$-$\frac{1}{2}$ probability distribution to every position, then the measure $\mu_{\sigma,\tau}$ induces the Lebesgue measure on $[0, 1]$. We can then deduce the measurability of a set $S \subseteq [0, 1]$ from the determinacy of the game $\Gamma(\phi^{-1}[S])$.

Some of the consequences of Theorem 4.3 are, that AD-Bl is not consistent with AC, and that the consistency of ZF + AD-Bl cannot be proven in ZFC. These results are all similar to results for AD. An open problem is that of the relationship between AD and AD-Bl, whether AD follows from AD-Bl, or vice versa, or even whether AD-Bl follows from a stronger version of AD such as $\mathrm{AD}_{\boldsymbol{R}}$. From a given game of Perfect Information, we can easily construct a Blackwell game that is 'equivalent', and assuming AD-Bl we can find an ϵ-optimal *mixed* strategy for that equivalent Blackwell-game. However, to derive AD from AD-Bl, we need to have a *pure* strategy, and even though we can interpret any mixed strategy as a probability distribution on pure strategies, there is no guarantee that any of these pure strategies will do as well as the mixed strategy against *all* counterstrategies.

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