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MARKOV SCULPTURE[1]

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Abstract

A stationary and non-transient Markov chain may be defined by a balanced weight function or by its transition probabilities, and these are mathematically equivalent. But starting with balanced weight functions gives rise to a different and somewhat liberating attitude. Because these functions are additive, complex models can be built rather easily, and such functions arise naturally from weighted circuits. The idea of a balanced weight function extends to a multi-particle system called an induced field, in which each particle is influenced by the other particles as a function of their locations, and yet the invariant distribution of the system is still immediately available.

This simplicity encourages the formulation of relatively complex models with the expectation that their invariant behavior will be available for interpretation and application.

Three different and somewhat novel processes are created this way and for each the invariant distribution is given explicitly. These results may be of some independent interest as well as illustrating the method.

1. Introduction. It is desired to create a Markov process as a model for some natural process. Starting with what may be a somewhat vague idea of its transition probability structure, it is possible, using a few simple mathematical ideas, to build a relatively complicated Markov process in such a way as to have an invariant weight function for the process at each stage, ending finally with relative weights giving the transition probability structure, more or less as desired, or perhaps one even more interesting than originally sought. Because the invariant weight function is available at each stage, it is available at the final stage, and thus one arrives at a process for which the invariant probabilities are known.

This approach, called "Markov sculpture", has a certain simplicity, and this frees up some amount of mathematical energy which can be spent on developing relatively complicated processes in expectation that the invariant distribution will be readily available for application.

There are several elements in Markov sculpture. These are (1), the use of a particle moving in a graph to represent the process, (2), the use of balanced weight functions on the graph to define the process, (3), the representation

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of a balanced weight function as the sum of weighted circuits, (4) the use of circuits to represent in a relatively simple way order-r Markov processes, and (5) the use of induced fields, which provide a generalization of balanced weights to situations involving several particles, where it is desired to have the particles influence the behavior of one another as a function of their locations.

These technical elements are described in Section 2. Several special processes are analyzed in Section 3 to illustrate the technical elements described in Section 2 and the general features of Markov sculpture as well. The invariant weights for the processes treated in Section 3 have not been obtained previously, and for this reason may be of independent interest.

2. Tools of Markov structure.

(1) Graphical representation. A Markov process is represented by the motion of a particle in a graph, this being a finite set of elements G, and a set of directed arcs which are ordered pairs r, s in G. The particle moves from element to element along the directed arcs, such a move being a transition in the usual parlance. The elements of G are usually called "states" but are also called variously vertices, locations, positions, places, or nodes.

The use of a graph in this way is a familiar device in the study of Markov processes, but usually, transition probabilities are placed on the arcs, whereas here "balanced weights" are used, as described below. These weights represent relative probabilities only, and it is a distinctive feature of Markov sculpture that such weights play a central role.

(2) Balanced weight functions. These are non-negative functions w on the arcs r, s of G satisfying

$$\sum_s w(s,r) = \sum_s w(r,s) \tag{2.1}$$

where the sum here and elsewhere below is over all s in the graph under consideration. This relationship is described by saying the total weight "out" of r, $\sum_s w(r,s)$, is equal to the total weight "into" r, $\sum_s w(s,r)$, hence the term balance.

The direction of an arc is from r to s if $w(r,s)$ is positive. Sometimes, for clarity in actual drawings, an arrow is placed on the arc pointing in that direction. Usually no arc is drawn if $w(r,s) = 0$.

The relationship between balanced weight functions and Markov chains in G is as follows: First suppose that $\sum_s w(r,s) = v(r) > 0$ for all r in G. An arc r, s is said to be available at r if $w(r,s) > 0$. If the particle is at r an available arc is selected with probability proportional to its weight, and if r, s is selected the particle moves to s. This description of how transitions are produced will be called the "transition rule".

Evidently if the particle is at r the probability it will move to s is

$$p(r,s) = w(r,s)/v(r), \tag{2.2}$$

and a sequence of transitions produced this way will form a Markov chain in G. But notice v will be an invariant weight function for this chain, that is,

$$\sum_s v(s)p(s,r) = \sum_s v(s)w(s,r)/v(s) = \sum_s w(s,r) = v(r) \tag{2.3}$$

with the last equality coming from the balance. It is this elementary fact that keeps the invariant weight functions at hand, at all times, as the sculpture proceeds.

The chain defined by (2.2) will not have any transient states because of the balance: There must be positive weight into a state if there is positive weight out. For this reason only non-transient chains are encountered in Markov sculpture.

But a balanced weight function and the transition rule may produce a Markov chain which has several irreducible sub-classes. In this case, v will give the unique stationary probability on each sub-class upon normalization on that class.

This gives rise to a converse of the invariance statement (2.3). It is well known (see Doob(1953),p. 172 ff.) that given a transition probability p for a chain without transient states, each irreducible sub-class of states A has a unique invariant probability distribution μ_A satisfying $\mu_A(r) = \sum_{s\in A} \mu_A(s)p(s,r)$, $r \in A$. Then $w_A(r,s) = \mu_A(r)p(r,s)$ is balanced on $A \times A$, and letting $b(A)$ be a non-negative number for each irreducible class $A, w(r,s) = \sum_A b(A)w_A(r,s)$ is balanced on $G \times G$.

The above summarizes the relationship between starting with transition probabilities on G and starting with invariant weights on G. They are mathematically equivalent in the case of non-transient chains.

But for building up processes in a constructive way, balanced weights have the great advantage of being closed under certain additive combinations, mainly, if w' and w'' are balanced and if a and b are non-negative, then $w = aw' + bw''$ is balanced.

Strictly speaking, it is only required that $aw' + bw''$ be non-negative so that transition probabilities are well defined. However, if desired, arbitrary scalar combinations $aw' + bw''$ can be used at the cost of transforming the resulting weights back to being non-negative where necessary. The obvious way to do this is to take any negative weight $w(x,y)$, add to it $-w(x,y)$, making it zero, and at the same time add $-w(x,y)$ to $w(y,x)$, which preserves balance and gets rid of at least one minus sign. This may be repeated until there are no negative weights. But, so far, no practical use has been found for balanced weight functions with negative values.

(3) Weighted Circuits. Circuits play an essential role in Markov sculpture. A circuit is a periodic function c on the integers into G. Thus there is a smallest integer $k = k(c)$ called the period or the length of c, such that $c(t) = c(t + k)$ for all integers t.

A circuit c is said to be elementary if either the period $k(c)$ is 1 or $k(c) > 1$ and the values of c for $t = 1, 2, \cdots, k(c)$ are all distinct. Only elementary circuits are used below.

If $r = c(t)$ and $s = c(t + 1)$ for some $t, 1 \leq t \leq k(c)$, then the pair r, s is an arc on or along c. There are $k(c)$ arcs along c.

A constant function c where $c(t) = r$ for all t, r being a fixed element in G, is a circuit and corresponds to a single arc (r, r), called a self-loop and which has period 1. These simplest of all circuits are surprisingly useful in Markov sculpture.

Now suppose for a given elementary circuit, a weight $a(c) > 0$ is placed on the arcs along the circuit. This weighted circuit defines a balanced weight function w_c where $w_c(r, s) = a(c)$ whenever (r, s) is an arc along c, and is zero otherwise. Such a weight function will be called a *circuit based* weight function.

Such functions play an important role here because they are easy to use, and because every balanced weight function w can be expressed as the sum of a finite number of these circuit based weight functions which use elementary circuits. (Non-elementary circuits can also be used in constructions, but the treatment is slightly more complicated because the weight in, and out, has to be counted at each recurrence, that is, each $t, 1 \leq t < k(c)$ such that $c(t)$ is equal to the repetitive element.)

Curiously, it is occasionally convenient to use a negative circuit weight in building up a balanced weight function for a process. This presents no problems as long as the total weight is positive, since it is the balanced weight function which is used to compute the transition probabilities, and these must be non-negative.

(4) Circuit representation of order r Markov chains. The representation of Markov chains in terms of weighted circuits extends to order-r Markov chains (MacQueen (1981)), as follows:

A circuit c will be r-elementary if each sequence of length r along c is distinct. More precisely let $h(t)$ be the sequence $(c(t), c(t+1), \cdots, c(t+r-1))$. Then the circuit is r-elementary if $h(t)$ is not equal $h(t + s)$ for any s, $1 \leq s < k(c)$.

Let h stand for a sequence $x_1, x_2, \cdots, x_r$ of r elements in G, called a history of length r. We say the circuit c passes through h if for some s, $c(s + t) = x_t$, $t = 1, 2, \cdots, r$. Let w_c be a non-negative number for each circuit c in a class of r-elementary circuits C. An order-r Markov chain in G, $X_1, X_2, \cdots$, is formed by letting the probability that $X_{n+1} = x$, given the history of the

last r steps is $h = (X_{n-r+1}, X_{n-r+2}, \cdots, X_n)$, $n \geq r$, be proportional to total weight of the circuits in C which pass through h and then through x. In an obvious notation,

$$p(h \to x) = \frac{\text{Sum of } w_c \text{ such that } c \text{ passes through } h, x}{\text{Sum of } w_c \text{ such that } c \text{ passes through } h} \tag{2.4}$$

This process may of course be thought of as a chain in G^r, the space of histories of length r, and it follows that in this space, the weight out of any history h is equal to the weight in, and therefore the total weight of the circuits which pass through h is an invariant weight function for the class of such histories.

Moreover, every stationary order r Markov chain in the space of such histories can be represented by a finite class of r-elementary circuits and weights w_c for each c in the class. The details of this approach to order-r chains are given in the paper just mentioned.

While every order-r chain has such a basis in the form of a finite class of circuits, it may be possible to find a particularly simple class and then subsequent analysis of the chain may be facilitated. This happened in the case of the order-2 simple chain which is treated in Section 3.

(5) The induced field particle system. Suppose there are N distinguished particles designated by the integers, $1, 2, \cdots, N$, each particle i moving in its own finite graph G_i. The state of the system will by an N-tuple, $x = (x_1, x_2, \cdots, x_N) \in G^N = G_1 \times G_2 \times \cdots \times G_N$ representing the locations of the N particles. Let x/i be an element in $G^N/G_i = G_1 \times \cdots G_{i-1} \times G_{i+1} \cdots \times G_N$ representing the locations of the $N-1$ particles excluding particle i. For each i and x/i, let $w = w(i, x/i, r, s)$ be a non-negative weight function on the arcs (r, s) in $G_i \times G_i$ which is balanced in r and s, that is,

$$\sum_s w(i, x/i, r, s) = \sum_s w(i, x/i, s, r) \tag{2.5}$$

for all x/i and $r \in G_i$. The function w so defined is called an "induced field".

If x is the state of the system at some given time, an arc (r, s) in G_i will be *available* to particle i if $x_i = r$ and $w(i, x/i, r, s) > 0$. Let $A_i = A_i(x/i)$ be the arcs available to particle i, and let $A = A(x)$ be the union of all the sets A_i.

Given that x is the state, a new state is produced by the transition rule, that is, one of the available arcs $A(x)$ is selected with probability proportional to its weight, and if (x_i, s) is selected, particle i is moved to s in G_i. This means that the probability (x_i, s) is selected is

$$w(i, x/i, x_i, s)/v(x), \tag{2.6}$$

where $v(x) = \sum_i \sum_s w(i, x/i, x_i, s)$ is the total induced weight out of x. A sequence of steps generated this way evidently produces a Markov chain in

G^N with transition probabilities given by (2.6). This process is the induced field particle system.

It is now almost immediate that v gives an invariant weight function for the process. This may be seen by thinking of the process as a single particle moving in a graph G where we have defined a weight function $w'(x,y)$ on arcs (x,y) in $G \times G$, x of the form $(x_1, x_2, \cdots, x_N)$ and y of the form $(x_1, x_2, \cdots, x_{i-1}, s, x_{i+1}, \cdots x_N)$, for some i and $s \in G_i$, which represents a move of one particle, and the weight $w'(x,y)$ on this arc is the induced weight $w(i, x/i, x_i, s)$. The probability of such a move is then $w'(x,y)/v(x)$ for y of the just indicated form, and the induced field process is a Markov chain using the balanced weight function w', just as described in Section 1.

To check that $w'(x,y)$ is balanced, note the total weight out of x is just $v(x)$ as we have seen, and the total into x, $\sum_y w'(y,x)$ is the sum of all the weights on arcs (y,x) of the form just defined, which can go to x in one step, that is, where y is of the form $(x_1, x_2, \cdots, x_{i-1}, s, x_{i+1}, \cdots x_N)$ for s in G_i, for some i. This total weight in is just $\sum_i \sum_s w(i, x/i, s, x_i) = v(x)$ by the assumption w is balanced in r and s.

It follows from the balance, of course, that v is an invariant function for the induced field process.

Induced fields w are also additive, but a new kind of weighted sum is available for construction. If $w(i, x/i, r, s)$ is balanced in r and s, so is $\alpha(x/i)w(i, x/i, r, s)$ where $\alpha = \alpha(i, x/i)$ is any non-negative function of the locations of the other particles, that is, the particles other than particle i.

This simple device is used repeatedly in the "meeting process" described in Section 3.

3. Examples.

3.1. The order-2 simple chain. The classical finite state simple chain is a process $X_1, X_2, \cdots$, on the integers $0, 1, \cdots, N$ where $X_{n+1} = X_n \pm 1$ a.s. with probabilities depending on the current state, and with reflection at 0 and N. This process is also known as a birth-and-death chain and the nearest-neighbor random walk. It has been treated by a good number of writers, e.g., Karlin(1956) and Gndenko(1962), and it is well understood.

The order-2 simple process also has $X_{n+1} = X_n \pm 1$, but now the transition probabilities depend on the last two values of the process so that the states are ordered pairs (x, x') with $x - x' = \pm 1$. Reflection at 0 and N is still in place.

The recurrence equations and arguments used in the order-1 case do not appear to apply immediately or easily to the order-2 case. For this reason the process offered an interesting challenge for the possibility of Markov sculpture. So it was decided to seek a basis in circuits for the process in hope of finding a particularly simple and useful one. The search proved successful, and the stationary distribution was then obtained immediately

in an explicit and surprisingly simple form. Also, with the circuits basis at hand it was possible to obtain formulas for analogues of the formulas already known for the order-1 case. These include formulas, in terms of the initial state, (an ordered pair) for the probability of hitting N before 1, the expected gain received up until hitting N for the first time, or hitting N or 1 for the first time, when a gain $g(x, x \pm 1)$ is received on going from x to $x \pm 1$. These further results are described in working papers (MacQueen,1981a, 1981b) and will not be given here.

It is convenient to take the parameters of the process to be the probabilities of changing direction. Write $p(x', x'' \to x)$ for $P(X_{n+1} = x \mid X_n = x'', X_{n-1} = X')$, a convenient and compact notation for circuit based probabilities. Then let

$$\begin{aligned} a_1 &= p(0, 1 \to 0) & b_1 &= p(2, 1 \to 2) \\ a_2 &= p(1, 2 \to 1) & b_2 &= p(3, 2 \to 3) \\ &\vdots & &\vdots \\ a_{N-1} &= p(N-2, N-1 \to N-2) & b_{N-1} &= p(N, N-1 \to N). \end{aligned}$$

In seeking a circuit basis, a promising intuition was that it might be possible as well as desirable to have a single circuit giving the numerator in each transition probability. This led to an economical circuit basis, which is shown schematically in Figure 1 for the case $N = 5$.

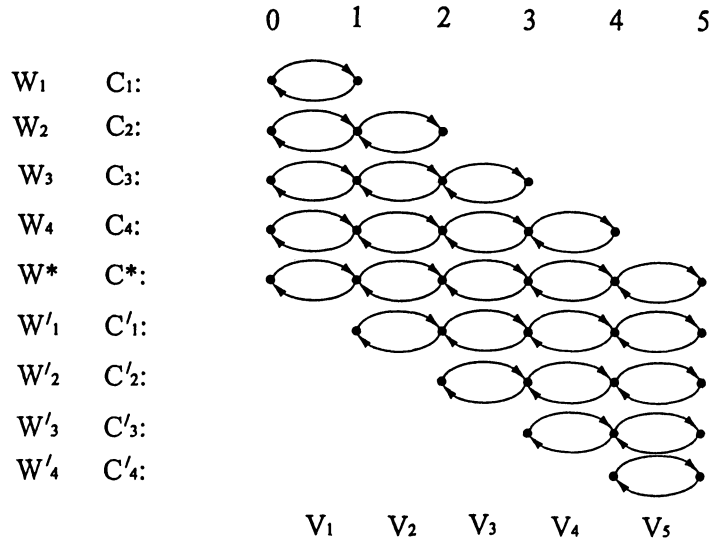


Figure 1.

As the drawing suggests, c_1 is the circuit $0, 1$, then back to 0, c_2 is the circuit $0, 1, 2, 1$, then back to 0, etc., while $c*$ goes all the way to N and back, and c_1' is the circuit $1, 2, 3, 4, 5, 4, 3, 2$ then back to 1, etc. The notation for the weights of these circuits is given in the left hand column. The variables

$V_1, V_2, \cdots, V_5$ are the totals of the weights of all the circuits which pass through the order-2 histories in the column above these variables. Thus $V_1 = w_1 + w_2 + w_3 + w_4 + w*, V_2 = w_2 + w_3 + w_4 + w_1' + w*$, etc.

Notice V_1 is the weight of all the circuits going through $0,1$ and at the same time it is the total weight going through $1,0$, and similarly for $1,2$ and $2,1$, etc. So V_i, $i = 1, 2, \cdots, N$ provides an invariant weight function for all the order-2 histories by virtue of the remark about the circuit basis in Section 2. Note the two histories $0,1$ and $1,0$ will have the same invariant weight, as will the two histories $1,2$ and $2,1$, etc. This is obvious in retrospect, since for every transition from x to $x+1$ there eventually has to be one from $x+1$ to x.

It remains then to calculate the V_i in terms of the transition probabilities a_i and b_i.

To do this notice that c_1 with weight w_1 is the only circuit that can lead to a change in direction when going up from 0 to 1, this being the only circuit through $0,1,0$. Then $p(0,1 \to 0) = a_1 = w_1/V_1$ by the transition rule, and in general,

$$a_n = w_n/V_n, \quad b_n = w_n'/V_{n+1}, \quad n = 1, 2, \cdots, N-1. \tag{3.1}$$

To solve for the w_n, the w_n' and $w*$, observe that $V_{n+1} = V_n - w_n + w_n'$, and from this, using (3.1), gives $V_{n+1} = V_n - a_n V_n + b_n V_{n+1}$, which yields the recursion $V_{n+1} = V_n(1 - a_n)/(1 - b_n)$, $n = 1, 2, \cdots, N-1$.

Since only ratios of weights matter, we are free to pick one weight or some sum of weights for convenience. So take $V_1 = 1$. Then the recursion gives

$$V_{n+1} = \frac{(1-a_1)(1-a_2)\cdots(1-a_n)}{(1-b_1)(1-b_2)\cdots(1-b_n)}. \tag{3.2}$$

With $V_1 = 1$ and V_n given by the above for $n = 2, 3, \cdots, N$, the weights w_n and w_n' can be found from (3.1). Finally, $V_1 = 1 = w_1 + w_2 + \cdots w_{N-1} + w*$ gives $w*$. Thus we have a set of circuits and weights which generate the original process, and (3.2) gives the invariant weights V_i for the order-2 histories.

Because the invariant probabilities for $x, x+1$ and $x+1, x$ are equal, as we have seen, it may be possible to solve the appropriate stationary equation directly as was done for the order 1 case. This is left as an exercise for the reader. Verification, at least, ought to be fairly easy, by just putting the above formula in the stationary equations.

Circuit bases have been found for a few other order-2 simple chains, mainly the case where staying at x is permitted, and the case where the values of the process are the "clock" of non-negative integers mod N.

3.2. The Meeting Process. Informally, the meeting process is as follows: Each of N individuals i, $i = 1, 2, \cdots, N$, are traveling from location

to location, in a Markovian way, in respective graphs G_i, $i = 1, 2, \cdots, N$. But there are special sets of locations called situations, to use the sociological expression, and if their travels should bring several of the individuals into a situation at the same time, they commence to have a meeting. This breaks up after a random time, depending on the situation and the individuals in it, who then continue on in their usual Markovian way. While a meeting is in process, the other individuals at other locations not involved in this meeting continue to move in their usual way, and in fact one or several groups of the other individuals may meet in some other situation and commence a meeting. So there is a constant movement of individuals in and out of meetings of different kinds in different places.

The meeting process developed below is a refined version of this process, with special and completely stationary structure, but it will be seen that it has many of the important qualitative features of the every day social reality of "meetings", which are ubiquitous and form the very core of social and economic life.

The stationary distribution for the meeting process is obtained in an explicit form in terms of the expected length of the interactions in each situation.

The main application of the meeting process, is as a carrier process for games, and in particular, time variable games, where the strategies of the players determine the length of play of the game as well as the utility outcomes. Such a game is placed in each situation, and its play constitutes the meeting there. While each situation has a unique and fixed game, the play will be different each time it is played because of the generic random features in the games. The expected times of play become the parameters needed for the stationary distribution of the meeting process, which then becomes a function of the strategy choices in the game. It turns out, as will be described elsewhere, that the stationary distribution of the meeting process is easily applied in this case, and this makes it possible to apply game theoretic ideas, such as Nash equilibria, to the resulting model for social and economic systems.

The meeting process is an application of the induced field process described in Section 2 although the term "individual" is used here rather than "particle" in view of the sociological interpretation.

The general setup is illustrated in Figure 2, where there are four individuals moving in separate graphs. As indicated in the figure, situations are disjoint sets S of locations

Let $k(S)$ be the number of locations in situation S and let $N(S)$ be the set of $k(S)$ individuals thus associated with situation S by virtue of the fact that each element in a situation belongs to the graph of one individual.

Also let $t(S)$ be the expected length of the meeting in S, which must

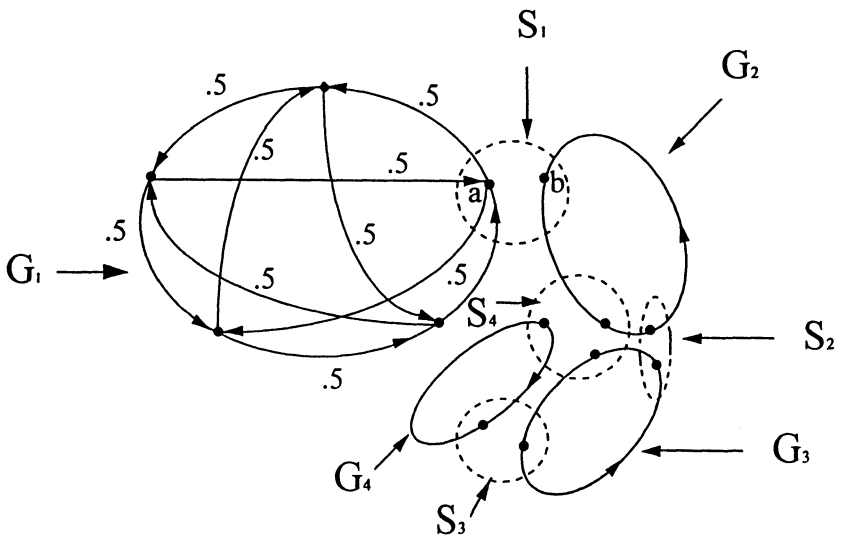


Figure 2.

satisfy $t(S) \geq 1/k(S)$. The reason for the latter condition will become clear shortly. The actual time each meeting requires is a random variable, whose distribution is given below.

The meeting process will be built up from a process representing the motion of the individuals when not in any meetings. These are defined by balanced weight functions $w_{0,i}$, $i = 1, 2, \cdots, N$,with $w_{0,i}(r, r) = 0$ for all i and r, which to say there are no self-loops, and

$$\sum_s w_{0,i}(r, s) = \sum_s w_{0,i}(s, r) = 1, i = 1, 2, \cdots, N. \tag{3.3}$$

Suppose the induced field was defined by $w_{0,i}$ alone, that is suppose $w(i, x/i, r, s)$ was equal to $w_{0,i}(r, s)$. Then the transition rule says individual i will have probability $1/N$ of being selected to move and if selected and if at r, will move to s with probability $w_{0,i}(r, s)$. This will be the meaning of moving "as usual" and this process will be called the w_0 process.

However, we are free to pick the time scale arbitrarily, and it is simplest to set it equal to $1/N$ units of time per step. This evidently means that in the w_0 process an individual would move on the average of once per unit time, since the expected number of steps between selection of each individual is N. This is an intuitively comfortable scale, since as N is varied the individuals do not change speed.

An important convention is adopted for the start and end of meetings, which is just that the meeting in S starts at the first step on which all the individuals who contribute locations to S are present. And the meeting ends at the step on which any one of the individuals leaves S.

But to illustrate how the meeting process will be developed, consider S_1 in Figure 2, which has just two elements, a and b, a from G_1 and b from G_2, so the presence of these two individual will initiate a meeting there. The induced field will be described for the special condition individuals 1 and 2 are at a and b respectively in a meeting, but in addition, individuals 3, 4, and 5 are *not* in any meeting.

For this condition there will be an induced self-loop with weight $2t(S_1) - 1$, for individual 1 at a, induced by individual 2 at b, which to say that if $x/1$ has the property that $x_2 = b$, then $w(1, x/1, a, a) = 2t(S_1) - 1$, and the reason for this specific quantity will be explained shortly. Similarly, if $x/2$ is such that $x_1 = a$, then $w(2, x/2, b, b) = 2t(S_1) - 1$. At the same time $w(1, x/2, a, r), r \neq a$, will be $w_{0,1}(a, r)$ and $w(2, x/2, b, r)$, $r \neq b$, will be $w_{0,2}(b, r)$. These last two statements mean there will some weight for individual 1 and 2 to leave the situation, thus ending the meeting.

Suppose that at the same time, $w(i, x/i, r, s) = \alpha(i, x/i)w_{0,i}(r, s)$, for $i = 3, 4$, and $r \neq s$, where α is chosen to keep these individual moving "as usual", that is, as in the w_0 process. It turns out that $\alpha = 2t(S_1)$ will work nicely.

This completes the description of the induced field for this one condition, and to illustrate how it works we calculate the expected time until the meeting ends on the assumption the condition is maintained until this happens.

The weights of the two self loops at a and b are $2t(S_1) - 1$ and the total weight out of a and out of the situation, for individual 1, is 1, and the weight out of b and out of the situation for individual 2 is 1 also, these each being the sum on s of $w_{0,1}(a, s)$ and of $w_{0,2}(b, s)$ which are both equal to 1 by the hypotheses on w_0. The total weight out of the locations of individuals 3 and 4, is $2(2t(S_1))$ because the sum on s of each $w_{0,i}(r, s), i = 3, 4$, is 1 and these are each multiplied by $\alpha = 2t(S_1)$. Altogether then, the weight for the condition we are considering is $2(2t(S_1) - 1) + 2 + 2(2t(S_1)) = 8t(S_1)$.

Since it is agreed the meeting in S_1 ends when any one individual leaves, and since the arcs for individuals 1 and 2 which have this result each have a weight of 1, the probability of one of them leaving at each step is $2/(8t(S_1)) = 1/(4t(S_1))$ as long as this condition continues. So under this condition the number of steps until one of the two individuals leaves would be a geometric random variable, and the expected number of steps until this happened would be just $4t(S_1)$ by the well known formula for the mean of the geometric distribution.

But the time scale was chosen above to be 1/4 a unit of time per step, so the expected time until the meeting ends is just $t(S_1)$ as desired.

The probability under this same condition that individuals 3 and 4, are selected to move are each $2t(S_1)/(8t(S_1)) = 1/4$ and so theseindividual are

moving at an average rate of 1 step per unit time because it takes on average 4 steps before each is selected to move. This we describe as moving at the "usual rate" and it is also as desired.

Of course, the above discussion does not cover the case where either the meeting in S_1 ends or another meeting involving the other individuals starts. It turns out that the induced field can be developed so that the probability a meeting ends can be kept constant just as in this instance, and with the appropriate parameter, so that even when other meetings start or stop the number of steps until a given meeting ends is as desired.

It will turn out, incidentally, that the individual who ends the meeting is equally likely to be any one of the individuals present, which seems quite in keeping with the programmatic description of the process given above.

Using the same idea as illustrated above, the general induced field for the meeting process will now be developed, but first several somewhat technical definitions are needed:

(1) For each i and x/i fixed, let $W(i, x/i)$ be the set of situations S which have at least one location from G_i, but which also have present all the individuals $N(S)$ except individual i, so that if individual i arrives a meeting would start. In a manner of speaking the individuals in a situation S in $W(i, x/i)$ are waiting for individual i, regardless of and in ignorance of, where individual i actually is. The set $W(i, x/i)$ can easily be empty.

(2) For each i and x/i, there is a set of situations $M(i, x/i)$ with meetings in process in each, none of which involve individual i, that is, there are no locations from G_i in any of these situations. This set of situations can have no overlap with $W(i, x/i)$ since if the locations in $S \in M(i, x/i)$ are all occupied, none of the individuals in it can be simultaneously at an element of $W(i, x/i)$ waiting for individual i to arrive. Let $\pi(i, x/i)$ stand for the product of $k(S)t(S)$ over S in $M(i, x/i)$, defined to be 1 if the set is empty.

(3) Let $M(x)$ be the set of situations where meeting are in process when x is the state of the system.

The definition of w is in two pieces: For a given individual i and for fixed x/i and for r in S and S in $W(i, x/i)$ not empty, let

$$w(i, x/i, r, r) = (k(S)t(S) - 1)\pi(i, x/i), \tag{3.4}$$

and if $r \neq s$, let

$$w(i, x/i, r, s) = w_{0,\,i}(r, s)\pi(i, x/i), \tag{3.5}$$

otherwise, that is, if r is not in S for any S in $W(i, x/i)$ or $W(i, x/i)$ is empty, let

$$w(i, x/i, r, s) = w_{0,\,i}(r, s)\pi(i, x/i) \tag{3.6}$$

for all $r \neq s$.

Notice that the effect of the above is that the induced field in a situation comes from all the individuals required for the meeting there except the one person who is absent. The self-loop field which keeps the meeting going does not operate, if less than $k(S)-1$ individuals are present, and this is why the meeting does not start until all its potential members are present.

Calculating the behavior using w much as was done above, will soon reveal that the meeting in each situation S has expected length $t(S)$ and given the meeting starts it is geometrically distributed and independent of the locations of the other individual until such time as it ends. Moreover, each individual not in a meeting continues to move as usual, that is with constant probability of $1/N$ of being the next person to move or attempt to move.

That w as defined by the above is balanced in r and s, is immediate. The expression (3.4) gives an induced self-loop and is automatically balanced. The terms (3.5) and (3.6) are obtained by multiplying the balanced weight function $w_{0,i}$ by a function α which depends only on x/i, and are also automatically balanced as was pointed out in Section 2.3.

But what is the invariant weight? For the induced field process, $v(x) = \sum_i \sum_s w(i, x/i, x_i, s)$ is invariant and so it is only necessary to calculate this from the above definition of w. The result is

$$v(x) = N\pi(x), \tag{3.7}$$

where $\pi(x)$ is the product of $k(S)t(S)$ for all S in $M(x)$, the set of all situations where there are meetings in process when x is the state.

To show this, consider x fixed, and consider the contribution to the total weight out of x from individual i who is in a meeting S. In this case individual i has arrived in $S \in W_i(x/i)$ and the available arcs (x_i, s) have their weight given by (3.4), for an immediate return, and by (3.5), to leave the situation. The total is $(k(S)t(S)-1)\pi(i, x/i)$ from (3.4) plus $\sum_s w_{0,i}(xi, s)\pi(i, x/i) = \pi(i, x/i)$ from (3.5), for a total of $k(S)t(S)\pi(i, x/i)$. But this is just $\pi(x)$ the product over all S in $M(x)$ since $\pi(i, x/i)$ is the product of $k(S)t(S)$ over the meetings in process excepting those involving individual i, or any other individual in the situation S under consideration.

But there are $k(S)$ individuals in S so their total contribution is $k(S)\pi(x)$.

The total weight out of locations of individuals not in any meeting is given directly by (3.6) and is just $\pi(x)$. The total then is $\sum_s k(S)\pi(x) + (N - \sum_s k(S))\pi(x))$ where the sum on S is taken over the situations where meetings are in process when x is the state. The second term in this sum is just the number of individual not in any meeting, multiplied by their contribution of $\pi(x)$ each to the total weight out of x. The total is evidently $N\pi(x)$ as was to be shown.

We note in passing that the discrete meeting process can easily be

changed to a continuous time Poisson process $X(t)$ in G, with transitions at Poisson times $t_1, t_2, \cdots$, with occurrence rate $1/N$, and with the same basic structure and the same stationary distribution. It is only necessary to associate i.i.d exponential random times with each individual and to take as the individual who next moves or attempts to move, the one whose time is a minimum, and let the actual move of this individual be governed by the induced field. Details are left to the reader.

The phenomenological properties of the continuous time version seem to be somewhat better than the discrete time process, because the sense that individual movements are independent of one another except in meetings is more vivid.

The stationary distribution for the meeting process gives a variety of statistics by direct and easy calculation using (3.7), $v(x) = N\pi(x)$, or more simply, using just $\pi(x) = \prod_{S \in M(x)} k(S)t(S)$ since N is a constant. For example, the stationary probability there is a meeting in S_1 but no other meeting going on at the same time, is proportional to $5k(S_1)t(S_1) = 10t(S_1)$, because there are 5 states x where this is the condition of the system, that is, a meeting in S_1 only. Similarly, the fraction of time there is a meeting in S_1 and a meeting in S_3 is proportional to $4t(S_1)t(S_3)$ since there is just 1 state x where this occurs. These relative weights will become probabilities on dividing by $D = 10t(S_1) + 20t(S_2) + 38t(S_3) + 4t(S_1)t(S_3) + 15t(S_4) + 80$. The probability there are no meetings going on is $80/D$.

3.3. Small circuit construction of h-diffusions. There is continuing interest in diffusions in many branches of science and an enormous mathematical literature on their construction and analysis. This section attempts to make a small contribution to this subject by finding the invariant distribution for a class of non-reversible processes called h-diffusions. The results may be of some interest to theorists since there is little available on the invariant behavior on non-reversible diffusions. Here a closed form solution for the invariant behavior of a class of two dimensional processes is given. The method is elementary, and extends easily to h-diffusions in several dimensions. The small circuits required in several dimensions are indicated briefly below.

Consider the set S_N of points (x, y) where $x = i/N, y = j/N$, $0 \leq i, j \leq N$, and let S_N^* be the subset of S_N where $0 < i, j < N$. The set $S = [0,1] \times [0,1]$ and its interior $S^* = (0,1) \times (0,1)$ also play important roles here.

We construct a Markov chain in S_N represented by a moving particle which starting at (x, y) goes either to an adjacent point such as $(x, y+h)$, $h = 1/N$, or returns immediately to (x, y), but in any case taking an amount of time h^2 for the step. This process will be called an h-diffusion, to distinguish it from the classical diffusion processes.

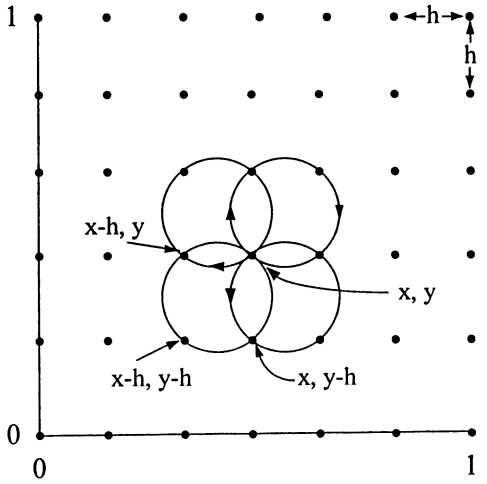
The construction is based on many small circuits in S_N, described below, and two non-negative weight functions w and s where w is twice differentiable and satisfies $w \geq \delta > 0$ on S and $s \geq 0$ is continuous. For each $h = 1/N$, the construction gives rise to "diffusion coefficients" d_h and σ_h^2 representing the expected velocity vector of the particle and the covariance matrix for a single step of the process expressed as usual on a per unit time basis. Thus technically, the construction may be regarded as a mapping C_h on the space of pairs w, s into the space of these diffusion coefficients d_h and σ_h^2. It is immediate from continuity that if $x, y \in S^*$, then as $h \to 0$, $d_h \to d$, $\sigma_h^2 \to \sigma^2$. So $C_h \to C$ in a sense easily made precise.

It is also shown that under certain conditions on a given pair, (d, σ^2) representing the mean velocity vector and covariance per unit time of an empirical process of interest, there is a unique pair w, s for which $C(w, s) = (d, \sigma^2)$, and then $C_h(w, s)$ defines for each $h = 1/N$ an h-diffusion with a known invariant weight function v_h.

This discrete process embodies the information in the given pair (d, σ^2) in a concrete form, suitable for interpretation and application to the process giving rise to d and σ^2.

The given pair (d, σ^2) may come from a number of sources, for example, as part of an approximation of another stochastic process, or it may be given as an axiom, postulated in hopes of getting better understanding of some natural process. So the pair does not have come from a mathematical diffusion, although this is one possibility. In any case the h-diffusion derived from the given pair (d, σ^2) is a possible model the process at hand, and its value stands or falls, as the case may be, on its interpretation and application.

To begin the construction, consider Figure 3 illustrating the set S_N.



The four circuits through x,y, drawn as circles for clarity

Figure 3.

The circuits for the construction are as follows:

First, there will be a clockwise and a counter clockwise circuit associated with each point (x, y), where $0 \leq x < 1$, $0 \leq y < 1$. A typical clockwise circuit is indicated in Figure 3. The circuit originates at (x, y), goes to $(x, y + h)$, then to $(x + h, y + h)$, then to $(x + h, y)$, then back to (x, y), and in that order. The counter clockwise circuit, not shown, follows the same points but in reverse order, that is, starting at (x, y) it goes through $(x + h, y)$ then $(x + h, y + h)$, then $(x, y + h)$, then back to (x, y).

There is also at each such point (x, y) a self-loop (not shown in Figure 3), that is, a circuit which goes from (x, y) back to (x, y) immediately. No other circuits are used in this construction.

The weight function w and s mentioned above are used to assign weight to each of the above circuits as follows: Let a, b be fixed numbers with $a + b = 1$. For each (x, y) in the above range where the circuits originate, that is, $0 \leq x < 1, 0 \leq y < 1$, we place weight $aw(x, y)$ on the clockwise circuit originating there (that is, on all of its arcs), place weight $bw(x, y)$ on the counter clockwise circuit originating there, and put weight $s(x, y)$ on the self-loop at (x, y).

Notice that all boundary points receive some weight. For example, the point $(1, 1)$ gets weight from the two circuits originating at $(1 - h, 1 - h)$.

The construction is given here for the set $[0, 1]^2$. But any set whose boundary is a sufficiently smooth closed curve can be used provided a certain convention is followed in weighting the circuits. For example, a piecewise linear boundary with a finite number of pieces will work easily. The convention is that the boundary be addressed by a method called "cutting and stitching": The boundary of the set is placed on a lattice graph with mesh $1/N$, and any point inside or on the boundary, and with an arc that is cut by the boundary, is declared to be a boundary point. Then the arcs of circuits originating at these points are stitched together where they were cut, leaving out that part of the circuit which is outside the boundary. The weights on these remaining arc segments are left just as they were. So each point inside the boundary is still balanced, and in fact has exactly the same weight as before. The solution worked out for the open interior S^* holds exactly as given for the case $S = [0, 1]^2$.

Consider now a point (x, y) in the interior of S_N, that is, S_N^*. As may be seen by inspection with the help of Figure 3, the total weight out of such an interior point comes from just four clockwise circuits, those originating at (x, y), at $(x - h, y)$, at $(x - h, y - h)$, and at $(x, y - h)$, plus the weight of the counter-clockwise circuits originating at these same four points, plus, finally, the weight on the self-loop. These nine circuits all go through such points x, y, and no other circuits do, as may be checked by inspection.

For $(x, y) \in S_N^*$ there is a certain total weight on each of the available

arcs leading to adjacent points. These weights to $(x, y+h)$, $(x+h, y)$, $(x, y-h)$, $(x-h, y)$, (x, y) are, respectively, $aw(x, y) + bw(x-h, y)$, $aw(x, y-h) + bw(x, y)$, $aw(x-h, y-h) + bw(x, y-h)$, $aw(x-h, y) + bw(x-h, y-h)$, and $s(x, y)$.

The total weight out of $(x, y) \in S_N^*$ from all nine circuits, considering $a + b = 1$ is

$$v_h(x, y) = s(x, y) + w(x, y) + w(x, y-h) + w(x-h, y) + w(x-h, y-h). \tag{3.8}$$

The above weights for available arcs out of (x, y) give transition probabilities on being divided by $v_h(x, y)$. For example the probability the particle will go to $x, y + h$ is $(aw(x, y) + bw(x-h, y))/v_h(x, y)$. Probabilities from boundary points have similar formulas. For example the probability of going from $(0, 0)$ to $(h, 0)$ is $aw(0, 0)/[aw(0, 0) + bw(0, 0) + s(0, 0)]$.

Because of the balance, v_h as defined by (3.8) above, is an invariant weight function on interior points, and similar invariant weights for boundary points are easily given. For example, $w(0, 0) + s(0, 0)$ is an invariant weight for $(0, 0)$ (on the same scale as v_h.)

It is not hard to check that the h-diffusion is not reversible unless $a = b$, by calculating the probability of the path $(x, y), (x + h, y)$, for example, and comparing it to the probability of the path $(x + h, y), (x, y)$. But if $a = b$ so the weight of the clockwise and counter clockwise circuits are equal, and the rotational effects of the circuits cancel out, then the process is easily seen to be reversible.

Notice that for h sufficiently small, either a or b may be negative without violating the positivity of the weight function v_h. If b, say, is negative, the fact that $a + b = 1$ means that the weight of out of x, y along any available arc, such as $w' = aw(x, y) + bw(x-h, y)$ above, is positive for sufficiently small h. To see this note $w(x-h, y) = w(x, y) - hw_1(x, y) - O(h^2)$, so w' is equal to $w(x, y) - hw_1(x, y) - O(h^2)$. Since w_1 is continuous on the compact set S and therefore bounded, and $w \geq \delta > 0$ on S, w' will be uniformly positive on S for h sufficiently small.

The self-loop weight s is bounded above by hypothesis, but may easily be zero at any or even all interior points.

Consider now the expected velocity vector d_h for a single step of the particle starting from an interior point x, y, and taking time h^2 for the step. The first coordinate $d_{1h}(x, y)$, defined as the component in the horizontal direction of the expected velocity vector for a single step, may be calculated using the probabilities indicated above, and is found to be

$$d_{1h}(x, y) = \frac{a((w(x, y-h) - w(x-h, y)) + b(w(x, y) - w(x-h, y-h))}{hv_h(x, y)}, \tag{3.9}$$

where v_h is given by (3.8). In this formula the division by h^2 has already been accomplished and a factor h canceled from the numerator and denominator. A similar formula holds for $d_{2h}(x,y)$, the expected velocity in the y direction.

From the transition probabilities, one can also calculate the covariance matrix for the next position x', y' starting from x, y. By definition this will be the covariance of the random vector $x, y+h'$ where h' takes on one of the five possible values $(0,0)$, $(h,0)$, $(-h,0)$, $(0,h)$, $(0,-h)$ with the corresponding transition probabilities. The covariance matrix per unit time for one step of the process, σ_h^2, is just the covariance of this random vector, but divided by h^2. The exact formula will not be given, since the rest of the argument requires only limiting values on S^*.

The limiting values of d_h and σ_h^2 are easily found for (x,y) in S^* by taking limits along convergent sequences in S_N^*. Specifically, in (3.9), adding and subtracting $w(x,y)$ and $w(x,y-h)$, from the terms within the parentheses, in an obvious way, treating d_{2h} likewise, and then letting $h \to 0$ and simplifying, we get

$$\begin{aligned} d_1 &= [w_1 - (a-b)w_2]/(s+4w), \\ d_2 &= [w_2 + (a-b)w_1]/(s+4w). \end{aligned} \tag{3.10}$$

A similar detailed calculation which is omitted, shows the variances in each direction are equal to each other in the limit, and the off diagonal term is zero. Let the common variance term be σ^2, which it turns out is given by

$$\sigma^2 = 2w/(s+4w). \tag{3.11}$$

The function $v = s+4w$ is the limiting value of v_h, the exact invariant weight function for each h, so this will be well approximated by v in the sense that sums on v_h over nice sets will be equal approximately to integrals of v over those sets. In effect, v_h converges weakly to v on S^*. Note $\sigma^2 \leq 1/2$.

An essential simplification is now possible. From (3.11), $s+4w$ is equal to $2w/\sigma^2$. Define the function u by setting $w = exp(2u)$, and let $k = a-b$. Putting this in (3.10) and simplifying, we get

$$\begin{aligned} d_1/\sigma^2 &= \alpha = u_1 - ku_2, \\ d_2/\sigma^2 &= \beta = u_2 + ku_1, \end{aligned} \tag{3.12}$$

where α and β are introduced to simplify notation.

Consider now the problem of recovering w and s, and the constant k, from some given d and covariance σ^2 having equal diagonal term and satisfying $\sigma^2 \leq 1/2$, and being strictly positive every where on S.

Rearranging (3.12) gives the gradient condition

$$\alpha + k\beta = u_1(1+k^2), \beta - k\alpha = u_2(1+k^2). \tag{3.13}$$

This must hold on S^* if α and β are to be derived from the circuit construction we are considering, since (3.12) was derived on this basis. But if

(3.13) holds then u can be computed by a standard formula for recovering a function from its gradient.

Differentiation shows it is both necessary and sufficient for (3.13) that for some k,

$$\alpha_2 + k\beta_2 = \beta_1 - k\alpha_1 \quad \text{or} \quad \alpha_2 - \beta_1 = -k(\alpha_1 + \beta_2). \tag{3.14}$$

There may be no such k and in this case the construction cannot produce the given d and σ^2.

Suppose then, there is at least one value of k such that (3,14) holds. Then u is given by

$$u = (\int_1 f + \int_2 g - \int_1 \int_2 f_2)/(1 + k^2) + c_0, \tag{3.15}$$

c_0 an arbitrary constant, $f = \alpha + k\beta$, $g = \beta - k\alpha$ and where for an integrable function R of x and y, $\int_1 R = \int_a^x R(s, y)ds$ and $\int_2 R = \int_b^y R(x, t)dt$, a and b being constants chosen for convenience.

But it is possible there are many values of k for which (3.14) holds. In fact,it turns out that the pair α, β falls into one of two classes depending on whether or not they are harmonic, by which is meant α and β satisfy the Cauchy-Riemann conditions,

$$\alpha_2 = \beta_1, \alpha_1 = -\beta_2. \tag{C}$$

If they do, then in (3.14) the coefficient of k is zero and k cannot be determined uniquely without further conditions.

For a specific example of this state of affairs, that is, the gradient condition (3.14) satisfied and α and β are harmonic, let $\sigma^2 = 1/2$, $\alpha = -2y$, and $\beta = -2x$. Then (3.14) holds for some k, and in fact for all k, but so does (C) as is easily checked. Then $f = -2y - 2kx$, $g = -2x + 2ky$ and with $\sigma^2 = 1/2$, (3.11) implies $s = 0$. Using the integration formula (3.15) gives $u = (-2kx^2 + 2ky^2 - 4xy)/(1 + k^2)$ plus an arbitrary constant and then from the definition of u, $w = \exp(2u)$ up to multiplication by a positive constant. So as k varies the h-diffusions have the same mean velocity vector but are clearly different in their behavior, as is evident from inspection of w.

In an effort to obtain a unique construction from d and σ^2, it is tempting to impose boundary conditions, although how to do so is not clear. The behavior on the boundary seems to have little effect on the interior. For example, a circuit with weight 1, say, could be passed through all the boundary points in a clockwise direction. The construction would still be possible, and the relative weight as between sets of interior points would not be changed at all.

Notice that in the harmonic case, differentiating the first line of (3.13) with respect to x and the second with respect y, and adding, shows u itself

is harmonic. If a boundary condition of the form $u = h$ on the boundary were made available, producing a classical Dirichlet problem, there would be a unique solution (up to an additive constant) equal to h on the boundary. But there is no obvious source for such a boundary condition.

So the meaning of this harmonic condition (C) when the gradient condition (3.14) is satisfied, is left for further study.

Now consider the condition where (3.14) is satisfied for some k but C fails at some point in S^*. Then in (3.14) not both $\alpha_2 - \beta_1$ and $\alpha_1 + \beta_2$ are zero at this point and a little consideration shows that (3.14) is then a simple linear condition determining a unique value of k. And in fact if (3.14) is satisfied for any k, it is satisfied with the same k over all of S^*, and this k is unique. Then the integration (3.15) determines u uniquely up to an additive constant. And then w and s are uniquely determined up to a multiplicative constant, and finally $s = 2w(1/\sigma^2 - 2)$.

It is not hard to see that if the given d and σ^2 satisfy the regularity conditions given above, having continuous derivatives on S, and σ^2 being bounded away from zero, then w and s will also satisfy their regularity conditions as given above. So the limits required by the construction will exist.

In summary, then, for a given h, the h-diffusion is available for interpretation and application, its exact invariant distribution is known, and its coefficients d_h and σ_h^2 converge to the given d and σ^2 as $h \to 0$.

Note this is all true without having to investigate the limiting process, as $h \to 0$, or even its existence.

An example. In the study of market dynamics processes are used in which the parameters of one diffusion process are themselves subject to diffusion. An alternative to the usual approach to such processes, usually via-stochastic differential equations (see, e.g. Cox and Ross(1985)) is provided by this example. The example is chosen in part because it is not a reversible process, which appears to be the case, almost inevitably, in the usual approaches.

Suppose the expected velocity or "drift" of a particle in one dimension is itself is subject to many small impulses and moves randomly with expected velocity depending on both its current value and the location of the particle. Such phenomena, it would seem, are very common, so creating such a process is a practical challenge as well as a mathematical one.

For the sake of the example, the set S taken as the square $[-B, B]^2$, B some large positive number.

We go directly to (3.12) taking σ^2 a constant equal to its maximum value of $1/2$ and interpret x as the position of a particle in $[-B, B]$.

More importantly, we set $d_1 = y$. This makes y itself random and by the construction and the transition rule, the value of y is moving as a component

of the h-diffusion just as is the particle. Then d_2, which is the expected velocity of the y coordinate of the process is in effect the expected rate of change of the expected velocity d_1 of the particle.

With $d_1 = y$ and the constant variance $\sigma^2 = 1/2$, so $\alpha = 2y$, it turns out that the gradient condition (3.14) will be satisfied for some k if $\beta = -2y + \phi(x + y)$, ϕ a function of a single variable, and then $k = 1$. So for the example we settle on a simple instance, which is $\phi = -(x + y)$. The harmonic condition (C) is not in effect since $\alpha_1 = 0$ and $\beta_1 \neq 0$. Then $\beta = -x - 3y$ and the functions f and g in (3.15) are $f = -x - y, g = -5y - x$ and u is given uniquely up to an additive constant by the integration (3.15). So taking the additive constant to be zero, $u = (-x^2 - 2xy - 5y^2)$ and then $w = exp(2u)$ is a bivariate normal. It can be written as $exp(-1/2(y + x/5)^2/(1/20))\exp(-1/2(x^2)/(5/18))$ to show the regression of y on x.

So under this invariant weight function, the expected value of y, the mean velocity of the particle, given x, is very close to $-x/5$ and would be exactly so as B, the edge of the square, goes to ∞. This expresses in another way what may be gleaned from $\beta = -3y - x$, which is that there is a stabilizing feed back tendency in the process. If the drift y of the particle itself were driven positive under the many highly energetic pulses, the acceleration β would go negative, and this would eventually turn the particle around and head it towards zero. The regression function $y = -x/5$ says this same thing with a certain precision.

In summary, the h-diffusion process promises to be easy to use in practice. All that is needed technically is a little calculus and the theory of finite state Markov chains. The concrete nature of the construction makes it clear what is going on, and should make it relatively easy to develop a model having given local behavior as described by diffusion coefficients, whatever their source.

Interacting particle systems formulated as h-diffusions can be treated easily in some instances by adopting the induced field model to this situation. Some examples of this are given in a working paper (MacQueen,1980).

Clearly, the h-diffusion construction is based on and constrained by the family of small circuits used. Families of small circuits quite different from the above are described briefly by MacQueen(1984,1985).

To extend the above construction to several dimensions, the above circuits can be used, but applying them to each plane. For example, with three dimensions, x, y, z, the clockwise circuits used above in the x, y plane are left as they are, but exactly analogous circuits are applied in the x, z plane and in the y, z plane as well. Thus in the x, z plane there is a clockwise circuit originating at (x, y, z), which goes through $(x, y, z + h)$, $(x + h, y, z + h)$, $(x + h, y, z)$, and back to (x, y, z). Counterclockwise circuits with weights bw, $a + b = 1$, are treated similarly, and a self-loop with weight $s = s(x, y, z)$

may be added if desired.

4. Acknowledgment.

David Blackwell has been a basic source of inspiration for the work described above and more specifically, he has been a steady source of inspiration in the direction of striving for phenomenological clarity. I learned from him to try and understand a problem as clearly as possible, so that formal proof is hardly necessary. I have heard it said, and have found it true, that Blackwell's explanations of mathematical results in lectures or in conversation, are often as good or even better than mathematical proofs in leading to rigorous understanding and at the very least with good understanding a formal proof is not far away. If there is any contribution to the understanding of Markov processes in the above work, it is largely due to my following his leadership in this regard.

My colleagues Tom Ferguson and Lester Dubins have also been important in this work. I owe them both a great deal.

REFERENCES

COX, J. C., INGERSOLL, J. E., AND ROSS, S. A. (1985). A theory of term structure of interest rates. *Econometrica* **53** 385-407.

DOOB, J. L. (1953). *Stochastic Processes.* New York, John Wiley & Sons, Inc.

GNENDENKO, B. V. (1962). *The Theory of Probability.* Chelsea Publishing Co., New York.

KARLIN, S. (1956). Classification of birth and death processes. *Trans. Amer. Math. Soc.*

MACQUEEN, J. (1980). Stationary distributions for interacting diffusions using induced fields. Notes for seminars given in the Statistical Laboratory, Dec. 5, 1979, and in the probability seminar in mathematics, UCLA, May 5, 1980, edited so as to be self contained. Privately distributed Working Paper, April 1981.

MACQUEEN, J. (1981). Circuit processes. *Annals of Probability* **9** 604-610.

MACQUEEN, J. (1981). The order two simple Markov chain: An analysis using circuit processes. Institute of Mathematical Statistics Bulletin, Abstract, September 1981. Privately distributed working paper with same title available on request from the author.

MACQUEEN, J. (1984). Construction of two-dimensional diffusions using the level curves of an analytic function. Institute of Mathematical Statistics Bulletin, September, 1984, Abstract, p.154.

MACQUEEN, J. (1985). Small circuit construction of market-like processes. Institute of Mathematical Statistics Bulletin, Abstract 85t-78.

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