Bayesian Robustness
IMS Lecture Notes - Monograph Series (1996) Volume 29

ASYMPTOTICS OF SOME LOCAL AND GLOBAL ROBUSTNESS MEASURES

BY S. SIVAGANESAN
University of Cincinnati

We study the asymptotic behavior of some local robustness measures in the context of ε−contamination classes. We find that when the contaminations are subjected to some reasonable constraints, the local robustness measures have similar asymptotic behavior as the global robustness measures.

1. Introduction Local robustness approach to studying sensitivity to small deviations from a specified (base) prior π_0 has received much attention in the recent past. Related references, among many others, are Basu(1994), Gustafson(1994), Gustafson, Wasserman and Srinivasan(1994), Meng and Sivaganesan(1995), Ruggeri and Wasserman(1991, 1993), and Sivaganesan(1993). An excellent review of various approaches and issues in Bayesian robustness is given in Berger(1994).

While the local robustness approach is recognized for its computational simplicity, and its potential use in multi-dimensional and similar complex problems where global robustness investigation may be difficult, there have also been (at least) two important issues of concern. The first issue is about the asymptotics. It is clear that in most cases the impact of prior on the posterior measure diminishes as the sample size n goes to infinity. In particular, it is known that for most, if not all, classes of priors considered in the literature, the global robustness measures, e.g., size of ranges, converge to 0 (asymptotically) as n goes to infinity. For details, see, e.g., Gustafson(1993), Sivaganesan(1988) and Pericchi and Walley(1991). In this sense, one would also expect the local robustness measure to converge to 0 asymptotically. However it is shown in Gustafson(1993) that for some general classes of priors the local robustness measures do not converge to 0, and even diverge for some multidimensional classes of priors. Similar results were also reported in Gustafson, Wasserman and Srinivasan(1994). These results are indeed troubling, and have raised some doubts about the use of local robustness measures as a tool in investigating robustness. In this paper, we show, for a wide classes of priors which satisfy some reasonable mild conditions, that the local robustness measures do indeed converge to 0 asymptotically, and that their asymptotic behavior is in concurrence with those of the global robustness measures. This finding is relevant and useful as it overcomes the problems alluded to earlier, and allays the doubts about the local robustness measures which resulted from such problems. The second issue is about the interpretation or calibration of the local robustness measures.

[1] *AMS Subject Classification:* Primary 62F15, secondary 62A15.
Key words: Local sensitivity, asymptotics, ε−contamination class.

In this paper, we address the first issue in the context of ε-contamination classes, using a local sensitivity measure defined below. (The issue of calibration is addressed in a separate paper.) For convenience, we only deal with one-dimensional problems. The results can also be extended to multidimensional problems with qualitatively similar results. However, in practice, it is perhaps useful to consider contaminations of only one-dimensional distributions (e.g., one-dimensional marginals or one dimensional conditionals) when the local robustness approach is envisaged, since (such) one-dimensional distributions are easier from elicitational considerations. In this sense, it may even be sufficient to focus on one-dimensional problems in the context of studying the asymptotic behavior of local robustness measures.

In section 2, we focus on the asymptotic behavior of the local sensitivity measures, and in section 3, we consider the asymptotics of some global robustness measures resulting from some ε-contamination classes.

We now introduce some notation, and define the local sensitivity measure that will be used in the next section. Throughout the rest of the paper, we let θ be one dimensional, and the class of priors be an $\varepsilon-$ contamination class given by

$$\Gamma = \{\pi_\varepsilon(\theta) = (1-\varepsilon)\pi_0 + \varepsilon q; q \in Q\}. \tag{1}$$

Let $g(\theta)$ be a function of interest, which we will assume to be differentiable. In the rest of the paper, we will also let D_π and ρ_π , respectively, be the marginal density, and posterior expectation of $g(\theta)$, w.r.t. prior π. In particular, we let $D_0 = D_{\pi_0}$ and $\rho_0 = \rho_{\pi_0}$. In fact, for convenience, we will assume $g(\theta) = \theta$ and that the parameter space is the whole real line; the results for more general functions will be similar when the derivative does not vanish in a neighborhood the true value of the parameter.

The local sensitivity of ρ_π w.r.t. priors $\pi \in \Gamma$, based on a sample of size n, may now be defined by

$$\begin{aligned} LS(n) &= \sup_q \left[\frac{d}{d\varepsilon}\rho_{\pi_\varepsilon}\right]_{\varepsilon=0} \\ &= \frac{1}{D_0}\sup_{q\in Q}\int(\theta-\rho_0)L(\theta)q(\theta)d\theta, \end{aligned}$$

where $L(\theta)$ is the likelihood function. Similarly, we may also define another local sensitivity measure obtained by taking the inf in the above. In a robust investigation, one may use the larger (in magnitude) of the two (or, both separately). However, for convenience in presentation, we will only focus on the local sensitivity measure defined as above; similar results hold for the other case.

2. Asymptotic properties of the local sensitivity measure. We begin by seeking the asymptotic behavior of $LS(n)$ for certain contamination classes. First, we need to introduce some more notation, and assumptions.

We let $\hat{\theta}$ be the MLE, and θ^* be the 'true' value of θ. Also, let $I(\theta)$ be the expected Fisher information number. We assume that the requirements for the asymptotic convergence of the posterior w.r.t. each $\pi \in \Gamma$ are valid. In particular, we assume that $\pi_0(\theta)$ is bounded, has bounded continuous derivatives, and $\pi_0(\theta^*) > 0$. We will also assume $\hat{\theta} \to \theta^*$, and

$$\hat{I} = -\frac{1}{n}\ell''(\hat{\theta}) \to I(\theta^*),$$

where $\ell(\cdot)$ is the log-likelihood. More details on this can be seen in, e.g., Chen(1985) and Gustafson(1994). In the following all statements of limits are to be interpreted as a.s., as the sample size n approaches ∞. In the following lemma we give some asymptotic results concerning D_0 and ρ_0. Although these results may not be entirely new, we provide the proof of both results as it helps with the presentation of the other proofs in the paper.

LEMMA 1 *Under the assumptions stated above,*

$$\sqrt{n}e^{-\ell(\hat{\theta})}D_0 \to \sqrt{2\pi}\frac{\pi_0(\theta^*)}{\sqrt{I(\theta^*)}},$$

and

$$n(\rho_0 - \hat{\theta}) \to \frac{\pi_0'(\theta^*)}{I(\theta^*)\pi_0(\theta^*)}.$$

PROOF We first outline the proof for the first part. Using the usual Taylor expansion of the likelihood around the MLE $\hat{\theta}$, we can approximate D_0 by

$$D(n) = e^{\ell(\hat{\theta})}\int e^{-\frac{1}{2}[-\ell''(\hat{\theta})](\theta-\hat{\theta})^2}\pi_0(\theta)d\theta,$$

which can be written as

$$e^{-\ell(\hat{\theta})}D(n) = \int e^{-\frac{1}{2}n\hat{I}(\theta-\hat{\theta})^2}\pi_0(\theta)d\theta.$$

Now, letting $t = \sqrt{n\hat{I}}(\theta - \hat{\theta})$, we have

$$\begin{aligned} e^{-\ell(\hat{\theta})}D(n) &= \sqrt{\frac{2\pi}{n\hat{I}}}\int \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\pi_0(\hat{\theta} + t/\sqrt{n\hat{I}})dt \\ &= \sqrt{\frac{2\pi}{n\hat{I}}}\left(\pi_0(\hat{\theta}) + \frac{1}{\sqrt{n\hat{I}}}\int \frac{1}{\sqrt{2\pi}}te^{-t^2/2}\pi_0'(\theta_t)dt\right), \end{aligned}$$

where $\hat{\theta} < \theta_t < \hat{\theta}+t/\sqrt{n\hat{I}}$. Now the desired conclusion follows by considering the limit of $\sqrt{n}e^{-\ell(\hat{\theta})}D(n)$.

To prove the second part, we first write $\rho_0 - \hat{\theta} = R_0/D_0$, where $R_0 = \int(\theta-\hat{\theta})L(\theta)\pi_0(\theta)d\theta$. Note that, as before, we can approximate R_0 by $R(n)$ where

$$\begin{aligned} e^{-\ell(\hat{\theta})}R(n) &= \frac{\sqrt{2\pi}}{n\hat{I}}\int \frac{te^{-t^2/2}}{\sqrt{2\pi}}\pi_0(\hat{\theta}+t/\sqrt{n})dt \\ &= \frac{\sqrt{2\pi}}{n\hat{I}}\frac{1}{\sqrt{n\hat{I}}}\int \frac{t^2e^{-t^2/2}}{\sqrt{2\pi}}\pi_0'(\theta_t)dt. \end{aligned}$$

Now, the second part of the Lemma follows by considering the limit of nR_0/D_0. □

In Gustafson(1994), the asymptotics of a (more general, but often equivalent) local sensitivity measure is studied for a variety of classes of a certain structure, one of which is the $\varepsilon-$contamination class with arbitrary contaminations. There, it is shown that the local sensitivity measure $LS(n)$ does not converge to zero, as n goes to ∞ when arbitrary contaminations are allowed. In the following theorem, we show that that $LS(n)$ does converge to 0, when the contaminations satisfy certain, mostly reasonable, conditions. In the following, we use the terminology that a class F of functions $f(\theta)$ on a set Θ is uniformly bounded if there is a $M < \infty$ such that $\sup_\theta f(\theta) < M$ for all $f \in F$.

THEOREM 2.1: *(i) When $q \in Q$ have uniformly bounded densities,*

$$LS(n) \text{ is } O(n^{-1/2}).$$

(ii) When $q \in Q$ have uniformly bounded densities with uniformly bounded derivatives,

$$LS(n) \text{ is } O(n^{-1}).$$

PROOF: We write $LS(n) = \sup_q N_q/D_0$. Thus,

$$\begin{aligned} N_q &= \int(\theta-\rho_0)L(\theta)q(\theta)d\theta \\ &= \int(\theta-\hat{\theta})L(\theta)q(\theta)d\theta - (\rho_0-\hat{\theta})D_q, \end{aligned}$$

where $D_q = \int L(\theta)q(\theta)d\theta$. Now, as in the proof of Lemma 1, we can approximate N_q by N_q^*, where

$$\begin{aligned} ne^{-\ell(\hat{\theta})}N_q^* = \\ \sqrt{2\pi}\hat{I}^{-1}\int \frac{te^{-t^2/2}}{\sqrt{2\pi}}q(\hat{\theta}+t/\sqrt{n\hat{I}})dt \\ -\sqrt{n}(\rho_0-\hat{\theta})\hat{I}^{-1/2}\int e^{-t^2/2}q(\hat{\theta}+t/\sqrt{n\hat{I}})dt. \end{aligned} \tag{2}$$

We note that the second term on the right side of the above converges to 0. Thus, letting A_q denote the first term, we have, for given $\delta > 0$,

$$A_q - \delta \le ne^{-\ell(\hat{\theta})} N_q^* \le A_q + \delta,$$

for sufficiently large n and all q. Thus, for sufficiently large n,

$$\sup_q A_q - \delta \le ne^{-\ell(\hat{\theta})} \sup_q N_q^* \le \sup_q A_q + \delta. \tag{3}$$

Now, since A_q are bounded for all n, $\sup_q A_q$ is $O(1)$, and hence,

$$e^{-\ell(\hat{\theta})} \sup_q N_q^* \text{ is of the order } O(1/n).$$

Now, by the first part of Lemma 1, $e^{-\ell(\hat{\theta})} D_0$ is $O(1/\sqrt{n})$. Combining these two results, and the fact $LS(n) = \sup_q N(q)/D_0$ lead to the desired conclusion.

To prove the second part of the theorem, we expand $q(\hat{\theta} + t/\sqrt{n\hat{I}})$ around $\hat{\theta}$ in (2), and get

$$\begin{aligned}
&[n\hat{I}]^{3/2} e^{-\ell(\hat{\theta})} N_q^* = \\
&\qquad \int t^2 e^{-t^2/2} q'(\theta_t)dt - n\hat{I}(\rho_0 - \hat{\theta})q(\hat{\theta}) \\
&\qquad -\sqrt{n}(\rho_0 - \hat{\theta})\hat{I}^{-1/2} \int e^{-t^2/2} q'(\theta_t)dt.
\end{aligned}$$

Thus, when $q(\cdot)$'s and $q'(\cdot)$'s are bounded, we get $e^{-\ell(\hat{\theta})} \sup_q N_q^*$, and hence $e^{-\ell(\hat{\theta})} \sup_q N_q$ are of order $O(n^{-3/2})$. Now the result follows using the first part of Lemma 1. □

The following theorem gives a specific class of contaminations for which $\sqrt{n} LS(n)$ converges to a non-zero value.

THEOREM 2.2: *Let Q be given by*

$$Q = \{ \text{ prob. density } q : L(\theta) \le q(\theta) \le U(\theta)\},$$

where L and U are bounded and continuous. Then,

$$\sqrt{n} LS(n) \text{ converges to } \sqrt{\frac{I(\theta^*)}{2\pi}} \frac{[U(\theta^*) - L(\theta^*)]}{\pi_0(\theta^*)}.$$

PROOF: Following the lines of the proof for Theorem 1, we get (see 3),

$$\begin{aligned}
\sup_q A_q &= \frac{1}{\hat{I}} \sup_q \int te^{-t^2/2} q(\hat{\theta} + t/\sqrt{n\hat{I}})dt \\
&= \frac{1}{\hat{I}} \int_{B_n} te^{-t^2/2} U(\hat{\theta} + t/\sqrt{n\hat{I}})dt \\
&\quad + \frac{1}{\hat{I}} \int_{B_n^c} te^{-t^2/2} L(\hat{\theta} + t/\sqrt{n\hat{I}}),
\end{aligned}$$

where $B_n = \{t : te^{-t^2/2} \geq c_n\}$ for some c_n such that

$$\int_{B_n} \frac{1}{\sqrt{n\hat{I}}} U(\hat{\theta} + t/\sqrt{n\hat{I}})dt + \int_{B_n^c} \frac{1}{\sqrt{n\hat{I}}} L(\hat{\theta} + t/\sqrt{n\hat{I}})dt = 1. \tag{4}$$

Thus,

$$\begin{aligned} \sup_q A_q &= \frac{1}{\hat{I}} \int_{B_n} te^{-t^2/2}(U(\hat{\theta} + t/\sqrt{n\hat{I}}) - L(\hat{\theta} + t/\sqrt{n\hat{I}}))dt + \\ &\quad \frac{1}{\hat{I}} \int te^{-t^2/2} L(\hat{\theta} + t/\sqrt{n\hat{I}})dt. \end{aligned} \tag{5}$$

Now, suppose $B_n = (a_n, b_n) \subset (0, \infty)$, (the other case where B_n is a union of two disjoint intervals can be considered similarly). Thus, letting $\int L(\theta)d\theta = k < 1$, we get from (4)

$$\int_{a_n}^{b_n} (U(\hat{\theta} + t/\sqrt{n\hat{I}}) - L(\hat{\theta} + t/\sqrt{n\hat{I}}))dt = \sqrt{n\hat{I}}(1 - k).$$

From the above, it is clear that b_n must approach ∞, and hence a_n must converge to 0 as n goes to ∞. Now, from (5), letting n go to ∞, we get

$$\sup_q A_q \to (U(\theta^*) - L(\theta^*))/I(\theta^*).$$

Now the desired result follows by using the first part of Lemma 1, and recalling (see Theorem 2.1) that $LS(n) = \sup_q N_q/D_0$, which is arbitrarily close to $\sup_q A_q/D_0$ for large enough n. □

REMARK:
(i) Usually, it would be the case that $L(\theta)$ in the above would be identically 0.
(ii) In particular, the above result would apply to density bounded classes whose bounds are of the form $L(\theta) = (1-\varepsilon)\pi_0(\theta)$ and $U(\theta) = (1-\varepsilon)\pi_0(\theta) + \varepsilon g(\theta)$ where $\pi_0(\theta)$ is a bounded differentiable density and $g(\theta)$ is a non-negative bounded function for a suitable (fixed) function $g(\theta)$.

Now, as a specific example for which $LS(n)$ converges to 0 at the rate of $1/n$, we show, for unimodal contaminations with some mode, say θ_0, that $nLS(n)$ converges to a certain non-zero value. This may also be of interest since the unimodal contaminations do not entirely satisfy the differentiability conditions of the theorem. We will assume that $\theta^* > \theta_0$, and $\pi_0'(\theta^*) < 0$.

THEOREM 2.3: *Suppose Q is the class of all unimodal densities with mode θ_0. Then,*

$$nLS(n) \to \frac{-\pi_0'(\theta^*)}{(\theta^* - \theta_0)I(\theta^*)\pi_0^2(\theta^*)}.$$

PROOF: For convenience, we let $\theta_0 = 0$. Using the notation in the above proofs, as before, we approximate $\int(\theta - \rho_0)L(\theta)q(\theta)d\theta$ by N_q^* as in (2). Now we use the result that a unimodal distribution with mode θ_0 can be expressed as a mixture of uniform distributions over intervals with one endpoint θ_0, and get

$$\sup_{q\in Q} e^{-\ell(\hat{\theta})} N_q^* = [n\hat{I}]^{-1/2} \sup_z \frac{1}{\sqrt{n\hat{I}}z} \int_{-\sqrt{n\hat{I}}\hat{\theta}}^{z_1} \left(t - \sqrt{n\hat{I}}(\rho_0 - \hat{\theta})\right) e^{-t^2/2} dt$$

where $z_1 = \sqrt{n\hat{I}}(z - \hat{\theta})$.

Thus we seek the supremum of

$$\frac{1}{\sqrt{n\hat{I}}z} \int_{-\sqrt{n\hat{I}}\hat{\theta}}^{z_1} f(t) dt$$

where $f(t) = \left(t - \sqrt{n\hat{I}}(\rho_0 - \hat{\theta})\right) e^{-t^2/2}$. By setting the derivative equal to zero, we see that the supremum is achieved at a z which satisfies

$$[z_1 - \sqrt{n\hat{I}}(\rho_0 - \hat{\theta})] e^{-z_1^2/2} = \frac{1}{\sqrt{n\hat{I}}z} \int_{-\sqrt{n\hat{I}}\hat{\theta}}^{z_1} f(t) dt. \tag{6}$$

It can be shown that there is a solution to (6) which corresponds to the supremum. Thus, the supremum is given by $f(z_1)$ where z is a solution of (6). In the rest of the proof, we will use z (and z_1) to represent the solution of (6).

Now, we note, from Lemma 1, that $\sqrt{n}(\rho_0 - \hat{\theta}) \to 0$ a.s., as $n \to \infty$, and hence, it is easy to verify that $z_1 > 0$ a.s. for large enough n. Now, we claim that $z_1 \to \infty$. To prove, suppose that $z_1 \le c$ for some constant c for infinitely many n. Now, we re-write (6) as follows.

$$(z_1 + \sqrt{n\hat{I}}\hat{\theta}) z_1 e^{-z_1^2/2} = \int_{-\sqrt{n\hat{I}}\hat{\theta}}^{z_1} t e^{-t^2/2} dt + g_n(z), \tag{7}$$

where

$$g_n(z) = \sqrt{n\hat{I}}(\hat{\theta} - \rho_0) \left[\int_{-\sqrt{n\hat{I}}\hat{\theta}}^{z_1} e^{-t^2/2} dt - \sqrt{n\hat{I}} z e^{-z_1^2/2} \right].$$

We observe that g_n must converge to a negative value as n goes to ∞. Thus, noting $\sqrt{n\hat{I}}\hat{\theta} \to \infty$, we see that, as n gets large, the left side of (7) would be non-negative, while the right side becomes negative. This is a contradiction, proving that $z_1 \to \infty$.

Now, we claim that $\delta = (z - \hat{\theta})$ converges to 0. To show this, we get from (7),

$$\begin{aligned} z_1 e^{-z_1^2/2} &= \frac{1}{\sqrt{n\hat{I}}z} \left(e^{-n\hat{\theta}^2/2} - e^{-z_1^2/2}\right) + \\ &\quad \sqrt{n\hat{I}}(\hat{\theta} - \rho_0) \{ \frac{1}{\sqrt{n\hat{I}}z} \int_{-\sqrt{n\hat{I}}\hat{\theta}}^{z_1} e^{-t^2/2} dt - e^{-z_1^2/2} \}. \end{aligned} \tag{8}$$

Thus, writing $z_1 = \sqrt{n\hat{I}}\delta$ and $z = \delta + \hat{\theta}$, we get after little algebra,

$$In^{3/2}\delta(\delta+\hat{\theta})e^{-n\hat{I}\delta^2/2} + (e^{-n\hat{I}\delta^2/2} - e^{-n\hat{I}\hat{\theta}^2/2})\sqrt{n} =$$

$$\sqrt{I}n(\hat{\theta}-\rho_0)\{\int_{-\sqrt{n\hat{I}}\hat{\theta}}^{\sqrt{n}\delta} e^{-t^2/2}dt - \sqrt{n\hat{I}}(\hat{\theta}+\delta)e^{-n\hat{I}\delta^2/2}\}. \tag{9}$$

First we note that $\delta \geq 0$ a.s. for large enough n. Now, suppose that δ converges to a positive number δ_0. Then, from (9), we see that the right side of (9) would converge to a non-zero number, while the left side would converge to 0, leading to a contradiction. Thus, δ converges to 0, and hence, z converges to θ^*.

From (8), it is also clear that $\sqrt{n}e^{-z_1^2/2} \to 0$. Now, we have from (6)

$$n\hat{I}f(z_1) = \frac{\sqrt{n\hat{I}}}{z}[e^{-n\hat{\theta}^2} - e^{-nz_1^2/2}] + n\hat{I}(\hat{\theta}-\rho_0)\frac{1}{z}\int_{-\sqrt{n\hat{I}}\hat{\theta}}^{z_1} e^{-t^2/2}dt.$$

Thus, using Lemma 1, and the observations made above, we have

$$n\hat{I}f(z_1) \to \frac{-\sqrt{2\pi}\pi_0'(\theta^*)}{\theta^*\pi_0(\theta^*)}.$$

Now,

$$nLS(n) = \frac{n\sup_q e^{-\ell(\hat{\theta})}N(q)}{e^{-\ell(\hat{\theta})}D_0} = \frac{nf(z_1)}{\sqrt{n\hat{I}}D_0},$$

which converges to $-\pi_0'(\theta^*)/\left(\theta^* I(\theta^*)\pi_0^2(\theta^*)\right)$ concluding the proof. □

3. Asymptotics of some global robustness measures. We now consider the asymptotic behavior of the range of ρ_π. For convenience, we will only consider $\sup_{\pi\in\Gamma}(\rho_\pi - \hat{\theta})$. The results for the inf are similar, and therefore will not be discussed.

THEOREM 3.1:

(i) When $q \in Q$ have uniformly bounded densities

$$\sup(\rho_\pi - \hat{\theta}) \text{ is } O(n^{-1/2}).$$

(ii) When $q \in Q$ have uniformly bounded densities with uniformly bounded derivatives,

$$\sup(\rho_\pi - \hat{\theta}) \text{ is } O(n^{-1}).$$

PROOF: We give an outline of the proof, which follows some of the steps of Theorem 2.1. First, we note, as in the proof of Theorem 4.1, that for $\pi = (1-\varepsilon)\pi_0 + \varepsilon q \in \Gamma$, $(\rho_\pi - \hat{\theta})$ can be approximated by

$$\Delta_q = \frac{\int (t/\sqrt{n\hat{I}})e^{-t^2/2}[(1-\varepsilon)\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}}) + \varepsilon q(\hat{\theta}+t/\sqrt{n\hat{I}})]dt}{\int e^{-t^2/2}[(1-\varepsilon)\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}}) + \varepsilon q(\hat{\theta}+t/\sqrt{n\hat{I}})]dt}.$$

Thus, letting λ be the sup of Δ_q over $q \in Q$, we get by the linearization algorithm, see Lavine, Wasserman and Wolpert (1993).

$$(1-\varepsilon)\int (t/\sqrt{n\hat{I}}-\lambda)e^{-t^2/2}\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}})dt+ \quad \varepsilon \sup_q \int (t/\sqrt{n\hat{I}}-\lambda)e^{-t^2/2}q(\hat{\theta}+t/\sqrt{n\hat{I}})dt = 0. \tag{10}$$

Assuming $q \in Q$ have bounded densities, we may conclude that the sup is achieved at some q, say, $\bar{q}$. Thus,

$$\sqrt{n}\lambda = \frac{\int te^{-t^2/2}[(1-\varepsilon)\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}})+\varepsilon\bar{q}(\hat{\theta}+t/\sqrt{n\hat{I}})]dt}{\int e^{-t^2/2}[(1-\varepsilon)\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}})+\varepsilon\bar{q}(\hat{\theta}+t/\sqrt{n\hat{I}})]dt}.$$

Thus, letting $\sup_\theta \max\{q(\theta), \pi_0(\theta)\} = M$,

$$|\sqrt{n}\lambda| \le \frac{2M}{(1-\varepsilon)\int e^{-t^2/2}\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}})dt}.$$

For given small $\delta > 0$, the denominator in the right side of the above is greater than $(1-\varepsilon)\sqrt{2\pi}(\pi_0(\theta^*)-\delta)$ for sufficiently large n, which concludes the proof of the first part of the theorem.

To prove the second part, we assume that $q \in Q$ have bounded densities with bounded derivatives. Thus, we get from (10)

$$(1-\varepsilon)\int \frac{t^2}{n\hat{I}}e^{-t^2/2}\pi_0'(\theta_t)dt - \lambda(1-\varepsilon)\int e^{-t^2/2}\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}})dt+ \quad \varepsilon\sup_q \int \frac{t^2}{n\hat{I}}e^{-t^2/2}q'(\theta_t^*)dt - \lambda\varepsilon \int e^{-t^2/2}q(\hat{\theta}+t/\sqrt{n\hat{I}})dt = 0.$$

Now, let the sup above is achieved at, say, $\bar{q}$. Then, letting $M' = \sup_\theta \max\{\pi_0'(\theta), q'(\theta)\}$, we have

$$|n\lambda| \le \frac{\sqrt{2\pi}M'}{(1-\varepsilon)\int e^{-t^2/2}\pi_0(\hat{\theta}+t/\sqrt{n\hat{I}})dt}.$$

Now, a similar argument as before concludes the proof. □

As before, we now consider a specific class of bounded contaminations.

THEOREM 3.2: *Let Q be given by*

$$Q = \{\text{ prob. density } q : L(\theta) \le q(\theta) \le U(\theta)\},$$

where L and U are bounded and continuous. Suppose also that

$$\sqrt{n\hat{I}}\sup(\rho_\pi - \hat{\theta}) \text{ converges to } c.$$

Then $c \neq 0$ whenever $U(\theta^*) \neq L(\theta^*)$.

PROOF: Following the proof of Theorem 2.2, we get from (10),

$$\begin{aligned}\varepsilon \sup_q \int (t - \sqrt{n\hat{I}}\lambda)e^{-t^2/2} q(\hat{\theta} + t/\sqrt{n\hat{I}})dt + \\ +(1-\varepsilon)\int (t - \sqrt{n\hat{I}}\lambda)e^{-t^2/2}\pi_0(\hat{\theta} + t/\sqrt{n\hat{I}})dt &= 0.\end{aligned}$$

Thus, we have, using the fact that the maximizing measure will be equal to U on some set, and to L otherwise, see e.g., Lavine, Wasserman and Wolpert (1993).

$$\begin{aligned}\int (t - \sqrt{n\hat{I}}\lambda)e^{-t^2/2}[(1-\varepsilon)\pi_0(\hat{\theta} + t/\sqrt{n\hat{I}}) + \varepsilon L(\hat{\theta} + t/\sqrt{n\hat{I}})]dt + \\ \varepsilon \int_{A_n} (t - \sqrt{n\hat{I}}\lambda)e^{-t^2/2}[U(\hat{\theta} + t/\sqrt{n\hat{I}}) - L(\hat{\theta} + t/\sqrt{n\hat{I}})]dt &= 0, \qquad (11)\end{aligned}$$

where $A_n = \{t : (t - \lambda\sqrt{n\hat{I}})e^{-t^2/2} > c_n\}$, and satisfies $\int_{A_n}(1/\sqrt{n\hat{I}})U(\hat{\theta} + t/\sqrt{n\hat{I}})dt + \int_{A_n^c}(1/\sqrt{n\hat{I}})L(\hat{\theta} + t/\sqrt{n\hat{I}})dt = 1$. As in Theorem 2.2, we may let $A_n = (a_n, b_n)$ where $a_n > \sqrt{n\hat{I}}\lambda$, and verify that b_n must approach to ∞, and $a_n - \sqrt{n\hat{I}}\lambda$ must converge to 0. Thus, assuming, $\sqrt{n\hat{I}}\lambda$ converges to c (note that c is a finite number by Theorem 2.2), we have that a_n converges to c. Moreover, from (11), we get by letting n go to ∞,

$$(U(\theta^*) - L(\theta^*))\int (t-c)e^{-t^2/2}dt = c\sqrt{2\pi}[(1-\varepsilon)\pi_0(\theta^*) + \varepsilon L(\theta^*)].$$

That $c \neq 0$ when $U(\theta^*) \neq L(\theta^*)$ follows from the above, which concludes the proof. □

The following theorem gives the asymptotic behavior of the sup of ρ_π for unimodal contaminations.

THEOREM 3.3 *Suppose Q is the class of all unimodal densities with mode θ_0, and that $\theta^* > \theta_0$. Then,*

$$n \sup(\rho_\pi - \hat{\theta}) \text{ converges to } \frac{(1-\varepsilon)(\theta^* - \theta_0)\pi_0'(\theta^*)}{(1-\varepsilon)(\theta^* - \theta_0)\pi_0(\theta^*) + \varepsilon}.$$

PROOF: For convenience, we assume $\theta_0 = 0$, and also will use symbols (for quantities that depend on n) which do not reflect (such) dependence on n. Thus, using the fact that the sup is achieved when $q(\cdot)$ is a uniform distribution over an interval of the form $(0, z)$, we can write (for a q of this form)

$$n\hat{I}(\rho_\pi - \hat{\theta}) = \frac{az + \sqrt{n\hat{I}}\int_{-\sqrt{n\hat{I}}\hat{\theta}}^{\sqrt{n\hat{I}}(z-\hat{\theta})} te^{-t^2/2}dt}{bz + \int_{-\sqrt{n\hat{I}}\hat{\theta}}^{\sqrt{n\hat{I}}(z-\hat{\theta})} e^{-t^2/2}dt}, \qquad (12)$$

where $a = \varepsilon^{-1}(1-\varepsilon)\int t^2 e^{-t^2/2}\pi_0'(\theta_t)dt$, and $b = \varepsilon^{-1}(1-\varepsilon)\int e^{-t^2/2}[\pi_0(\hat\theta) + t\pi_0'(\theta_t)/\sqrt{n}]dt$ and $\hat\theta < \theta_t < \hat\theta + t/\sqrt{n\hat I}$.

Let $\lambda = \sup(\rho_\pi - \hat\theta)$. Then, we can see, e.g., by the linearization algorithm, that

$$\sup_z \frac{1}{\sqrt{n\hat I}z}\int_{-\sqrt{n\hat I}\hat\theta}^{\sqrt{n\hat I}(z-\hat\theta)}(t-\sqrt{n\hat I}\lambda)e^{-t^2/2}dt > 0$$

and hence $z_1 = \sqrt{n\hat I}(z-\hat\theta) > 0$ for (the) z which corresponds to the sup above.

Moreover, we get (by setting the derivative of the right side of (12) equal to 0) that the sup of $(\rho_\pi - \hat\theta)$ is achieved at a z which satisfies

$$bzz_1e^{-z_1^2/2} + \frac{a}{n\hat I}\int_{-\sqrt{n\hat I}\hat\theta}^{z_1} e^{-t^2/2}dt + z_1e^{-z_1^2/2}\int_{-\sqrt{n\hat I}\hat\theta}^{z_1} e^{-t^2/2}dt =$$

$$\frac{az}{n\hat I}e^{-z_1^2/2} + \frac{1}{\sqrt{n\hat I}}(b + e^{-z_1^2/2})(e^{-n\hat I\hat\theta^2} - e^{-z_1^2/2}). \tag{13}$$

For convenience, we will use z (and z_1 as above) to represent the solution of (13) corresponding to the sup.

Now, we can show that z_1 diverges as n goes to ∞. To show this, first note from (13) that z_1 cannot converge to some $c \neq 0$. Also, by multiplying (13) by $\sqrt{n}$ and letting n go to ∞ we note that z_1 does not converge to 0, thus arriving at the desired conclusion.

Now, letting $u = z - \hat\theta = z_1/\sqrt{n\hat I}$, and replacing z and z_1 above in terms of u, we have

$$b + \left(b\hat\theta + \int_{-\sqrt{n\hat I}\hat\theta}^{\sqrt{n\hat I}u} e^{-t^2/2}dt\right)\frac{1}{u} + a\frac{e^{n\hat Iu^2/2}}{n^{3/2}u^2}\int_{-\sqrt{n\hat I}\hat\theta}^{\sqrt{n\hat I}u} e^{-t^2/2}dt =$$

$$\frac{a\hat\theta}{n^{3/2}u^2} + \frac{a}{n^{3/2}u} - \left(\frac{b}{nu^2} + \frac{e^{-n\hat Iu^2/2}}{nu^2}\right) + be^{-n\hat I\hat\theta^2}\left(\frac{e^{n\hat Iu^2}}{nu^2} + \frac{1}{nu^2}\right). \tag{14}$$

We first note that $a \to \varepsilon^{-1}(1-\varepsilon)\sqrt{2\pi}\pi_0'(\theta^*)$ and $b \to \varepsilon^{-1}(1-\varepsilon)\sqrt{2\pi}\pi_0(\theta^*)$. Now, we note from (14) that u remains bounded a.s. as n goes to ∞. Else, the left side would approach ∞ while the right side is negative as n goes to ∞. Similarly, we get, by multiplying the above by u^2, and letting n go to infinity, that the limit of u can't be non-zero. Hence, u converges to 0, and hence z converges to θ^*. Now, multiplying (14) by $u^2e^{-nu^2/2}$ and replacing u in terms of z_1, we can verify that $\sqrt{n}z_1e^{-z_1^2/2}$ converges to 0, and hence so does $\sqrt{n}e^{-y^2/2}$. Now the result follows from (12) by taking the limit, and using the above results. □

We have shown above that when the contamination class is chosen to have reasonable shape constraints, the local sensitivity measure (LS) converges to 0 at a reasonable rate, and that its asymptotic behavior is similar to that of the global robustness measure ($\sup \rho_\pi$). This, in particular, means that the concern about the asymptotic behavior of LS may be resolved, in the context of ε-contamination class, by choosing the contamination classes with reasonable shape restrictions.

Acknowledgements: I would like to express my sincere appreciation to the organizers of the 2nd International Workshop on Bayesian Robustness held in Rimini, Italy for their efforts and support. I also thank the referees for their careful reading of the manuscript and their suggestions which improved the presentation.

REFERENCES

BASU, S. (1994). Local sensitivity, Functional derivatives and nonlinear posterior quantities. Research report, Department of Mathematical Science, University of Arkansas.

BERGER, J. (1994). An overview of Robust Bayesian Analysis (with discussion). *Test*,3 No. 1, pp5-124.

CHEN, C. F. (1985). On asymptotic normality of limiting density functions with Bayesian implications. *J. Roy. Statist. Soc.* B97, 540-546.

GUSTAFSON, P. (1994). Posterior Expectations. Ph.D Thesis, Carnegie Mellon University.

GUSTAFSON, P. WASSERMAN, L AND SRINIVASAN, C. (1994). Local sensitivity. (with discussion) To appear in *Bayesian Statistics 5*.

LAVINE, M. WASSERMAN, L. AND WOLPERT, R. L. (1993). Linearization of Bayesian robustness problems. *Journal of Statistical planning and inference* 37, 307-316.

MENG, Q. AND SIVAGANESAN, S. (1995) Local sensitivity of Density bounded priors. To appear in *Statistics and Probability Letters*.

RUGGERI, F. AND WASSERMAN, L. (1991). Density based classes of priors: infinitesimal properties and approximations. Technical Report #528, Carnegie Mellon University.

RUGGERI, F. AND WASSERMAN, L. (1993). Infinitesimal sensitivity of posterior distributions, *Cand. J. Statist., 21, 195-203.*

SIVAGANESAN, S. (1993). Robust Bayesian diagnostics. *J. Statist. Plann. Inf.*, 35, 171-188.

WASSERMAN, L. (1992). Recent methodological advances in robust Bayesian inference. *Bayesian Statistics 4* (J.M. Bernardo, J. O. Berger, A. P. David and A.F.M. Smith, eds.). Oxford University Press, 483-502.

S. SIVAGANESAN
DEPARTMENT OF MATH SCIENCE
UNIVERSITY OF CINCINNATI
P.O. BOX 210025
CINCINNATI, OH 45221-0025

Asymptotics of some Local Robustness measures

discussion by
FABRIZIO RUGGERI
CNR-IAMI, Milano, Italy

I wish to thank Siva Sivaganesan for his paper, which, I believe, will help in boosting the research in the area of local sensitivity. Despite being one of the "hottest" topics in Bayesian robustness (and the number of talks on such a subject at the Workshop in Rimini confirms it), a major concern about it was expressed in very recent years by some authors, mainly Gustafson, Srinivasan and Wasserman. The work of such authors pointed out that the asymptotic behaviour of the quantities used in local sensitivity analysis was very surprising: the quantities showed very high sensitivity of the posterior to the prior, despite the increase in the sample size. By using Sivaganesan's notation, they showed that there contaminating classes such that $LS(n) \not\to 0$, as $n \to \infty$. The problem might have been with the measures of sensitivity and/or the width of the class of the priors. In the first case, all the approach should have been re-thought, being the measures very questionable. Sivaganesan's paper gives an encouraging answer: the problem is in the class and it may be solved if give up some (often unreasonable) priors. His results apply to well-known, quite general classes. He removes point masses from the contaminating priors but, as pointed out in Gustafson, Wasserman and Srinivasan (1995), that is not enough to get $LS(n) \to 0$. Therefore something else must be removed, like in the paper by Sivaganesan.

By analogy with the "classical" theory of robustness, we might look for a breakdown point (actually, a class Γ^*), such that $LS(n)$ converges to 0 in Γ^* but not in larger classes. Problems should arise immediately, about the possibility of providing a "simple" description of such class, its uniqueness and its invariance with respect to changes in the measure of sensitivity.

By considering more regular contaminating priors, i.e. asking for uniformly bounded derivatives, besides uniformly bounded densities, Sivaganesan has been able to find a larger rate of convergence. I am wondering if he has any idea about other requirements (e.g. bounds on derivatives of higher orders) which might give larger rates. For the proof he gave us, no improvement can be obtained by considering bounds on the second derivative.

If we consider the parameter space to be a bounded interval, then it can be seen that $\sqrt{n}LS(n)$ converges to the same quantity, given in Theorem 2.2, when the contaminating priors are chosen in a density bounded class whose bounds have been shifted by the same quantity k (provided that the class is not empty).

Finally, I want to mention that the current version of the paper is different from the one I discussed in Rimini, not only because Sivaganesan has modified the paper by accepting some technical remarks I made, but also because he has kept the most relevant part of his paper, dropping the first and the third parts, on which I had expressed some concerns. The removed parts were not so complete as the left one, but they contained very interesting suggestions, which Sivaganesan wants to further develop in the future.

The first part was mainly about a contamination class for multi-dimensional priors, given, in the bidimensional case, by

$$\Gamma = \{\pi(\theta_1,\theta_2) = (1-\varepsilon_1-\varepsilon_2)\pi_0(\theta_1,\theta_2)+\varepsilon_1\pi_0(\theta_1|\theta_2)q_2(\theta_2)+\varepsilon_2\pi_0(\theta_2|\theta_1)q_1(\theta_1) :$$

$$q_1 \in \mathcal{Q}_1, q_2 \in \mathcal{Q}_2\},$$

for suitable choices of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. The class allows, according to Sivaganesan, for a reasonable flexibility in the shape of the marginals and it is "small", being tightly centered around π_0.

I agree that the class is an useful one, but for slightly different reasons. Rewriting the class as

$$\Gamma = \{\pi(\theta_1,\theta_2) = \pi_0(\theta_1,\theta_2)h_{\varepsilon_1,\varepsilon_2}(\theta_1,\theta_2)\}$$

where

$$h_{\varepsilon_1,\varepsilon_2}(\theta_1,\theta_2) = (1-\varepsilon_1-\varepsilon_2)+\varepsilon_1\frac{q_2(\theta_2)}{\pi_0(\theta_2)}+\varepsilon_2\frac{q_1(\theta_1)}{\pi_0(\theta_1)},$$

then it follows that it is very easy to control the shape of the marginals but the contaminated prior could be quite far from π_0. Besides, the new form makes even easier and "natural" the specification of the class Γ (actually, $\mathcal{Q}_1$ and $\mathcal{Q}_2$), because it is possible to give more (or less) weight to some subsets of the parameter space.

The last missing part is about calibration, another crucial item in local sensitivity analysis, i.e. how to interpret the values of the measures of sensitivity. Sivaganesan suggests to consider the change LS^* in the posterior expected value (w.r.t. the one given by π_0) caused by taking the contaminating prior which gives the supremum of the measure LS of the local sensitivity. We can write $LS^* = W \cdot LS$. It is easy to show that for the same value of LS^* (or LS) there exist very different W's so that very different values of LS (or LS^*) are obtained, making LS^* not very effective as a measure of calibration. I think that the suggestion by Sivaganesan is an important attempt to face the problem of calibration, as well as the paper by Ruggeri and Wasserman (1995), but both approaches are still far from giving a satisfactory solution.

REJOINDER

S. Sivaganesan

I am very much grateful to Fabrizio Ruggeri for his interesting and useful comments. Indeed, I agree with all his comments and observations on the issue of the asymptotics of the local sensitivity measures. The idea of seeking a breakdown point is an interesting one. Ruggeri asks if higher order convergence can be achieved in cases where higher (than second) order derivatives are uniformly bounded. Although I am not absolutely sure, I very much doubt that one can achieve higher order convergence by bounding higher order derivatives; it may be possible with some other restrictions on the structure of the problem.

The rest of the comments by Ruggeri concern two other issues that were addressed in the initial draft of the paper, and in the talk in Rimini. They are not included in the current version of the paper partly due to space limitations, and also to keep the focus on one issue. The comments and suggestions by Ruggeri on the theses other issues are indeed very interesting and intriguing. I hope to address them elsewhere, in a more suitable context. I would like to end this by again thanking the discussant for his valuable comments, and for the wonderful workshop he and his colleagues organized.