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POINT PROCESSES WITHOUT TOPOLOGY

MARK FINKELSTEIN, HOWARD G. TUCKER, AND JERRY ALAN VEEH

University of California Irvine, University of California Irvine, and Auburn University

Abstract

The basic theory of point processes, including the theory of marked Poisson processes, is developed here under the sole assumption that the mean measure of the process is sigma finite. No other measure theoretic assumption is made. No topological structure is imposed on the state space of the process.

To David Blackwell,
who with his characteristically concise sentences taught me, among other things, how to write a mathematics paper, how to look at mathematics, how to welcome responsibility and how to face one's more mature years, this paper is affectionately dedicated. [H.G.T.]

1. Introduction. The natural mathematical framework for the theory of point processes, or, more generally, for the theory of random measures, is one in which only measure theoretic considerations play a role. Some of the existing theory of point processes, however, seems to depend on a combination of both measure theoretic and topological conditions. The objective of this paper is to introduce a mathematical setting for the theory of point processes in which no topology is needed on the state space of the process. Some of the most basic aspects of the theory, including the theory of marked Poisson processes, can in fact be developed in this more general and natural setting without any additional effort. References for the usual theory include Kallenberg (1983) and Resnick (1987), both of whom utilize to some extent a metric space structure on the state space when proving theorems like the ones below. Kingman (1993) develops the theory of Poisson processes, including the theory of marked Poisson processes, in a setting very much like ours, but with additional conditions imposed on both the state space sigma algebra and the mean measure.

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2. Point Processes. Suppose E is a non-empty set and $\mathcal{E}$ is a sigma algebra of subsets of E. A **point measure** ω over the measurable space $(E, \mathcal{E})$ is a measure determined by a finite or denumerable sequence $\{x_n\}$ of not necessarily distinct points in E such that, for every $F \in \mathcal{E}$, $\omega(F)$ is the number of points of the sequence that belong to F. Let M denote a non-empty set of point measures over $\mathcal{E}$, and let $\mathcal{M}$ be the σ-algebra of subsets of M defined by

$$\mathcal{M} = \sigma\{\{\omega \in M : \omega(F) \leq k\} : F \in \mathcal{E}, 0 \leq k \leq \infty\}.$$

Let $\mathcal{F}$ be the set of extended real valued non-negative measurable functions with domain E. For arbitrary $f \in \mathcal{F}$ and $\omega \in M$ denote

$$<f, \omega> = \int_E f \, d\omega = \sum_n f(x_n) \leq \infty,$$

where $\{x_n\}$ is a sequence determining (or determined by) ω. Note that for each $f \in \mathcal{F}$ the mapping $\omega \mapsto <f, \omega>$ is $\mathcal{M}$-measurable.

Associated with a probability measure P on the space $(M, \mathcal{M})$ is its **mean measure** μ_P defined by

$$\mu_P(F) = \int_E \omega(F) P(d\omega)$$

for $F \in \mathcal{E}$. The mean measure will be seen to play an important role in the development below. In fact, the usual topological requirement of σ-compactness of the state space will be seen to be replaced by the requirement of the σ-finiteness of the mean measure of the point process.

Definition. A probability measure P on $(M, \mathcal{M})$ is said to be a **point process** over $(E, \mathcal{E})$ if its mean measure μ_P is σ-finite.

Sometimes we shall just say that $(M, \mathcal{M}, P)$ is a point process when P is a point process over $(E, \mathcal{E})$.

Note that this definition is *not* equivalent to the definition of a point process as a stochastic process $\{N(A), A \in \mathcal{E}\}$ defined on some probability space $(\Omega, \mathcal{A}, P)$, where for every $\omega \in \Omega$, $N(\cdot)(\omega)$ is a non-negative integer-valued measure over $\mathcal{E}$. This is because this alternate definition does not imply the σ-finiteness of the mean measure of the process N. For example, let K be any non-negative integer valued random variable having infinite expectation and let e be any fixed point in E. The stochastic process $\{K(\omega)\delta_e(A), A \in \mathcal{E}\}$ clearly satisfies this alternate definition, but fails to have a σ-finite mean measure.

In this section the Laplace functional $\hat{P}$ of a point process $(M, \mathcal{M}, P)$ will be defined, and we shall prove that $\hat{P}$ uniquely determines P.

We recall two definitions and some useful results connected with them.

Definition. A subset $\mathcal{I}$ of $\mathcal{E}$ is called a **π-system** if $\mathcal{I}$ is closed under intersections.

Definition. A subset $\mathcal{J}$ of $\mathcal{E}$ is called a **λ-system** if $\mathcal{J}$ satisfies:

(1) $E \in \mathcal{J}$,
(2) if $A, B \in \mathcal{J}$ and $A \subset B$ then $B \setminus A \in \mathcal{J}$, and
(3) if $A_n \in \mathcal{J}$ and $A_n \subset A_{n+1}$ for $n = 1, 2, \ldots$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{J}$.

Dynkin's Theorem. *If $\mathcal{P}$ is a π-system, if $\mathcal{L}$ is a λ-system, and if $\mathcal{P} \subset \mathcal{L}$, then $\sigma\{\mathcal{P}\} \subset \mathcal{L}$.*

Corollary to Dynkin's Theorem. *If $\mathcal{P}$ is a π-system, if P and Q are probability measures over $\sigma\{\mathcal{P}\}$, and if $P(A) = Q(A)$ for all $A \in \mathcal{P}$, then $P = Q$ over $\sigma\{\mathcal{P}\}$.*

Proofs of the preceding two results can be found in Billingsley (1986).

Lemma 1. *Let $(M, \mathcal{M}, P)$ be a point process, and suppose that $\mathcal{I} \subset \mathcal{E}$ satisfies*

(1) *$\mathcal{I}$ is a π-system,*
(2) *$\sigma\{\mathcal{I}\} = \mathcal{E}$, and*
(3) *there exists a sequence $\{E_n\}$ in $\mathcal{I}$ such that $E_n \subset E_{n+1}$, $\mu_P(E_n) < \infty$ for $n = 1, 2, \ldots$, and $\cup_{n=1}^{\infty} E_n = E$.*

Let $\mathcal{N} = \sigma\{\{\omega \in M : \omega(I) \leq k\}, I \in \mathcal{I}, k \geq 0\}$, and let $(M, \overline{\mathcal{M}}, \overline{P})$ and $(M, \overline{\mathcal{N}}, \overline{\overline{P}})$ be the unique completions of $(M, \mathcal{M}, P)$ and $(M, \mathcal{N}, P|_{\mathcal{N}})$ respectively. Then $\overline{\mathcal{M}} = \overline{\mathcal{N}}$.

Proof. It is immediate from the definitions that $\mathcal{N} \subset \mathcal{M}$, and so $\overline{\mathcal{N}} \subset \overline{\mathcal{M}}$. We shall prove $\overline{\mathcal{M}} \subset \overline{\mathcal{N}}$. Denote

$$\mathcal{G}_n = \{F \in \mathcal{E} : \{\omega \in M : \omega(F \cap E_n) \leq k\} \in \overline{\mathcal{N}} \text{ for all } k \geq 0\}.$$

We prove a sequence of claims.

Claim 1: For all n, $\mathcal{I} \subset \mathcal{G}_n$.
Proof: Let $H \in \mathcal{I}$. By hypothesis, $E_n \in \mathcal{I}$, and $\mathcal{I}$ is a π-system. Hence $H \cap E_n \in \mathcal{I}$. By the definition of $\mathcal{N}$, $\{\omega \in M : \omega(H \cap E_n) \leq k\} \in \mathcal{N} \subset \overline{\mathcal{N}}$ for all $k \geq 0$. Hence by the definition of $\mathcal{G}_n$, $H \in \mathcal{G}_n$, which proves Claim 1.

Claim 2: $E \in \mathcal{G}_n$ for all n.
Proof: Since $E_n \subset E$, we have $\{\omega \in M : \omega(E \cap E_n) \leq k\} = \{\omega \in M : \omega(E_n) \leq k\}$. By hypothesis, $E_n \in \mathcal{I}$, and by the definition of $\mathcal{N}$, $\{\omega \in M : \omega(E_n) \leq k\} \in \mathcal{N} \subset \overline{\mathcal{N}}$ for all $k \geq 0$. Hence by the definition of $\mathcal{G}_n$, $E \in \mathcal{G}_n$, which proves Claim 2.

Claim 3: Let $M_1 = \{\omega \in M : \omega(E_n) < \infty, \text{ for all } n\}$. Then $M_1 \in \mathcal{N}$ and $P(M_1) = 1$.

Proof: By hypothesis, $E_n \in \mathcal{I}$ for all n. Thus for each n we have (by the definition of $\mathcal{N}$) that $\{\omega \in M : \omega(E_n) \leq k\} \in \mathcal{N}$. Since $\mathcal{N}$ is a σ-algebra, we have

$$\bigcup_{k=1}^{\infty} \{\omega \in M : \omega(E_n) \leq k\} = \{\omega \in M : \omega(E_n) < \infty\} \in \mathcal{N}.$$

For the same reason

$$\bigcap_{n=1}^{\infty} \{\omega \in M : \omega(E_n) < \infty\} = \{\omega : \omega(E_n) < \infty \text{ for all } n\} \in \mathcal{N}.$$

Hence $M_1 \in \mathcal{N}$. In order to prove $P(M_1) = 1$, we use the hypothesis $\mu_P(E_n) < \infty$ for all n and the fact that $\mu_P(E_n) = \int_M \omega(E_n)P(d\omega)$ to obtain $P(\{\omega \in M : \omega(E_n) < \infty\}) = 1$. Hence $P(M_1) = 1$, which proves Claim 3.

Claim 4: For every n, $\mathcal{G}_n$ is closed under proper differences.
Proof: Let F_1 and F_2 be members of $\mathcal{G}_n$, and such that $F_1 \subset F_2$. We wish to prove that $F_2 \setminus F_1 \in \mathcal{G}_n$, i.e., that $\{\omega \in M : \omega((F_2 \setminus F_1) \cap E_n) \leq k\} \in \overline{\mathcal{N}}$ for all $k \geq 0$. For all $\omega \in M$,

$$\omega(F_2 \cap E_n) = \omega((F_2 \setminus F_1) \cap E_n) + \omega(F_1 \cap E_n).$$

Note that for all $\omega \in M_1$, we may write

$$\omega((F_2 \setminus F_1) \cap E_n) = \omega(F_2 \cap E_n) - \omega(F_1 \cap C_n),$$

since all three terms are finite by the definition of M_1. Since $\{\omega \in M : \omega(F_i \cap E_n) \leq k\} \in \overline{\mathcal{N}}$ for $i = 1, 2$, and since by Claim 3, $M_1 \in \overline{\mathcal{N}}$, it follows that $M_1 \cap \{\omega \in M : \omega(F_2 \setminus F_1) \leq k\} \in \overline{\mathcal{N}}$, for all $k \geq 0$. Since $P(M_1) = 1$ by Claim 3, it follows that $(M \setminus M_1) \cap \{\omega \in M : \omega(F_2 \cap F_1) \leq k\} \in \overline{\mathcal{N}}$, since this is a subset of $M \setminus M_1 \in \mathcal{N}$ and $P(M \setminus M_1) = 0$. Hence

$$\{\omega \in M : \omega((F_2 \setminus F_1) \cap E_n) \leq k\} \in \overline{\mathcal{N}}$$

for all $k \geq 0$. Therefore, $F_2 \setminus F_1 \in \mathcal{G}_n$, proving Claim 4.

Claim 5: If $A_k \in \mathcal{G}_n$, and if $A_k \subset A_{k+1}$ for $k = 1, 2, \ldots$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{G}_n$.
Proof: By the definition of $\mathcal{G}_n$, since $A_k \in \mathcal{G}_n$, then $A_k \in \mathcal{E}$ and $\{\omega \in M : \omega(A_k \cap E_n) \leq r\} \in \overline{\mathcal{N}}$ for all $r \geq 0$. Hence the function $\omega \mapsto \omega(A_k \cap E_n)$ is $\overline{\mathcal{N}}$-measurable. Since each $\omega \in M$ is a measure, and since $A_k \subset A_{k+1}$ for each k, it follows that $\omega(A_k \cap E_n) \uparrow \omega((\cup_{j=1}^{\infty} A_j) \cap E_n)$ as $k \to \infty$. Now, the monotone limit of a sequence of $\overline{\mathcal{N}}$-measurable functions is $\overline{\mathcal{N}}$-measurable,

so $\{\omega \in M : \omega((\cup_{k=1}^{\infty} A_k) \cap E_n) \leq r\} \in \overline{\mathcal{N}}$ for all $r \geq 0$. Since $\cup_{k=1}^{\infty} A_k \in \mathcal{E}$, it follows that $\cup_{k=1}^{\infty} A_k \in \mathcal{G}_n$, which proves Claim 5.

We now prove the lemma. By Claims 2, 4, and 5, $\mathcal{G}_n$ is a λ-system. By Claim 1, $\mathcal{I} \subset \mathcal{G}_n$, and by Dynkin's theorem, $\sigma\{\mathcal{I}\} \subset \mathcal{G}_n$. By hypothesis, $\sigma\{\mathcal{I}\} = \mathcal{E}$, so $\mathcal{E} \subset \mathcal{G}_n$. But also $\mathcal{G}_n \subset \mathcal{E}$. Hence $\mathcal{G}_n = \mathcal{E}$. This implies that for every $F \in \mathcal{E}$, $\{\omega \in M : \omega(F \cap E_n) \leq k\} \in \overline{\mathcal{N}}$ for all $k \geq 0$. Let $F \in \mathcal{E}$ be fixed, and define $f_n : M \to \{0, 1, 2, \ldots, \infty\}$ by $f_n(\omega) = \omega(F \cap E_n)$. Thus f_n is $\overline{\mathcal{N}}$-measurable. Since $E_n \subset E_{n+1}$ for all n and since each $\omega \in M$ is a measure, it follows that $f_n(\omega) \uparrow f(\omega)$ as $n \to \infty$ for all $\omega \in M$, where $f(\omega) = \omega(F \cap (\cup_{n=1}^{\infty} E_n)) = \omega(F \cap E) = \omega(F)$. Hence f is $\overline{\mathcal{N}}$-measurable for every $F \in \mathcal{E}$. This means $\{\omega \in M : \omega(F) \leq k\} \in \overline{\mathcal{N}}$. Thus $\sigma\{\{\omega \in M : \omega(F) \leq k\}, F \in \mathcal{E}, k \geq 0\} \subset \overline{\mathcal{N}}$, and $\mathcal{M} \subset \overline{\mathcal{N}}$. From this it follows that $\overline{\mathcal{M}} \subset \overline{\mathcal{N}}$. □

Theorem 1. *Let $(M, \mathcal{M}, P)$ be a point process over $(E, \mathcal{E})$, and suppose $\mathcal{I} \subset \mathcal{E}$ satisfies*

(1) *$\mathcal{I}$ is a π-system,*
(2) *$\sigma\{\mathcal{I}\} = \mathcal{E}$, and*
(3) *there exists a sequence $\{E_n\} \subset \mathcal{I}$ such that $E_n \subset E_{n+1}$, $\mu_P(E_n) < \infty$ for all n, and $\cup_{n=1}^{\infty} E_n = E$.*

Let Q be a probability measure over $(M, \mathcal{M})$ which satisfies

$$P(\cap_{j=1}^{k}\{\omega \in M : \omega(I_j) \leq n_j\}) = Q(\cap_{j=1}^{k}\{\omega \in M : \omega(I_j) \leq n_j\})$$

for every $k \geq 1$ and every $I_1, \ldots, I_k$ in $\mathcal{I}$ and $n_1 \geq 0, \ldots, n_k \geq 0$. Then $P = Q$ over $\mathcal{M}$.

Proof. Let us define

$$\mathcal{J} = \{\cap_{j=1}^{k}\{\omega \in M : \omega(I_j) \leq n_j\} : I_j \in \mathcal{I}, n_j \geq 0, 1 \leq j \leq k, k \geq 1\}.$$

Clearly, $\mathcal{J}$ is a π-system. Let $\mathcal{N} = \sigma\{\mathcal{J}\}$. Now since P and Q are probability measures over $(M, \mathcal{M})$ by hypothesis, by the Corollary to Dynkin's theorem their restrictions to $\mathcal{N}$ are probability measures over $(M, \mathcal{N})$, so $P(F) = Q(F)$ for all $F \in \mathcal{N}$. Hence $P = Q$ over $\overline{\mathcal{N}}$, and since by Lemma 1, $\overline{\mathcal{N}} = \overline{\mathcal{M}}$, it follows that $P = Q$ over $\overline{\mathcal{M}}$. Since $\mathcal{M} \subset \overline{\mathcal{M}}$, then $P = Q$ over $\mathcal{M}$. □

If $(M, \mathcal{M}, P)$ is a point process, we define its **Laplace functional** $\hat{P} : \mathcal{F} \to [0, 1]$ by $\hat{P}(f) = \int_M e^{-<f,\omega>} P(d\omega)$ for all $f \in \mathcal{F}$. We now show that if two point processes have the same Laplace functional, they are identical.

Theorem 2. *Let $(M, \mathcal{M}, P)$ and $(M, \mathcal{M}, Q)$ be point processes and suppose $\hat{P}(f) = \hat{Q}(f)$ for every $f \in \mathcal{F}$. Then $P = Q$ over $\mathcal{M}$.*

Proof. Let $\mathcal{I} = \{F \in \mathcal{E} : \mu_P(F) < \infty$ and $\mu_Q(F) < \infty\}$. Since both P and Q are point processes, μ_P and μ_Q are σ-finite. Thus with little effort one can show that there exists a sequence $\{E_n\}$ in $\mathcal{E}$ such that $E_n \subset$

E_{n+1}, $\bigcup_{n=1}^{\infty} E_n = E$, $\mu_P(E_n) < \infty$, and $\mu_Q(E_n) < \infty$, for all n. Thus $\{E_n\} \subset \mathcal{I}$. Clearly, $\mathcal{I}$ is a π-system. Also, $\sigma\{\mathcal{I}\} = \mathcal{E}$. (Take any $F \in \mathcal{E}$. Then $F = \cup_{n=1}^{\infty}(F \cap E_n)$. But $F \cap E_n \in \mathcal{I}$ for each n since $\mu_P(F \cap E_n)\mu_Q(F \cap E_n) \leq \mu_P(E_n)\mu_Q(E_n) < \infty$ for all n, so $F \in \sigma\{\mathcal{I}\}$.) Now let $F_1, \ldots, F_k$ be any finite collection of elements of $\mathcal{I}$, and let $f \in \mathcal{F}$ be defined by $f(x) = \sum_{j=1}^{k} \lambda_j I_{F_j}(x)$ for $\lambda_j \geq 0$, $1 \leq j \leq k$. For every $\omega \in M$, $<f, \omega> = \int_E f(x)\omega(dx) = \sum_{i=1}^{k} \lambda_i \omega(F_i)$. By hypothesis, $\hat{P}(f) = \hat{Q}(f)$, so

$$\int_M e^{-\sum_{j=1}^{k} \lambda_j \omega(F_j)} P(d\omega) = \int_M e^{-\sum_{j=1}^{k} \lambda_j \omega(F_j)} Q(d\omega).$$

Since $\mu_P(F_i)\mu_Q(F_i) < \infty$, $1 \leq i \leq k$, it follows that the random variables $\omega \mapsto \omega(F_i)$ are finite a.e. $[P]$ and $[Q]$. Hence by the uniqueness of Laplace transforms over $\mathbf{R}^k$

$$P(\cap_{j=1}^{k}\{\omega \in M : \ \omega(F_j) \leq n_j\}) = Q(\cap_{j=1}^{k}\{\omega \in M : \ \omega(F_j) \leq n_j\})$$

for all $k \geq 1$ and $F_1, \ldots, F_k$ in $\mathcal{I}$. Now apply Theorem 1 to conclude that $P = Q$ over $\mathcal{M}$. □

3. Poisson Random Measures. Let E, $\mathcal{E}$, M, $\mathcal{M}$, and $\mathcal{F}$ be as above. Each $F \in \mathcal{E}$ determines an extended-valued random variable $X_F(\omega) = \omega(F)$ for all $\omega \in M$. Thus the set $\{X_F : F \in \mathcal{E}\}$ is a set of measurable functions over $(M, \mathcal{M})$.

Definition. A point process $(M, \mathcal{M}, P)$ is called a **Poisson random measure** with mean measure μ_P if

(1) for every $k \geq 2$, and for arbitrary disjoint sets $F_1, \ldots, F_k$ in $\mathcal{E}$, the random variables $X_{F_1}, \ldots, X_{F_k}$ are independent, and
(2) the distribution of X_F is Poisson with expectation $\mu_P(F)$ for all $F \in \mathcal{E}$.

In exactly the same way as for a topological state space E one obtains that if $(M, \mathcal{M}, P)$ is a Poisson random measure, then its Laplace functional $\hat{P}(f)$ is

$$\hat{P}(f) = \exp\{-\int_E (1 - e^{-f(x)})\mu_P(dx)\}$$

for $f \in \mathcal{F}$. This is proved in Resnick (1987), as is the following converse: If $\phi : \mathcal{F} \to [0, 1]$ is a function defined by

$$\phi(f) = \exp\{-\int_E (1 - e^{-f(x)})\mu(dx)\}.$$

for some σ-finite measure μ over $(E, \mathcal{E})$, then there exists a set of point measures M over $(E, \mathcal{E})$ and a probability measure P over $(M, \mathcal{M})$ such that $(M, \mathcal{M}, P)$ is a Poisson random measure with mean measure $\mu_P = \mu$.

The converse above was obtained by the following construction in two cases.

Case I: $0 < \mu(E) < \infty$. Standard methods establish the existence of a probability space $(\Omega, \mathcal{A}, \Pi)$ for which the following are true. There is a Poisson distributed $(\mu(E))$ random variable Y defined over it; independent of Y there is then defined a sequence of independent and identically distributed random elements $\{X_j\}$ taking values in E such that for each $F \in \mathcal{E}$, $\Pi(\{\omega \in \Omega : X_1(\omega) \in F\}) = \mu(F)/\mu(E)$. Now for each $\omega \in \Omega$, define

$$\omega(F) = \sum_{j=1}^{Y} I_{[X_j \in F]}(\omega).$$

The set Ω is thus a set M of point measures, with $\mathcal{M} \subset \mathcal{A}$. The probability Π becomes the Poisson random measure P with mean measure μ.

Case II: $\mu(E) = \infty$. Write E as a disjoint union $E = \cup_{n=1}^{\infty} E_n$ where each $E_n \in \mathcal{E}$ and $0 < \mu(E_n) < \infty$. For each E_n let $\mathcal{E}_n = \{F \cap E_n : F \in \mathcal{E}\}$ and let μ_n be a measure defined over $(E_n, \mathcal{E}_n)$ by $\mu_n(F \cap E_n) = \mu(F \cap E_n)$. For each $(E_n, \mathcal{E}_n, \mu_n)$, define as in Case I above $(\Omega_n, \mathcal{A}_n, \Pi_n)$, Y_n, and $\{X_{n,j},\ j \geq 1\}$. Then over the product measure space $(\Omega, \mathcal{A}, \Pi) = \prod_{n=1}^{\infty}(\Omega_n, \mathcal{A}_n, \Pi_n)$, define for each $\omega = (\omega_1, \omega_2, \dots) \in \prod_{n=1}^{\infty} \Omega_n$ and each $F \in \mathcal{E}$, $\omega(F) = \sum_{n=1}^{\infty} \omega_n(F \cap E_n) = \sum_{n=1}^{\infty} \sum_{j=1}^{Y_n} I_{[X_{n,j} \in F \cap E_n][Y_n \geq 1]}(\omega)$. Thus Ω becomes a set of point measures. Denoting this set by M, letting $\mathcal{M}$ be as previously defined, and letting P be the restriction of Π to $\mathcal{M}$, $(M, \mathcal{M}, P)$ is a Poisson random measure with Laplace functional $\phi(\cdot)$ and mean measure μ. A proof of all this with no topological assumptions on E is the same as that proved by Resnick, with topological assumptions.

What is of interest to us here is a representation theorem for the point process. Kallenberg (1983) proved that any point process over $(E, \mathcal{E})$, where E has suitable topological structure, can be represented as

$$\omega(F) = \sum_{j=1}^{K} I_{[X_j \in F]}(\omega)$$

for all $\omega \in \Omega$ for some fixed denumerable sequence of random elements X_j taking values in E, and where K is a non-negative, integer-valued random variable. The method of proof used by Kallenberg depended heavily on the topological structure of E.

What we do here is make the following representation: Let $(M, \mathcal{M}, Q)$ be a Poisson random measure with mean measure μ over $(E, \mathcal{E})$, and represent this random measure as the stochastic process $\{X_F,\ F \in \mathcal{E}\}$, where as above, $X_F(\omega) = \omega(F)$ for all $F \in \mathcal{E}$. Then there exists a Poisson random measure $\{N(A),\ A \in \mathcal{E}\}$ defined over some probability space $(\Omega, \mathcal{A}, P)$ with mean measure μ such that the joint distributions of $\{X_F : F \in \mathcal{E}\}$ and

$\{N(A) : A \in \mathcal{E}\}$ are the same, and there exists a denumerable sequence of random elements $\{X_n\}$ defined over Ω and taking values in E, and a random variable Y taking values in $\{0, 1, 2, \dots, \infty\}$ such that

$$N(A)(\omega) = \sum_{n=1}^{Y} I_{[X_n \in A][Y \geq 1]}(\omega)$$

for all $\omega \in \Omega$. Another way of stating this is as follows.

Theorem 3. *For every σ-finite measure μ over $(E, \mathcal{E})$, there is a Poisson random measure $(M, \mathcal{M}, P)$ with mean measure μ, and a denumerable sequence of random elements $\{X_n\}$ taking values in E defined over M, and there exists a random variable Y defined over M, taking values in $\{0, 1, 2, \dots, \infty\}$ such that*

$$X_F = \sum_{n=1}^{Y} I_{[X_n \in F][Y \geq 1]}$$

for all $F \in \mathcal{E}$.

Proof. In case $0 < \mu(E) < \infty$, the proof is exactly the same as in Resnick. But suppose $\mu(E) = \infty$. Since μ is σ-finite, there exists a denumerable sequence $\{E_n\}$ of disjoint members of $\mathcal{E}$ such that $E = \cup_{n=1}^{\infty} E_n$ and $0 < \mu(E_n) < \infty$ for all n. Let $(\Omega, \mathcal{A}, \Pi)$, $\{E_n\}$, $\{\{X_{nj}\}\}$, and $\{Y_n\}$ be as outlined above, so that $\{X_F,\ F \in \mathcal{E}\}$ as defined by

$$X_F = \sum_{n=1}^{\infty} \sum_{n=1}^{Y_n} I_{[X_{nj} \in F \cap E_n][Y_n \geq 1]}$$

for all $F \in \mathcal{E}$ is the desired Poisson random measure. We shall construct $\{X_n\}$ and prove that $X_F = \sum_{n=1}^{\infty} I_{[X_n \in F]}$ for all $F \in \mathcal{E}$. In the construction above, all of the random elements in $\{\{Y_n, X_{n1}, X_{n2}, \dots\}, n = 1, 2, \dots\}$ are, by construction, independent. One consequence of the independence of the Y's is that $\sum_{n=1}^{\infty} Y_n = \infty$ a.e. This follows from the fact that $1 - e^{-x} \geq (x \wedge 1)/e$ for $x \geq 0$ and the computation, using the fact that the Y's are Poisson, $\sum_{n=1}^{\infty} P(Y_n \geq 1) = \sum_{n=1}^{\infty}(1 - e^{-\mu(E_n)}) \geq \sum_{n=1}^{\infty}(1 \wedge \mu(E_n))/e$. If only finitely many of $\{\mu(E_n)\}$ are larger than 1, this last sum is comparable to $\sum_{n=1}^{\infty} \mu(E_n) = \infty$, while if infinitely many of $\{\mu(E_n)\}$ are greater than 1 this last sum is infinite too. Thus $[Y_n \geq 1$ infinitely often$]$ is an event of probability 1, by the Borel-Cantelli Lemma. In the remainder of this discussion we shall assume that the associated set of zero probability has been discarded. Now, define non-negative integer-valued random variables $\{r_n\}$ and $\{m_n\}$ as follows:

$$r_n = \max\{t : \sum_{j=1}^{t-1} Y_j < n\}$$

and

$$m_n = n - \sum_{j=1}^{r_n - 1} Y_j.$$

We observe that $1 \le m_n(\omega) \le n$ for all $\omega \in \Omega$. Define $X_n = X_{r_n, m_n}$, i.e., for every $\omega \in \Omega$, $X_n(\omega) = X_{r_n(\omega), m_n(\omega)}(\omega)$, and observe that X_n as defined is measurable. Our object now is to prove that

$$\sum_{n=1}^{\infty} I_{[X_{r_n,m_n} \in F]} = \sum_{n=1}^{\infty} \sum_{j=1}^{Y_n} I_{[X_{nj} \in F][Y_n \ge 1]}.$$

This will give the desired representation of the theorem with $Y = \infty$. In order to show this, we make some preliminary observations about r_n. From the definition of r_n it follows that for any $j \ge 1$, $[r_n = j] \cap [Y_j = 0] = \emptyset$. Also,

$$[r_n = j] \cap [Y_j \ge 1] = \left[1 + \sum_{l=1}^{j-1} Y_l \le n \le \sum_{l=1}^{j} Y_l\right] \cap [Y_j \ge 1].$$

These facts suggest that when computing the left hand sum above, the integers should be partitioned into bins consisting of the integers between 1 and Y_1, between $Y_1 + 1$ and $Y_1 + Y_2$, ... Of course there will be no bin corresponding to any Y_j which is 0. Since $\sum_{n=1}^{\infty} Y_n = \infty$ it follows that each positive integer will fall into exactly one of the bins. The details of the computation are as follows.

$$\begin{aligned}
\sum_{n=1}^{\infty} I_{[X_{r_n,m_n} \in F]} &= \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} I_{[X_{r_l,m_l} \in F]} I_{[r_l = j]} I_{[Y_j \ge 1]} \\
&= \sum_{j=1}^{\infty} \left(\sum_{l=1+\sum_{k=1}^{j-1} Y_k}^{\sum_{k=1}^{j} Y_k} I_{[X_{r_l,m_l} \in F]} I_{[r_l = j]} \right) I_{[Y_j \ge 1]} \\
&= \sum_{j=1}^{\infty} \left(\sum_{l=1+\sum_{k=1}^{j-1} Y_k}^{\sum_{k=1}^{j} Y_k} I_{[X_{j,m_l} \in F]} I_{[r_l = j]} \right) I_{[Y_j \ge 1]} \\
&= \sum_{j=1}^{\infty} \sum_{l=1}^{Y_j} I_{[X_{j,l} \in F]} I_{[Y_j \ge 1]}
\end{aligned}$$

where the last equality follows from the definition of m_l. This completes the proof of the theorem. □

4. The Marking Theorem. The development of point processes over a state space that has no topology and the representation theorem for Poisson random measures over the same kind of state space given above lead to an approach to a proof of the Marking Theorem. This extends and in a sense completes the development provided by Resnick. There is much structure to this theorem, so it is necessary to isolate the hypotheses.

The Marking Theorem Conditions. Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two measurable spaces, and let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $\{X_n\}$ and $\{J_n\}$ be two denumerable sequences of random elements defined over Ω, and taking values in E_1 and E_2 respectively. Let H be a non-negative extended integer-valued random variable which is possibly infinite with positive probability. Assume that the stochastic process $\{N(F),\ F \in \mathcal{E}_1\}$ defined by $N(F) = \sum_{n=1}^{H} I_{[X_n \in F]}$ is a Poisson random measure with σ-finite mean measure μ. Further assume that H and the sequences $\{X_n\}$ and $\{J_n\}$ are related by

$$P([J_i \in F] | \{X_n\}, \{J_\alpha : \alpha \neq i\}, H) \overset{\text{a.s.}}{=} K(X_i, F)$$

for all $F \in \mathcal{E}_2$ and all $i \geq 1$, where $K : E_1 \times \mathcal{E}_2 \to [0,1]$ is a function satisfying

(1) $K(\cdot, F)$ is $\mathcal{E}_1$-measurable for all $F \in \mathcal{E}_2$, and
(2) $K(x, \cdot)$ is a probability measure over $\mathcal{E}_2$ for every $x \in E_1$.

Six lemmas provide the background for our proof of the Marking Theorem. Lemma 1 is a general result about conditioning; Lemmas 2 through 6 are based on hypotheses which include the Marking Theorem Conditions.

Lemma 1. *Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ be sub-sigma algebras of $\mathcal{A}$ in $(\Omega, \mathcal{A}, P)$, and suppose that $P(A|\mathcal{A}_2, \mathcal{A}_3) \overset{\text{a.s.}}{=} P(A|\mathcal{A}_2)$ for all $A \in \mathcal{A}_1$. Then*

$$P(D|\mathcal{A}_2, \mathcal{A}_3) \overset{\text{a.s.}}{=} P(D|\mathcal{A}_2)$$

for all $D \in \sigma\{\mathcal{A}_1, \mathcal{A}_2\}$.

Proof. Let $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ be arbitrary. Then by hypothesis and by properties of conditional expectation,

$$\begin{aligned} P(A_1 \cap A_2 | \mathcal{A}_2, \mathcal{A}_3) &\overset{\text{a.s.}}{=} I_{A_2} P(A_1 | \mathcal{A}_2, \mathcal{A}_3) \\ &\overset{\text{a.s.}}{=} I_{A_2} P(A_1 | \mathcal{A}_2) \\ &\overset{\text{a.s.}}{=} P(A_1 \cap A_2 | \mathcal{A}_2). \end{aligned}$$

For arbitrary $F \in \sigma\{\mathcal{A}_2, \mathcal{A}_3\}$ and $D \in \sigma\{\mathcal{A}_1, \mathcal{A}_2\}$, let us define $\phi(F, D) = \int_F P(D|\mathcal{A}_2, \mathcal{A}_3)\, dP$ and $\psi(F, D) = \int_F P(D|\mathcal{A}_2)\, dP$. We obtain by the identity above that $\phi(F, A_1 \cap A_2) = \psi(F, A_1 \cap A_2)$. Thus, $\phi(F, \cdot) = \psi(F, \cdot)$ over a π-system that generates $\sigma\{\mathcal{A}_1, \mathcal{A}_2\}$. Hence we may apply the corollary to Dynkin's theorem to obtain $\phi(F, D) = \psi(F, D)$ for any $F \in \sigma\{\mathcal{A}_2, \mathcal{A}_3\}$ and all $D \in \sigma\{\mathcal{A}_1, \mathcal{A}_2\}$. Hence for all $F \in \sigma\{\mathcal{A}_2, \mathcal{A}_3\}$ and all $D \in \sigma\{\mathcal{A}_1, \mathcal{A}_2\}$ it follows that $\int_F P(D|\mathcal{A}_2, \mathcal{A}_3)\, dP = \int_F P(D|\mathcal{A}_2)\, dP$. Since both integrands are $\sigma\{\mathcal{A}_2, \mathcal{A}_3\}$-measurable, the uniqueness of the Radon-Nikodym derivative implies that $P(D|\mathcal{A}_2, \mathcal{A}_3) \overset{\text{a.s.}}{=} P(D|\mathcal{A}_2)$. □

Lemma 2. *Under the conditions of the Marking Theorem, if* $f : E_1 \times E_2 \mapsto [0, 1]$ *is* $\mathcal{E}_1 \times \mathcal{E}_2$*-measurable, then*

$$E(f(X_i, J_i)|\{X_n : n = 1, 2, \dots\}, H) \overset{\text{a.s.}}{=} \int_{E_2} f(X_i, y) K(X_i, dy)$$

for all i.

Proof. From the identity

$$P([J_i \in F]|\{X_n\}, \{J_\alpha : \alpha \neq i\}, H) \overset{\text{a.s.}}{=} K(X_i, F)$$

for all $F \in \mathcal{E}_2$ and all i, we obtain, upon taking conditional expectations of both sides, given X_i, that

$$P([J_i \in F]|X_i) \overset{\text{a.s.}}{=} K(X_i, F).$$

Hence for all i and for all $F \in \mathcal{E}_2$,

$$P([J_i \in F]|\{X_n\}, \{J_\alpha : \alpha \neq i\}, H) \overset{\text{a.s.}}{=} P([J_i \in F]|X_i).$$

Let $\mathcal{A}_1 = \sigma\{J_i\}$, $\mathcal{A}_2 = \sigma\{X_i\}$ and $\mathcal{A}_3 = \sigma\{\{X_n : n \neq i\}, \{J_\alpha : \alpha \neq i\}, H\}$. Then the last identity above becomes

$$P([J_i \in F]|\mathcal{A}_2, \mathcal{A}_3) \overset{\text{a.s.}}{=} P([J_i \in F]|\mathcal{A}_2).$$

Applying Lemma 1 yields $P(D|\mathcal{A}_2, \mathcal{A}_3) \overset{\text{a.s.}}{=} P(D|\mathcal{A}_2)$ for all $D \in \sigma\{\mathcal{A}_1, \mathcal{A}_2\}$. Now $f(X_i, J_i)$ is measurable with respect to $\sigma\{\mathcal{A}_1, \mathcal{A}_2\}$ and is non-negative. Hence $f(X_i, J_i)$ is an everywhere monotone limit of a sequence of non-negative linear combinations of indicators of events in $\sigma\{\mathcal{A}_1, \mathcal{A}_2\}$, call it $\{Y_r\}$, where $Y_r = \sum_{j=1}^{r2^r} c_{rj} I_{G_{rj}}$ and $G_{rj} \in \sigma\{\mathcal{A}_1, \mathcal{A}_2\}$. Thus by the conditional form of the monotone convergence theorem

$$\begin{aligned}
E(f(X_i, J_i)|\{X_n\}, H) &= E(\lim_{r\to\infty} Y_r|\{X_n\}, H) \\
&\overset{\text{a.s.}}{=} \lim_{r\to\infty} E(Y_r|\{X_n\}, H) \\
&\overset{\text{a.s.}}{=} \lim_{r\to\infty} \sum_{j=1}^{r2^r} c_{rj} P(G_{rj}|\mathcal{A}_2, \mathcal{A}_3) \\
&\overset{\text{a.s.}}{=} \lim_{r\to\infty} \sum_{j=1}^{r2^r} c_{rj} P(G_{rj}|\mathcal{A}_2) \\
&\overset{\text{a.s.}}{=} \lim_{r\to\infty} \sum_{j=1}^{r2^r} c_{rj} K(X_i, G_{rj}), \\
&= \int_{E_2} f(X_i, y) K(X_i, dy)
\end{aligned}$$

where the last equality holds since for each ω, $K(X_i(\omega),\cdot)$ is a probability measure. □

Lemma 3. *Under the conditions of the Marking Theorem,*

$$P([J_i \in F]|X_i, H) \overset{\text{a.s.}}{=} P([J_i \in F]|\{X_n\}, H) \overset{\text{a.s.}}{=} K(X_i, F)$$

for all i and all $F \in \mathcal{E}_2$.

Proof. Take the conditional expectation of both sides of

$$P([J_i \in F]|\{X_n\}, \{J_\alpha : \alpha \neq i\}, H) \overset{\text{a.s.}}{=} K(X_i, F)$$

with respect to $\sigma\{\{X_n\}, H\}$ to obtain

$$P([J_i \in F]|\{X_n\}, H) \overset{\text{a.s.}}{=} K(X_i, F).$$

Then take the conditional expectation of both sides of this resulting equation, given $\sigma\{X_i, H\}$, to obtain

$$P([J_i \in F]|X_i, H) \overset{\text{a.s.}}{=} K(X_i, F). \quad \square$$

Lemma 4. *Under the conditions of the Marking Theorem, the random elements $J_1, J_2, \ldots$ are conditionally independent given $\sigma\{\{X_n\}, H\}$, i.e.,*

$$P(\bigcap_{j=1}^{k}[J_j \in F_j]|\{X_n\}, H) \overset{\text{a.s.}}{=} \prod_{j=1}^{k} P([J_j \in F_j]|\{X_n\}, H)$$

for all $k \geq 1$ and arbitrary $F_1, \ldots, F_k$ in $\mathcal{E}_2$.

Proof. We use induction on k. The result is trivial for $k = 1$. Suppose it is true for some $k \geq 1$. Then

$$\begin{aligned}
&P(\cap_{j=1}^{k+1}[J_j \in F_j]|\{X_n\}, H) \\
&\overset{\text{a.s.}}{=} E(P(\cap_{j=1}^{k+1}[J_j \in F_j]|\{X_n\}, \{J_\alpha : \alpha \neq k+1\}, H)|\{X_n\}, H) \\
&\overset{\text{a.s.}}{=} E\big((\textstyle\prod_{j=1}^{k} I_{[J_j \in F_j]})P([J_{k+1} \in F_{k+1}]|\{X_n\}, \{J_\alpha : \alpha \neq k+1\}, H)|\{X_n\}, H\big) \\
&\overset{\text{a.s.}}{=} E\big((\textstyle\prod_{j=1}^{k} I_{[J_j \in F_j]})K(X_{k+1}, F_{k+1})|\{X_n\}, H\big) \\
&\overset{\text{a.s.}}{=} P([J_{k+1} \in F_{k+1}]|\{X_n\}, H)\textstyle\prod_{j=1}^{k} P([J_j \in F_j]|\{X_n\}, H) \\
&\overset{\text{a.s.}}{=} \prod_{j=1}^{k+1} P([J_j \in F_j]|\{X_n\}, H),
\end{aligned}$$

by use of the Marking Theorem Conditions, Lemma 3, and the induction hypothesis. □

Lemma 5. *Under the conditions of the Marking Theorem,*

$$P(\bigcap_{j=1}^{k}[J_j \in F_j]|\{X_n\}, H) \overset{\text{a.s.}}{=} \prod_{j=1}^{k} P([J_j \in F_j]|X_j, H)$$

for all $k \geq 1$, $F_1, \ldots, F_k \in \mathcal{E}_2$.

Proof. This follows from Lemmas 3 and 4. □

Lemma 6. *Let* $g : E_1 \times E_2 \to [0,1]$ *be* $\mathcal{E}_1 \times \mathcal{E}_2$*-measurable and assume that the conditions of the Marking Theorem hold. Then, for* $1 \leq r \leq \infty$,

$$E(\prod_{i=1}^{r} g(X_i, J_i)|\{X_n\}, H) \overset{\text{a.s.}}{=} \prod_{i=1}^{r} E(g(X_i, J_i)|\{X_n\}, H).$$

Proof. Assume first that $r < \infty$. Let $A_i \in \mathcal{E}_1$, $B_i \in \mathcal{E}_2$, for $1 \leq i \leq r$. Then by Lemma 4,

$$\begin{aligned} E(\prod_{i=1}^{r} I_{[X_i \in A_i][J_i \in B_i]}|\{X_n\}, H) &= (\prod_{i=1}^{r} I_{[X_i \in A_i]}) E(\prod_{i=1}^{r} I_{[J_i \in B_i]}|\{X_n\}, H) \\ &= (\prod_{i=1}^{r} I_{[X_i \in A_i]}) \prod_{i=1}^{r} E(I_{[J_i \in B_i]}|\{X_n\}, H) \\ &= \prod_{i=1}^{r} E(I_{[X_i \in A_i][J_i \in B_i]}|\{X_n\}, H). \end{aligned}$$

For $1 \leq i \leq r$ let $C_i \in \mathcal{E}_1 \times \mathcal{E}_2$, and let $D \in \sigma\{\{X_n\}, H\}$. Then define

$$\Psi(C_1, \ldots, C_r, D) = \int_D E(\prod_{i=1}^{r} I_{[(X_i, J_i) \in C_i]}|\{X_n\}, H)\, dP$$

and

$$\rho(C_1, \ldots, C_r, D) = \int_D \prod_{i=1}^{r} E(I_{[(X_i, J_i) \in C_i]}|\{X_n\}, H)\, dP.$$

For fixed $A_2, \ldots, A_r$ in $\mathcal{E}_1$, $B_2, \ldots, B_r$ in $\mathcal{E}_2$, and $D \in \sigma\{\{X_n\}, H\}$, let $C_j = A_j \times B_j$, $2 \leq j \leq r$. Then by the identity obtained above, $\Psi(C_1, \ldots, C_r, D) = \rho(C_1, \ldots, C_r, D)$ for all C_1 of the form $C_1 = A_1 \times B_1$, where $A_1 \in \mathcal{E}_1$ and $A_2 \in \mathcal{E}_2$. Since Ψ and ρ are measures in $C_1 \in \mathcal{E}_1 \times \mathcal{E}_2$, and since for all fixed $A_2, \ldots, A_r$, $B_2, \ldots, B_r, D$ they are equal over the π-system $\{A_1 \times B_1 : A_1 \in \mathcal{E}_1, B_1 \in \mathcal{E}_2\}$, it follows from the corollary to Dynkin's Theorem that

$$\Psi(C_1, \ldots, C_r, D) = \rho(C_1, \ldots, C_r, D)$$

for all $C_1 \in \mathcal{E}_1 \times \mathcal{E}_2$. For $i = 2, \dots, r$, treating each C_i in turn as we treated C_1 above, we obtain the equality of Ψ and ρ for all $C_i \in \mathcal{E}_1 \times \mathcal{E}_2$, $1 \le i \le r$. Since this last equation holds for all $D \in \sigma\{\{X_n\}, H\}$, and since both integrands in the definitions of Ψ and ρ above are measureable with respect to $\sigma\{\{X_n\}, H\}$, it follows that both integrands are equal a.s., i.e.,

$$E(\prod_{i=1}^{r} I_{[(X_i,J_i)\in C_i]} | \{X_n\}, H) \overset{\text{a.s.}}{=} \prod_{i=1}^{r} E(I_{[(X_i,J_i)\in C_i]} | \{X_n\}, H)$$

for $C_i \in \mathcal{E}_1 \times \mathcal{E}_2$, $1 \le i \le r$. Now g can be written as the monotone limit of a sequence of non-negative functions $\{g_n\}$ defined over $E_1 \times E_2$ of the form $g_n = \sum_{j=1}^{2^n} a_{nj} I_{C_{nj}}$ where each $C_{nj} \in \mathcal{E}_1 \times \mathcal{E}_2$. By linearity and this last identity it follows that

$$E(\prod_{i=1}^{r} g_m(X_i, J_i) | \{X_n\}, H) \overset{\text{a.s.}}{=} \prod_{i=1}^{r} E(g_m(X_i, J_i) | \{X_n\}, H)$$

for each m. Each side is non-decreasing in m a.s., and bounded above by 1, so by the conditional form of the Lebesgue monotone convergence theorem we have

$$E(\prod_{i=1}^{r} g(X_i, J_i) | \{X_n\}, H) \overset{\text{a.s.}}{=} \prod_{i=1}^{r} E(g(X_i, J_i) | \{X_n\}, H).$$

Thus the lemma is true for finite values of r. In case $r = \infty$, we note that the right hand side of the above equation is a.s. non-increasing and bounded below by 0, so the limit as $r \uparrow \infty$ exists a.s. Applying the conditional form of the Lebesgue dominated convergence theorem to the left hand side of the above, we obtain the lemma in this case. □

Remark. In the above lemma, if $g_i : E_1 \times E_2 \to [0, 1]$ is $\mathcal{E}_1 \times \mathcal{E}_2$-measurable for $1 \le i \le r$, then

$$E(\prod_{i=1}^{r} g_i(X_i, J_i) | \{X_n\}, H) \overset{\text{a.s.}}{=} \prod_{i=1}^{r} E(g_i(X_i, J_i) | \{X_n\}, H)$$

for $1 \le r \le \infty$.

The Marking Theorem. *Under the conditions of the Marking Theorem the stochastic process $\{N^*(A), A \in \mathcal{E}_1 \times \mathcal{E}_2\}$, defined by*

$$N^*(A) = \sum_{j=1}^{H} I_{[(X_j,J_j)\in A]}$$

is a Poisson random measure with mean measure μ^* defined by

$$\mu^*(A) = \int_{E_1} K(x, A_x)\mu(dx).$$

Proof. We first note that for H as defined, $1 = \sum_{j=0}^{\infty} I_{[H=j]} + I_{[H=\infty]}$. Next, for any nonnegative $\mathcal{E}_1 \times \mathcal{E}_2$-measurable function f defined over $E_1 \times E_2$, compute the Laplace functional $\hat{N}^*(f)$ of $N^*(\cdot)$ as follows:

$$\begin{aligned}
\hat{N}^*(f) &= E(e^{-N^*(f)}) \\
&= E(\exp\{-\sum_{j=1}^{H} f(X_j, J_j)\}) \\
&= E\left\{ E((\sum_{k=0}^{\infty} I_{[H=k]} + I_{[H=\infty]})\exp\{-\sum_{j=1}^{H} f(X_j, J_j)\}|\{X_n\}, H) \right\}.
\end{aligned}$$

Now by the conditional form of the Lebesgue monotone convergence theorem, Lemma 6, and Lemma 2,

$$\begin{aligned}
\hat{N}^*(f) = & E\left\{ \sum_{k=0}^{\infty} I_{[H=k]} E\Big(\prod_{j=1}^{k} e^{-f(X_j,J_j)}|\{X_n\}, H\Big) \right\} \\
& + E\left\{ I_{[H=\infty]} E\Big(\prod_{j=1}^{\infty} e^{-f(X_j,J_j)}|\{X_n\}, H\Big) \right\} \\
= & E\left\{ \sum_{k=0}^{\infty} I_{[H=k]} \prod_{j=1}^{k} E(e^{-f(X_j,J_j)}|\{X_n\}, H) \right\} \\
& + E\left\{ I_{[H=\infty]} \prod_{j=1}^{\infty} E(e^{-f(X_j,J_j)}|\{X_n\}, H) \right\} \\
= & E\left\{ \sum_{k=0}^{\infty} I_{[H=k]} \prod_{j=1}^{k} \int_{E_2} e^{-f(X_j,y)} K(X_j, dy) \right\} \\
& + E\left\{ I_{[H=\infty]} \prod_{j=1}^{\infty} \int_{E_2} e^{-f(X_j,y)} K(X_j, dy) \right\}.
\end{aligned}$$

Now let us define

$$\theta(x) = \int_{E_2} e^{-f(x,y)} K(x, dy)$$

for all $x \in E_1$. Then

$$\hat{N}^*(f) = E\left\{\sum_{k=0}^{\infty} I_{[H=k]} \prod_{j=1}^{k} \theta(X_j) + I_{[H=\infty]} \prod_{j=1}^{\infty} \theta(X_j)\right\}.$$

Since $0 < \theta(x) \le 1$ for all $x \in E_1, \log\theta(x)$ is well defined, and thus

$$\begin{aligned}
\hat{N}^*(f) &= E\left\{\sum_{k=0}^{\infty} I_{[H=k]} e^{-\sum_{j=1}^{k}(-\log\theta(X_j))} + I_{[H=\infty]} e^{-\sum_{j=1}^{\infty}(-\log\theta(X_j))}\right\} \\
&= E\left\{I_{[H<\infty]} e^{-\sum_{j=1}^{H}(-\log\theta(X_j))} + I_{[H=\infty]} e^{-\sum_{j=1}^{\infty}(-\log\theta(X_j))}\right\} \\
&= E(\exp\{-\sum_{j=1}^{H}(-\log\theta(X_j)\}) \\
&= E(\exp\{-\int_{E_1}(-\log\theta(x))N(dx)\}) \\
&= \hat{N}(-\log(\theta(\cdot))).
\end{aligned}$$

But by hypothesis N is a Poisson random measure with mean measure μ, and so

$$\begin{aligned}
\hat{N}^*(f) &= \hat{N}(-\log\theta(\cdot)) \\
&= \exp\left\{-\int_{E_1}(1 - e^{-(-\log\theta(x))})\mu(dx)\right\} \\
&= \exp\left\{-\int_{E_1}(1 - \int_{E_2} e^{-f(x,y)}K(x,dy))\mu(dx)\right\} \\
&= \exp\left\{-\int_{E_1}\left(\int_{E_2}(1 - e^{-f(x,y)})K(x,dy)\right)\mu(dx)\right\}.
\end{aligned}$$

Since $\hat{N}^*(f)$ is nonnegative and finite, we may apply the Fubini theorem, as stated in the appendix, to obtain

$$\hat{N}^*(f) = \exp\left\{-\int_{E_1\times E_2}(1 - e^{-f(z)})\,\mu^*(dz)\right\},$$

where $\mu^*(C) = \int_{E_1}\left(\int_{C_x} K(x,dy)\right)\mu(dx)$ for all $C \in \mathcal{E}_1 \times \mathcal{E}_2$ and where $C_x = \{y \in E_2 : (x,y) \in C\}$. By the representation of the Laplace functional of a Poisson random measure, we have shown that N^* is such a process. □

5. Appendix. The computation above made use of the following somewhat unconventional form of Fubini's Theorem. The necessary result is stated precisely here. A proof of this result can be constructed by using Dynkin's Theorem in a manner similar to the way in which Dynkin's Theorem was used to prove other results earlier.

Fubini's Theorem. *Let $(E_1, \mathcal{F}_1)$, and $(E_2, \mathcal{F}_2)$ be measurable spaces, let $\mathcal{F}_1 \times \mathcal{F}_2$ denote the σ-algebra generated by $\{A \times B : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$, and let μ denote a σ-finite measure on $(E_1, \mathcal{F}_1)$. Let $K : E_1 \times \mathcal{F}_2 \to [0,1]$ satisfy:*

(1) *for every $x \in E_1$, the set function $K(x, \cdot)$ is a probability measure over $\mathcal{F}_2$, and*
(2) *for every $F \in \mathcal{F}_2$, the function $K(\cdot, F)$ is $\mathcal{F}_1$-measurable.*

For each $C \in \mathcal{F}_1 \times \mathcal{F}_2$, let $C_x = \{y \in E_2 : (x,y) \in C\}$. Then

(1) *$K(x, A_x)$ is $\mathcal{F}_1$-measurable, and*
(2) *μ_* as defined by $\mu_*(C) = \int_{E_1} K(x, C_x)\mu(dx)$ is a measure over $\mathcal{F}_1 \times \mathcal{F}_2$.*
(3) *If $f : E_1 \times E_2 \to \mathbf{R}^1$ is $\mathcal{F}_1 \times \mathcal{F}_2$-measurable, and if f satisfies the condition $\int_{E_1} \left(\int_{E_2} |f(x,y)| K(x, dy)\right) \mu(dx) < \infty$, then*

$$\int_{E_1} \left(\int_{E_2} f(x,y) K(x, dy) \right) \mu(dx) = \int_{E_1 \times E_2} f d\mu_*.$$

REFERENCES

PATRICK BILLINGSLEY (1986). *Probability and Measure.* 2nd edition, John Wiley and Sons, New York.

OLAV KALLENBERG (1983). *Random Measures.* 3rd revised and enlarged edition, Academic Press, New York.

J. F. C. KINGMAN (1993). *Poisson Processes.* Oxford University Press, New York.

SIDNEY RESNICK. *Extreme values, regular variation, and point processes.* Springer Verlag, New York.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
IRVINE, CA 92717
htucker@math.uci.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
IRVINE, CA 92717
mfinkels@math.uci.edu

DEPARTMENT OF DISCRETE AND STATISTICAL SCIENCES
AUBURN UNIVERSITY
ALABAMA, 36849
veehjer@mallard.duc.auburn.edu