Chapter 6

Stochastic differential equations on Φ' driven by Poisson random measures

Stochastic differential equations (SDE's) on infinite dimensional spaces arise from such diverse fields as nonlinear filtering theory, infinite particle systems, neurophysiology, etc. In this chapter, we study SDE's on duals of nuclear spaces driven by Poisson random measures. Namely, we consider the following SDE

$$X_t = X_0 + \int_0^t A(s, X_s)ds + \int_0^t \int_U G(s, X_{s-}, u)\tilde{N}(duds) \qquad (6.0.1)$$

on the dual of a CHNS Φ, where $A : \mathbf{R}_+ \times \Phi' \to \Phi'$, $G : \mathbf{R}_+ \times \Phi' \times U \to \Phi'$, $(U, \mathcal{E}, \mu)$ is a σ-finite standard measure space, N(duds) is a Poisson random measure on $\mathbf{R}_+ \times U$ with characteristic measure $\mu(du)$ and $\tilde{N}(duds)$ is the compensated random measure of N(duds). Motivated by neurophysiological problems, such equations were first considered by Kallianpur and Wolpert [27] [28] for finite dimensional equations (corresponding to the case when the neuron can be regarded as a single point) and for infinite dimensional linear equations. The general case was studied by Hardy, Kallianpur, Ramasubramanian and Xiong [13], most of the results of this chapter being taken from that paper.

The following assumption will be made throughout the rest of this book: There exists a sequence $\{\phi_i\}$ of elements in Φ, such that $\{\phi_i\}$ is a CONS in Φ_0 and is a COS in each space $\Phi_n, n \in \mathbf{Z}$.

Let $\phi_i^n \equiv \phi_i \|\phi_i\|_n^{-1}$, $n \in \mathbf{Z}$, $i \in \mathbf{N}^+$. It is easy to see that $\{\phi_i^n\}$ is a CONS in Φ_n.

6.1 Weak convergence theorems

In this section we establish the existence of a weak solution of (6.0.1) by weak convergence technique for Φ'-valued stochastic process sequences. The idea is as follows: Consider a sequence of Φ'-valued process $\{X^n\}$ governed by a sequence of SDE's of the type of (6.0.1) with coefficients (A^n, G^n) tending to (A, G) in some sense (cf. Assumption (A2) below). Under suitable conditions, show that the distribution sequence $\{\mathcal{L}(X^n)\}$ is tight and its cluster points are solutions to the martingale problem corresponding to (6.0.1). By making use of the representation theorem for purely-discontinuous Φ'-valued martingales introduced in Chapter 3, we then obtain a weak solution of (6.0.1) from the solutions of the martingale problem.

We define the weak solution of (6.0.1) first.

Definition 6.1.1 *A probability measure λ on $D([0,T],\Phi')$ is called a* **weak solution** *on [0,T] of the SDE (6.0.1) with initial distribution λ_0 on the Borel sets of Φ' if there exists a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ and a Poisson random measure N with σ-finite characteristic measure μ, a Φ'-valued process X such that λ and λ_0 are the distributions of X and X_0 respectively and for any $\phi \in \Phi$, $t \in [0,T]$, we have*

$$X_t[\phi] = X_0[\phi] + \int_0^t A(s, X_s)[\phi] ds + \int_0^t \int_U G(s, X_{s-}, u)[\phi] \tilde{N}(duds) \quad a.s. \tag{6.1.1}$$

If [0,T] can be changed to $[0,\infty)$ and (6.1.1) holds for any $t \geq 0$, then we call λ on $D([0,\infty),\Phi')$ a **weak solution** *of (6.0.1).*

The next lemma is useful in calculating the norm $||\phi||_{-r}$ for $r \geq 0$.

Lemma 6.1.1 *For any $r \geq 0$ and $j \geq 1$, we have*

$$||\phi_j||_r ||\phi_j||_{-r} = 1. \tag{6.1.2}$$

Proof: Note that

$$\begin{aligned} 1 &= \phi_j[\phi_j] \leq ||\phi_j||_{-r} ||\phi_j||_r \\ &= ||\phi_j||_r \sup\{\phi_j[\phi] : \phi \in \Phi, ||\phi||_r = 1\} \\ &= ||\phi_j||_r \sup\left\{\phi_j \left[\sum_k < \phi, \phi_k^r >_r \phi_k^r\right] : \phi \in \Phi, ||\phi||_r = 1\right\} \\ &= \sup\{< \phi, \phi_j^r >_r : \phi \in \Phi, ||\phi||_r = 1\} \leq 1. \end{aligned}$$

∎

To show the existence of a weak solution of (6.0.1), we impose the following assumptions (I) for (A, G, μ): $\forall T > 0$, $\exists p_0 = p_0(T) \in \mathbf{N}^+$, such that, $\forall p \geq p_0$, $\exists q \geq p$ and a constant $K = K(p, q, T)$ such that

(I1) (Continuity) $\forall t \in [0,T]$, the maps $v \in \Phi_{-p} \to A(t,v) \in \Phi_{-q}$ and $v \in \Phi_{-p} \to G(t,v,\cdot) \in L^2(U,\mu;\Phi_{-p})$ are continuous.
(I2) (Coercivity) $\forall t \in [0,T]$ and $\phi \in \Phi$,

$$2A(t,\phi)[\theta_p\phi] \leq K(1+\|\phi\|_{-p}^2); \tag{6.1.3}$$

(I3) (Growth) $\forall t \in [0,T]$ and $v \in \Phi_{-p}$, we have

$$\|A(t,v)\|_{-q}^2 \leq K(1+\|v\|_{-p}^2)$$

and

$$\int_U \|G(t,v,u)\|_{-p}^2\mu(du) \leq K(1+\|v\|_{-p}^2).$$

Remark 6.1.1 *The left hand side of (6.1.3) is well-defined as $\theta_p\Phi \subset \Phi$.*

Proof: We only need to show that for any $p,r \geq 0$ and $\phi \in \Phi$, we have $\theta_p\phi \in \Phi_r$. Note that

$$\begin{aligned}
\theta_p\phi &= \theta_p\left\{\sum_j <\phi,\phi_j^r>_r \phi_j^r\right\} \\
&= \theta_p\left\{\sum_j <\phi,\phi_j^r>_r \|\phi_j\|_p^{-1}\|\phi_j\|_r^{-1}\phi_j^{-p}\right\} \\
&= \sum_j <\phi,\phi_j^r>_r \|\phi_j\|_p^{-1}\|\phi_j\|_r^{-1}\phi_j^p \\
&= \sum_j <\phi,\phi_j^r>_r \|\phi_j\|_p^{-2}\phi_j^r
\end{aligned}$$

and

$$\begin{aligned}
\sum_j\{<\phi,\phi_j^r>_r \|\phi_j\|_p^{-2}\}^2 &\leq \sum_j <\phi,\phi_j^r>_r^2 \\
&= \|\phi\|_r^2 < \infty.
\end{aligned}$$

Therefore $\theta_p\phi \in \Phi_r$. ∎

Now we consider a sequence of Φ'-valued processes $\{X^n\}$ satisfying SDE's of the type of (6.0.1) with coefficients A^n, G^n, characteristic measures μ^n and initial distributions λ_0^n. We shall give conditions such that this sequence is relatively compact and its cluster points are characterized by the SDE (6.0.1). We fix $T > 0$ and consider Φ'-valued processes on [0,T].

We make the following assumptions (A) for the sequence $(A^n, G^n, \mu^n, \lambda_0^n)$:
(A1)(1°) The assumptions (I) are satisfied by (A^n, G^n, μ^n) for each n. Furthermore, the continuity in (I1) is uniform in n, the indexes p, q, p_0 and the

constant K in (I) are independent of n.
(2°) For each $n \geq 1$, the following SDE

$$X_t = X_0 + \int_0^t A^n(s, X_s)ds + \int_0^t \int_U G^n(s, X_{s-}, u)\tilde{N}^n(duds)$$

has a weak solution λ^n on [0,T] with initial distribution λ_0^n. Let X^n be a Φ'-valued process on a stochastic basis $(\Omega^n, \mathcal{F}^n, P^n, (\mathcal{F}_t^n))$ corresponding to the weak solution λ^n. We further assume that there exists an index $p = p(T) \geq p_0$ and a constant $\tilde{K} > 0$ independent of n such that $X_t^n \in \Phi_{-p}$, P^n-a.s. $\forall t \in [0, T]$ and

$$E^{P^n} \sup_{0 \leq t \leq T} \|X_t^n\|_{-p}^2 \leq \tilde{K}.$$

(A2)(1°) $\mu^n = \mu$;
(2°) $\forall t \in [0, T]$, $v \in \Phi_{-p_1}$ and $\phi \in \Phi$, we have

$$A^n(t, v)[\phi] \to A(t, v)[\phi];$$

(3°) $\forall t \in [0, T]$, $v \in \Phi_{-p_1}$, we have

$$\int_U \|G^n(t, v, u) - G(t, v, u)\|_{-p_1}^2 \mu(du) \to 0.$$

We need the following definition and Theorem 6.1.1 about real-valued stochastic processes taken from the book of Jacod and Shiryaev ([22], p317, Corollary 3.33 and p322, Theorem 4.13).

Definition 6.1.2 *A sequence of probability measures $\{\lambda^n\}$ on $D([0, T], \mathbf{R})$ is* **C-tight** *if it is tight and all cluster points are supported on $C([0, T], \mathbf{R})$.*

Theorem 6.1.1 *For each n, let λ^n be a probability measure on $D([0, T], \mathbf{R})$ induced by a real-valued semimartingale $\xi_0^n + M_t^n + A_t^n$ on a stochastic basis $(\Omega^n, \mathcal{F}^n, P^n, (\mathcal{F}_t^n))$, where ξ_0^n is a random variable, $M^n \in \mathcal{M}^2(\mathbf{R})$ and $A^n \in \mathcal{A}$. If $\{\xi_0^n\}$ is tight in $\mathbf{R}$, $\{< M^n >\}$ and $\{A^n\}$ are* **C***-tight, then $\{\lambda^n\}$ is tight.*

Let $p_1 = p_1(T) \geq p$ be an index such that the canonical injection from Φ_{-p} into Φ_{-p_1} is Hilbert-Schmidt.

Lemma 6.1.2 *Under assumption (A1), $\{\lambda^n\}$ is tight in $D([0, T], \Phi_{-p_1})$.*

Proof: For any $\phi \in \Phi$, let

$$C_t^n = \int_0^t A^n(s, X_s^n)[\phi]ds$$

and

$$M_t^n = \int_0^t \int_U G^n(s, X_{s-}^n, u)[\phi]\tilde{N}^n(duds).$$

Then $\forall \epsilon > 0$, $\exists \delta = \delta_\epsilon > 0$ such that

$$\begin{aligned}
&\sup_n P^n\left(\sup_{0<\beta-\alpha<\delta} |C_\alpha^n - C_\beta^n| > \epsilon\right) \\
= &\sup_n P^n\left(\sup_{0<\beta-\alpha<\delta} \left|\int_\alpha^\beta A^n(s, X_s^n)[\phi]ds\right| > \epsilon\right) \\
\leq &\sup_n \frac{1}{\epsilon^2} E^{P^n}\left(\delta^2 \sup_{0\leq s\leq T} |A^n(s, X_s^n)[\phi]|^2\right) \\
\leq &\sup_n \left(\frac{\delta}{\epsilon}\right)^2 E^{P^n}\left(K\left(1 + \sup_{0\leq s\leq T} \|X_s^n\|_{-p}^2\right)\|\phi\|_q^2\right) \\
\leq &K\delta^2\|\phi\|_q^2(1+\tilde{K})/\epsilon^2 < \epsilon.
\end{aligned}$$

The set

$$K_\epsilon = \left\{ f \in C([0,T], \mathbf{R}) : \begin{array}{l} f(0) = 0,\ |f(s) - f(t)| < \epsilon 2^{-m}, \\ \forall m \geq 1 \text{ and } |s-t| \leq \delta_{\epsilon 2^{-m}} \end{array} \right\}$$

is relatively compact and

$$P^n(C^n \notin K_\epsilon) \leq \sum_{m=1}^\infty \epsilon 2^{-m} = \epsilon,\ \forall n \geq 1,$$

i.e. $\{C^n\}$ is **C**-tight. Similarly we can prove the **C**-tightness for $\{< M^n >\}$. Furthermore, the sequence $\{X_0^n[\phi]\}$ is tight in **R** as

$$P^n\left\{|X_0^n[\phi]|^2 > \frac{\tilde{K}\|\phi\|_p^2}{\epsilon}\right\} \leq \frac{\epsilon}{\tilde{K}\|\phi\|_p^2} E|X_0^n[\phi]|^2 \leq \epsilon.$$

Hence, it follows from Theorem 6.1.1 that, $\forall \phi \in \Phi$, the sequence of semimartingales $X_t^n[\phi] = X_0^n[\phi] + C_t^n + M_t^n$ is tight in D([0,T],**R**). It then follows from assumption (A1)(2°) and Theorem 2.5.2 that $\{\lambda^n\}$ is tight in $D([0,T], \Phi_{-p_1})$. ∎

Let λ^* be a cluster point of $\{\lambda^n\}$ in $D([0,T], \Phi_{-p_1})$. To characterize λ^*, we need a connecting idea which is the martingale problem formulated below. Let

$$\mathcal{D}_0^\infty(\Phi') = \left\{ F : \Phi' \to \mathbf{R}\ \begin{array}{l} \exists h \in C_0^\infty(\mathbf{R}) \text{ and } \phi \in \Phi \text{ such} \\ \text{that } F(v) = h(v[\phi]),\ \forall v \in \Phi' \end{array} \right\}.$$

For $F \in \mathcal{D}_0^\infty(\Phi')$, consider the map $\mathcal{L}_s F : \Phi' \to \mathbf{R}$ defined by

$$\begin{aligned}\mathcal{L}_s F(v) &= A(s,v)[\phi]h'(v[\phi])\\ &+ \int_U \{h(v[\phi]+G(s,v,u)[\phi]) - h(v[\phi]) - G(s,v,u)[\phi]h'(v[\phi])\}\mu(du).\end{aligned}$$

For $Z \in D([0,T],\Phi')$, let

$$M^F(Z)_t = F(Z(t)) - F(Z(0)) - \int_0^t \mathcal{L}_s F(Z(s))ds. \tag{6.1.4}$$

Definition 6.1.3 *A probability measure λ on $D([0,T],\Phi')$ is called a solution on [0,T] of the $\mathcal{L}$-* **martingale problem** *with initial distribution λ_0 if, $\forall F \in \mathcal{D}_0^\infty(\Phi')$, $\{M^F(Z)_t, 0 \le t \le T\}$ is a λ-martingale and $\lambda \circ Z(0)^{-1} = \lambda_0$. If λ is a probability measure on $D([0,\infty),\Phi')$ such that $\forall F \in \mathcal{D}_0^\infty(\Phi')$, $\{M^F(Z)_t, 0 \le t < \infty\}$ is a λ-martingale and $\lambda \circ Z(0)^{-1} = \lambda_0$, we call λ a solution of the $\mathcal{L}$-martingale problem with initial distribution λ_0.*

Now, we proceed to prove that $\{M^F(Z)_t, 0 \le t \le T\}$ is a λ^*-martingale for every $F \in \mathcal{D}_0^\infty(\Phi')$. We define $M_n^F(Z)_t$ in a similar fashion as in (6.1.4). From assumption (A1) and Itô's formula, it is easy to see that $\{M_n^F(Z)_t, 0 \le t \le T\}$ is a λ^n-martingale. To pass to the limit, we need the following Lemmas.

Lemma 6.1.3 *Under assumption (A1), M_n^F is a λ^n-martingale and*

$$E^{\lambda^n}|M_n^F(Z)_t|^2 \le \|h'\|_\infty^2 K\|\phi\|_p^2(\tilde{K}+1)T, \ \forall F \in \mathcal{D}_0^\infty(\Phi') \text{ and } n \ge 1,$$

where $\|h'\|_\infty = \sup_{x\in\mathbf{R}}|h'(x)|$.

Proof: Applying the Itô's formula (Theorem 3.4.4) to (6.0.1), we have

$$\begin{aligned}&M_n^F(X^n)_t\\ = &\int_0^t\int_U \{h(X_{s-}^n[\phi]+G^n(s,X_{s-}^n,u)[\phi]) - h(X_{s-}^n[\phi])\}\tilde{N}^n(duds).\end{aligned}$$

Therefore $M_n^F(X^n)$ is a P^n-martingale and hence, $M_n^F(Z)$ is a λ^n-martingale. Further

$$\begin{aligned}&E^{\lambda^n}|M_n^F(Z)_t|^2\\ = &E^{P^n}\int_0^t\int_U |h(X_{s-}^n[\phi]+G^n(s,X_s^n,u)[\phi]) - h(X_{s-}^n[\phi])|^2\mu^n(du)ds\\ \le &\|h'\|_\infty^2 E^{P^n}\int_0^t\int_U \|G^n(s,X_s^n,u)[\phi]\|^2\mu^n(du)ds\\ \le &\|h'\|_\infty^2 E^{P^n}\int_0^t\int_U \|G^n(s,X_s^n,u)\|_{-p}^2\|\phi\|_p^2\mu^n(du)ds\\ \le &\|h'\|_\infty^2 K\|\phi\|_p^2(\tilde{K}+1)T.\end{aligned}$$

∎

Lemma 6.1.4 *Under assumption (A1), we have*

$$E^{\lambda^*} \sup_{0\le t\le T} \|Z_t\|^2_{-p_1} \le \tilde{K}. \tag{6.1.5}$$

Proof: As λ^* is a cluster point of $\{\lambda^n\}$, without loss of generality, we may assume that λ^n converges to λ^* weakly. By Skorohod's Theorem, there exists a probability space $(\Omega, \mathcal{F}, P)$ and $D([0,T], \Phi_{-p_1})$-valued random variables ξ^n and ξ on it, such that ξ^n and ξ have distributions λ^n and λ^* respectively, and ξ^n converges to ξ a.s. It follows from (A1) that

$$E \sup_{0\le t\le T} \|\xi^n_t\|^2_{-p_1} \le E \sup_{0\le t\le T} \|\xi^n_t\|^2_{-p} = E^{P^n} \sup_{0\le t\le T} \|X^n_t\|^2_{-p} \le \tilde{K}.$$

Let $n \to \infty$, using Fatou's Lemma, we have

$$\begin{aligned} E^{\lambda^*} \sup_{0\le t\le T} \|Z_t\|^2_{-p_1} &= E \sup_{0\le t\le T} \|\xi_t\|^2_{-p_1} \\ &\le \liminf_{n\to\infty} E \sup_{0\le t\le T} \|\xi^n_t\|^2_{-p_1} \\ &\le \liminf_{n\to\infty} E^{P^n} \sup_{0\le t\le T} \|X^n_t\|^2_{-p} \le \tilde{K}. \end{aligned}$$ ∎

The following corollary will be used in Chapters 8 and 9.

Corollary 6.1.1 *Under assumption (A1), we have $Z_t \in \Phi_{-p}$, λ^*-a.s. $\forall t \in [0,T]$. Further*

$$E^{\lambda^*} \sup_{0\le t\le T} \|Z_t\|^2_{-p} \le \tilde{K}.$$

Proof: Using the notations of Lemma 6.1.4, we have

$$\begin{aligned} E^{\lambda^*} \sup_{0\le t\le T} \sum_{j=1}^{\infty} Z_t[\phi^p_j]^2 &= E \sup_{0\le t\le T} \sum_{j=1}^{\infty} \xi_t[\phi^p_j]^2 \\ &\le \liminf_{n\to\infty} E \sup_{0\le t\le T} \sum_{j=1}^{\infty} \xi^n_t[\phi^p_j]^2 \\ &= \liminf_{n\to\infty} E^{P^n} \sup_{0\le t\le T} \|X^n_t\|^2_{-p} \le \tilde{K}. \end{aligned}$$ ∎

The following two lemmas are elementary and we leave their proofs to the reader.

Lemma 6.1.5 *For $h \in C^\infty_0(\mathbf{R})$, let*

$$H(x,y) = h(x+y) - h(x) - h'(x)y, \ \forall x,y \in \mathbf{R}.$$

Then, for any x, y, x_1, x_2, y_1 and $y_2 \in \mathbf{R}$, we have the following inequalities:

$$|H(x,y)| \leq \|h''\|_\infty y^2;$$

$$|H(x_1,y) - H(x_2,y)| \leq \|h'''\|_\infty y^2 |x_1 - x_2|; \tag{6.1.6}$$

$$|H(x,y_1) - H(x,y_2)| \leq \|h''\|_\infty (|y_1| + |y_2|)|y_1 - y_2|. \tag{6.1.7}$$

Lemma 6.1.6 *Let C_0 be a compact subset of Φ_{-p_1}. Under assumptions (A), we have that for any $t \in [0,T]$ and $\phi \in \Phi$,*

$$\sup_{v \in C_0} |(A^n(t,v) - A(t,v))[\phi]| \to 0,$$

and

$$\sup_{v \in C_0} \int_U \|G^n(t,v,u) - G(s,v,u)\|_{-p_1}^2 \mu(du) \to 0.$$

The following lemma is the major step in passing to the limit.

Lemma 6.1.7 *Suppose that (A, G, μ) satisfies assumptions (I) and $\{(A^n, G^n, \mu^n)\}$ satisfies assumptions (A). Let ξ^n and ξ be $D([0,T], \Phi_{-p_1})$-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that ξ^n converges to ξ a.s.*

Then, for $F \in \mathcal{D}_0^\infty(\Phi')$ and $t \in [0,T] \setminus \mathcal{N}$, $M_n^F(\xi^n)_t$ converges to $M^F(\xi)_t$ in probability, where $\mathcal{N} = \{t : P(\omega : \xi_t \neq \xi_{t-}) > 0\}$.

Proof: As ξ^n converges to ξ, then, for any $\epsilon > 0$, there exists a compact subset C of $D([0,T], \Phi_{-p_1})$ such that

$$P(\omega : \xi^n \in C) > 1 - \epsilon \quad \text{and} \quad P(\omega : \xi \in C) > 1 - \epsilon. \tag{6.1.8}$$

It follows from Theorem 2.4.3 that there exists a compact subset C_0 of Φ_{-p_1} such that

$$C \subset \{Z \in D([0,T], \Phi_{-p_1}) : Z_s \in C_0, \ \forall s \in [0,T]\}.$$

Let $M > 0$ be such that

$$C_0 \subset \{x \in \Phi_{-p_1} : \|x\|_{-p_1} \leq M\}.$$

For $F \in \mathcal{D}_0^\infty(\Phi')$, let $h \in C_0^\infty(\mathbf{R})$ and $\phi \in \Phi$ such that $F(v) = h(v[\phi])$ for $v \in \Phi'$. By the definition of $M_n^F(Z)_t$ and $M^F(Z)_t$, for ω such that $\xi^n(\omega)$ and $\xi(\omega) \in C$, we have (suppressing ω for convenience)

$$\begin{aligned}
& |M_n^F(\xi^n)_t - M^F(\xi)_t| \\
\leq \ & |h(\xi_t^n[\phi]) - h(\xi_t[\phi]) - h(\xi_0^n[\phi]) + h(\xi_0[\phi])| \\
& + \int_0^t |A^n(s,\xi_s^n)[\phi] h'(\xi_s^n[\phi]) - A(s,\xi_s)[\phi] h'(\xi_s[\phi])| ds \\
& + \int_0^t \int_U |H(\xi_s^n[\phi], G^n(s,\xi_s^n,u)[\phi]) - H(\xi_s[\phi], G(s,\xi_s,u)[\phi])| \mu(du) ds \\
\equiv \ & I_1 + I_2 + I_3.
\end{aligned}$$

Note that

$$\begin{aligned}
I_3 &\leq \int_0^t \int_U |H(\xi_s^n[\phi], G^n(s,\xi_s^n,u)[\phi]) - H(\xi_s[\phi], G^n(s,\xi_s^n,u)[\phi])|\mu(du)ds \\
&\quad + \int_0^t \int_U |H(\xi_s[\phi], G^n(s,\xi_s^n,u)[\phi]) - H(\xi_s[\phi], G(s,\xi_s^n,u)[\phi])|\mu(du)ds \\
&\quad + \int_0^t \int_U |H(\xi_s[\phi], G(s,\xi_s^n,u)[\phi]) - H(\xi_s[\phi], G(s,\xi_s,u)[\phi])|\mu(du)ds \\
&\leq \int_0^t \int_U \|h'''\|_\infty |G^n(s,\xi_s^n,u)[\phi]|^2 |\xi_s^n[\phi] - \xi_s[\phi]|\mu(du)ds \\
&\quad + \int_0^t \int_U \|h''\|_\infty (|G^n(s,\xi_s^n,u)[\phi]| + |G(s,\xi_s^n,u)[\phi]|) \\
&\qquad |G^n(s,\xi_s^n,u)[\phi] - G(s,\xi_s^n,u)[\phi]|\mu(du)ds \\
&\quad + \int_0^t \int_U \|h''\|_\infty (|G(s,\xi_s^n,u)[\phi]| + |G(s,\xi_s,u)[\phi]|) \\
&\qquad |G(s,\xi_s^n,u)[\phi] - G(s,\xi_s,u)[\phi]|\mu(du)ds \\
&= I_{31} + I_{32} + I_{33}, \qquad \text{say,}
\end{aligned}$$

where the second inequality follows from (6.1.6) and (6.1.7). For ω such that $\xi^n(\omega)$ and $\xi(\omega) \in C$, we have (again suppressing ω),

$$I_{31} \leq \|h'''\|_\infty K(1+M^2)\|\phi\|_{p_1}^2 \int_0^t |\xi_s^n[\phi] - \xi_s[\phi]|ds \to 0, \ a.s.;$$

$$\begin{aligned}
I_{32}^2 &\leq \|h''\|_\infty \int_0^t \int_U (|G^n(s,\xi_s^n,u)[\phi]| + |G(s,\xi_s^n,u)[\phi]|)^2 \mu(du)ds \\
&\qquad \int_0^t \int_U |G^n(s,\xi_s^n,u)[\phi] - G(s,\xi_s^n,u)[\phi]|^2 \mu(du)ds \\
&\leq \|h''\|_\infty 4KT(1+M^2)\|\phi\|_{p_1}^4 \\
&\qquad \int_0^t \sup_{v\in C_0} \int_U \|G^n(s,v,u) - G(s,v,u)\|_{-p_1}^2 \mu(du)ds \to 0;
\end{aligned}$$

and

$$\begin{aligned}
I_{33}^2 &\leq \|h''\|_\infty 4KT(1+M^2)\|\phi\|_{p_1}^4 \\
&\qquad \int_0^t \int_U \|G(s,\xi_s^n,u) - G(s,\xi_s,u)\|_{-p_1}^2 \mu(du)ds \to 0.
\end{aligned}$$

Hence, for ω such that $\xi^n(\omega)$ and $\xi(\omega) \in C$, we have $I_3 \to 0$. The same arguments yield that $I_2 \to 0$. It is easy to see that, for $t \notin \mathcal{N}$, we have that $I_1 \to 0$ a.s. So, combining with (6.1.8), we see that, for $t \notin \mathcal{N}$, $M_n^F(\xi^n)_t$ converges to $M^F(\xi)_t$ in probability. ∎

The next result characterizes λ^*.

Theorem 6.1.2 *Suppose that (A, G, μ) satisfies assumptions (I) and $\{(A^n, G^n, \mu^n)\}$ satisfies assumptions (A). Then λ^* is a solution on [0,T] of the $\mathcal{L}$-martingale problem.*

Proof: Let ξ_n and ξ be as given in the proof of Lemma 6.1.4. By Lemma 6.1.3, for fixed t, we can easily see that $\{M_n^F(\xi^n)_t\}_{n\in\mathbf{N}}$ is uniformly integrable. Hence, for any bounded continuous $\mathcal{B}_s$-measurable function f on $D([0,T], \Phi_{-p_1})$, we have that $\{f(\xi^n)M_n^F(\xi^n)_t\}_{n\in\mathbf{N}}$ is uniformly integrable. So, by Lemma 6.1.7, for $t, s \notin \mathcal{N}$ and $s < t$, we have

$$\begin{aligned} E^{\lambda^*}M^F(Z)_t f(Z) &= EM^F(\xi)_t f(\xi) = \lim_n EM_n^F(\xi^n)_t f(\xi^n) \\ &= \lim_n E^{\lambda^n}M_n^F(Z)_t f(Z) = \lim_n E^{\lambda^n}M_n^F(Z)_s f(Z) \\ &= \lim_n EM_n^F(\xi^n)_s f(\xi^n) = EM^F(\xi)_s f(\xi) \\ &= E^{\lambda^*}M^F(Z)_s f(Z). \end{aligned}$$

i.e.

$$E^{\lambda^*}M^F(Z)_t f(Z) = E^{\lambda^*}M^F(Z)_s f(Z). \tag{6.1.9}$$

For general $s < t$, as $\mathcal{N}$ is countable, we can find two sequences s_n and t_n decreasing to s and t respectively such that $s_n, t_n \notin \mathcal{N}$ and $s_n < t_n$. Then, (6.1.9) still holds with (s,t) replaced by (s_n, t_n) as f is also $\mathcal{B}_{s_n}$-measurable. By the right continuity and the uniform integrability of $M^F(Z)_{t_n} f(Z)$ and $M^F(Z)_{s_n} f(Z)$, passing to limit, we see that (6.1.9) still holds for any $t > s$. Define two signed measures on $\mathcal{B}_s$ by

$$\mathcal{V}_t(A) = E^{\lambda^*}M^F(Z)_t 1_A(Z) \quad \text{and} \quad \mathcal{V}_s(A) = E^{\lambda^*}M^F(Z)_s 1_A(Z).$$

Then, from the above, we see that the integrals of f with respect to signed measures $\mathcal{V}_t$ and $\mathcal{V}_s$ coincide for any bounded continuous $\mathcal{B}_s$-measurable functions f. Hence $\mathcal{V}_t = \mathcal{V}_s$ on $\mathcal{B}_s$. i.e. $\{M^F(Z)_t\}$ is a λ^*-martingale. ∎

It remains to prove that λ^* is a weak solution on [0,T] of the SDE (6.0.1). The idea is to show that the martingale $M_\phi(t, Z)$, defined in Lemma 6.1.9 below, can be represented as a stochastic integral with respect to a Poisson random measure. We do this by proving that $M_\phi(t, Z)$ is purely-discontinuous in Theorem 6.1.3 and characterizing the jump process $\Delta M_\phi(t, Z)$ in Lemma 6.1.11.

Lemma 6.1.8 *There exist two sequences of real functions $\{\rho_m\}$, $\{g_m\}$ on $\mathbf{R}$ and a constant L such that, $\forall m \in \mathbf{N}$, $\rho_m \in C_0^\infty(\mathbf{R})$ and*
(1) $\rho_m(x) = x$ when $|x| \le m-1$ and $|\rho_m(x)| \le L|x|$ for any $x \in \mathbf{R}$;
(2) $\|\rho_m'\|_\infty \le L$, $\|\rho_m''\|_\infty \le L/m$, and $\|\rho_m\rho_m''\|_\infty \le L$;
(3) $g_m \in C_0(\mathbf{R})$ are nonnegative functions increasing to x^2 as m tends to ∞.
Furthermore, for each m, there exists d_m such that $g_m(x) = 0$ when $|x| \le d_m$.

Proof: Let $\tilde{\rho}_m$ be a sequence of odd functions defined in $\mathbf{R}$ as follows:

$$\tilde{\rho}_m(x) = \begin{cases} x & \text{as } 0 \le x \le m \\ 0 & \text{as } x \ge 2m \\ -\frac{(x-2m)^3}{m^2}\left(\frac{9(x-m)^2}{m^2} + \frac{4(x-m)}{m} + 1\right) & \text{as } m \le x \le 2m. \end{cases}$$

Then $\tilde{\rho}_m \in C_0^2(\mathbf{R})$ and for any $x \in \mathbf{R}$,

$$|\tilde{\rho}_m(x)| \le 14|x|,\ |\tilde{\rho}_m'(x)| \le 64 \text{ and } |\tilde{\rho}_m''(x)| \le \frac{234}{m}. \tag{6.1.10}$$

Let J be the Friedrichs mollifier given by

$$J(x) = \begin{cases} k \cdot \exp\{-(1-x^2)^{-1}\} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \ge 1, \end{cases}$$

where k is a constant such that $\int J(x)dx = 1$. Let

$$\rho_m(x) = (\tilde{\rho}_m * J)(x) \equiv \int J(x-y)\tilde{\rho}_m(y)dy.$$

Then $\rho_m \in C_0^\infty(\mathbf{R})$. As $\tilde{\rho}_m \in C_0^2(\mathbf{R})$, integrating by parts, we have

$$\rho_m'(x) = \int J(x-y)\tilde{\rho}_m'(y)dy \tag{6.1.11}$$

and

$$\rho_m''(x) = \int J(x-y)\tilde{\rho}_m''(y)dy.$$

Then, for $|x| \le m-1$,

$$\begin{aligned} \rho_m'(x) &= \int J(x-y)\tilde{\rho}_m'(y)dy = \int J(y)\tilde{\rho}_m'(x-y)dy \\ &= \int J(y)dy = 1. \end{aligned}$$

As $\rho_m(0) = 0$, we have

$$\rho_m(x) = x \qquad \text{as } |x| \le m-1.$$

In addition, by (6.1.10) and (6.1.11), we have

$$\|\rho_m'\|_\infty \le 64 \quad \text{and} \quad \|\rho_m''\|_\infty \le \frac{234}{m}.$$

Furthermore, by (6.1.11) again, $\rho_m''(x) = 0$ as $|x| \ge 2m+1$. Hence

$$\|\rho_m\rho_m''\|_\infty \le \sup_{|x|\le 2m+1} 64|x|\frac{234}{m} \le 64|2m+1|\frac{234}{m} \le L.$$

(3) Let g_m be an even function given by

$$g_m(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{m} \text{ or } x \ge m; \\ (x - m^{-1})^2 & \text{if } m^{-1} \le x \le m-1; \\ (m - m^{-1} - 1)^2(m - x) & \text{if } m-1 \le x \le m. \end{cases}$$

It is easy to check the condition (3) for g_m. ∎

Lemma 6.1.9 *$\forall \phi \in \Phi$, let*

$$M_\phi(t, Z) = Z_t[\phi] - Z_0[\phi] - \int_0^t A(s, Z_s)[\phi]ds.$$

Under the conditions of Theorem 6.1.2, $\{M_\phi(t,Z)\}_{t\le T}$ is a λ^*-square integrable martingale.

Proof: Let ρ_m be given by Lemma 6.1.8. Let $F_m \in \mathcal{D}_0^\infty(\Phi')$ be given by $F_m(v) = \rho_m(v[\phi])$. Let

$$\mathcal{X} \equiv \{Z \in D([0,T], \Phi_{-p_1}) : \|Z_t\|_{-p_1} \le (m-1)\|\phi\|_{p_1}^{-1},\ \forall t \in [0,T]\}.$$

Then, for $Z \in \mathcal{X}$, we have $|Z_s[\phi]| \le m - 1$ and hence,

$$M^{F_m}(Z)_t = M_\phi(t, Z) - \int_0^t \int_U H_m(Z_s[\phi], G(s, Z_s, u)[\phi])\mu(du)ds, \quad (6.1.12)$$

where H_m is defined as in Lemma 6.1.5 with h replaced by ρ_m. Hence, by (6.1.12), Lemma 6.1.5, assumption (I3) and (6.1.5), we have

$$\begin{aligned} & E^{\lambda^*}|M^{F_m}(Z)_t - M_\phi(t,Z)|1_{\mathcal{X}}(Z) \\ \le\ & E^{\lambda^*}\int_0^t \int_U \|\rho_m''\|_\infty |G(s, Z_s, u)[\phi]|^2 \mu(du)ds \\ \le\ & \frac{L}{m} tK(1+\tilde{K})\|\phi\|_{p_1}^2 \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

On the other hand,

$$\lambda^*(\mathcal{X}^c) \le \frac{1}{(m-1)^2\|\phi\|_{p_1}^{-2}} E^{\lambda^*}\left(\sup_{0\le t\le T} \|Z_t\|_{-p_1}^2\right) \le \frac{\|\phi\|_{p_1}^2}{(m-1)^2}\tilde{K} \to 0,$$

as $m \to \infty$. So, $\forall \epsilon > 0$, we have

$$\begin{aligned} & \lambda^*\left\{Z \in D([0,T], \Phi_{-p_1}) : |M^{F_m}(Z)_t - M_\phi(t,Z)| > \epsilon\right\} \\ \le\ & \lambda^*(\mathcal{X}^c) + \frac{1}{\epsilon} E^{\lambda^*}|M^{F_m}(Z)_t - M_\phi(t,Z)|1_{\mathcal{X}}(Z) \to 0. \end{aligned}$$

i.e.

$$M^{F_m}(Z)_t \to M_\phi(t,Z) \quad \text{in probability } \lambda^*. \tag{6.1.13}$$

Next, by assumptions (I) and the properties of ρ_m, it is easy to show that there exists a constant C' independent of m such that

$$\left|M^{F_m}(Z)_t\right| \le C'\left(1+\sup_{0\le t\le T}\|Z_t\|_{-p_1}^2\right). \tag{6.1.14}$$

Hence, by Lemma 6.1.4, the left hand side of (6.1.14) is integrable with respect to λ^* uniformly in m. Then, by (6.1.13),

$$E^{\lambda^*}\left|M^{F_m}(Z)_t - M_\phi(t,Z)\right| \to 0.$$

But $\{M^{F_m}(Z)_t\}$ are λ^*-martingales, so $\{M_\phi(t,Z)\}$ is a λ^*-martingale. Finally, by assumptions (I), it is easy to see that there exists a constant C'' such that

$$|M_\phi(t,Z)|^2 \le C''\left(1+\sup_{0\le t\le T}\|Z_t\|_{-p_1}^2\right).$$

Hence, by Lemma 6.1.4 again, $\{M_\phi(t,Z)\}$ is a λ^*-square-integrable-martingale. ∎

Lemma 6.1.10 *Let $< M_\phi > (t,Z)$ be the quadratic variation process of the square integrable martingale M_ϕ. Under the conditions of Theorem 6.1.2, we have*

$$< M_\phi > (t,Z) = \int_0^t \int_U (G(s,Z_s,u)[\phi])^2 \mu(du)ds. \tag{6.1.15}$$

Proof: $\forall \phi \in \Phi$, let

$$\begin{aligned} N_\phi(t,Z) &= Z_t[\phi]^2 - Z_0[\phi]^2 - 2\int_0^t A(s,Z_s)[\phi]Z_s[\phi]ds \\ &\quad - \int_0^t \int_U (G(s,Z_s,u)[\phi])^2\mu(du)ds. \end{aligned}$$

Then, by a similar argument as in the proof of Lemma 6.1.9, $\{N_\phi(t,Z)\}_{t\le T}$ is a λ^*-martingale. By the definition of M_ϕ, it is easy to see that

$$\Delta Z_s[\phi] = \Delta M_\phi(s,Z) \quad \text{and} \quad < M_\phi^c >_t = < Z[\phi]^c >_t, \tag{6.1.16}$$

where M_ϕ^c and $Z[\phi]^c$ are the continuous parts of the semimartingales M_ϕ and $Z[\phi]$ respectively. It follows from Theorem 3.4.2 that

$$[Z[\phi]]_t = \sum_{s\le t}(\Delta Z_s[\phi])^2 + < Z[\phi]^c >_t = [M_\phi]_t. \tag{6.1.17}$$

By (6.1.16), (6.1.17) and Itô's formula, it is easy to show that

$$\begin{aligned} Z_t[\phi]^2 &= Z_0[\phi]^2 + 2\int_0^t A(s, Z_s)[\phi] Z_s[\phi] ds \\ &\quad + 2\int_0^t Z_{s-}[\phi] dM_\phi(s) + [Z[\phi]]_t. \end{aligned} \tag{6.1.18}$$

Hence, by the definition of $N_\phi(t, Z)$ and (6.1.18), we have

$$\begin{aligned} &N_\phi(t, Z) \\ = \; & 2\int_0^t Z_{s-}[\phi] dM_\phi(s) + [Z[\phi]]_t - \int_0^t \int_U (G(s, Z_s, u)[\phi])^2 \mu(du) ds \\ = \; & 2\int_0^t Z_{s-}[\phi] dM_\phi(s) + [M_\phi]_t - \int_0^t \int_U (G(s, Z_s, u)[\phi])^2 \mu(du) ds. \end{aligned}$$

Therefore

$$\begin{aligned} & < M_\phi > (t, Z) - \int_0^t \int_U (G(s, Z_s, u)[\phi])^2 \mu(du) ds \qquad (6.1.19) \\ = \; & (< M_\phi > (t, Z) - [M_\phi]_t) + N_\phi(t, Z) - 2\int_0^t Z_{s-}[\phi] dM_\phi(s). \end{aligned}$$

The right hand side of (6.1.19) is a martingale as all three terms are martingales. On the other hand, the left hand side of (6.1.19) is in $\mathcal{A}$ and predictable. (6.1.15) then follows from the Doob-Meyer decomposition theorem. ∎

Theorem 6.1.3 *Under the conditions of Theorem 6.1.2, $M_\phi(t, Z)$ is purely-discontinuous.*

Proof: Let $g \in C_0(\mathbf{R})$ be non-negative and such that $g(x) = 0$ when $|x| \leq a$ for some $a > 0$. Let Y^n and F^n be functionals defined on $D([0, T], \Phi_{-p_1})$ by

$$Y^n(Z) = \int_0^t \int_U g(G^n(s, Z_s, u)[\phi]) \mu(du) ds$$

and

$$F^n(Z) = \sum_{0<s\leq t} g(\Delta Z_s[\phi]) - Y^n(Z)$$

Similarly, we define functionals Y and F on $D([0, T], \Phi_{-p_1})$. Let ξ^n and ξ be as given in the proof of Lemma 6.1.4. By the same arguments as in the proof of Lemma 6.1.7 it follows that $Y^n(\xi^n)$ converges to $Y(\xi)$ in probability. By Corollary 2.4.2

$$\sum_{0<s\leq t} g(\Delta \xi_s^n[\phi]) \to \sum_{0<s\leq t} g(\Delta \xi_s[\phi]) \qquad \text{a.s.,}$$

and hence, $F^n(\xi^n)$ converges to $F(\xi)$ in probability.

On the other hand, from

$$X_t^n[\phi] = X_0^n[\phi] + \int_0^t A^n(s, X_s^n)[\phi]ds + \int_0^t \int_U G^n(s, X_{s-}^n, u)[\phi]\tilde{N}^n(duds)$$

we have

$$\Delta X_s^n[\phi] = G^n(s, X_{s-}^n, p^n(s))[\phi]1_{D^n}(s)$$

where $p^n(\cdot)$, D^n are the point processes and jump sets corresponding to the Poisson random measures N^n. Hence

$$\begin{aligned} \sum_{0<s\leq t} g(\Delta X_s^n[\phi]) &= \sum_{0<s\leq t} g(G^n(s, X_{s-}^n, p^n(s))[\phi]1_{D^n}(s)) \\ &= \sum_{0<s\leq t} g(G^n(s, X_{s-}^n, p^n(s))[\phi])1_{D^n}(s) \\ &= \int_0^t \int_U g(G^n(s, X_{s-}^n, u)[\phi])N^n(duds). \end{aligned}$$

So

$$F^n(X^n) = \int_0^t \int_U g(G^n(s, X_{s-}^n, u)[\phi])\tilde{N}^n(duds).$$

Hence

$$E\{F^n(\xi^n)\} = E^{P^n}\{F^n(X^n)\} = 0$$

and

$$\begin{aligned} E\{F^n(\xi^n)^2\} &= E^{P^n}\{F^n(X^n)^2\} \\ &= E^{P^n}\int_0^t \int_U g^2(G^n(s, X_s^n, u)[\phi])\mu(du)ds \\ &\leq E^{P^n}\int_0^t \int_U K_g(G^n(s, X_s^n, u)[\phi])^2\mu(du)ds \\ &\leq K_g E^{P^n}\int_0^t \int_U \|G^n(s, X_s^n, u)\|_{-p_1}^2 \|\phi\|_{p_1}^2 \mu(du)ds \\ &\leq K_g\|\phi\|_{p_1}^2 K(1+\tilde{K})T, \end{aligned}$$

where K_g is a constant such that $|g^2(x)| \leq K_g x^2$. So, $\{F^n(\xi^n)\}$ is uniformly integrable and, passing to the limit, we have $E\{F(\xi)\} = 0$. i.e.

$$E\sum_{0<s\leq t} g(\Delta\xi_s[\phi]) = E\int_0^t \int_U g(G(s, \xi_s, u)[\phi])\mu(du)ds.$$

So

$$E^{\lambda^*}\sum_{0<s\leq t} g(\Delta Z_s[\phi]) = E^{\lambda^*}\int_0^t \int_U g(G(s, Z_{s-}, u)[\phi])\mu(du)ds. \qquad (6.1.20)$$

Let g_m be given by Lemma 6.1.8, then (6.1.20) still holds with g replaced by g_m. As $g_m(x) \uparrow x^2$ when $m \uparrow \infty$, it follows from the monotone convergence theorem and Lemma 6.1.10 that

$$\begin{aligned} E^{\lambda^*} \sum_{0<s\leq t} (\Delta M_\phi(s))^2 &= E^{\lambda^*} \sum_{0<s\leq t} (\Delta Z_s[\phi])^2 \\ &= E^{\lambda^*} \int_0^t \int_U (G(s, Z_s, u)[\phi])^2 \mu(du) ds \\ &= E^{\lambda^*} < M_\phi > (t, Z) = E^{\lambda^*}[M_\phi](t, Z). \end{aligned}$$

Hence, by (6.1.16) and (6.1.17)

$$E^{\lambda^*} < M_\phi^c > (t, Z) = 0,$$

i.e. $\forall t$, $< M_\phi^c > (t, Z) = 0$ a.s. Then, by the continuity of $< M_\phi^c > (t, Z)$ in t, we get $< M_\phi^c > (t, Z) = 0$ $\forall t$, a.s.. This proves that $M_\phi(t, Z)$ is purely-discontinuous. ∎

We next identify the compensator of the point process ΔZ_s.

Lemma 6.1.11 *Let*

$$\Gamma = \left\{ A \in \mathcal{B}(\Phi_{-p_1} \setminus \{0\}) : E^{\lambda^*} \sum_{0<s\leq t} 1_A(\Delta Z_s) < \infty, \ \forall 0 < t \leq T \right\}.$$

Then, for $A \in \Gamma$,

$$\sum_{0<s\leq t} 1_A(\Delta Z_s) - \int_0^t \int_U 1_A(G(s, Z_s, u)) \mu(du) ds$$

is a λ^*-martingale.

Proof: Let h be a bounded non-negative continuous $\mathcal{B}_s$-measurable function on $D([0, T], \Phi_{-p_1})$ and let f be a smooth function on $\mathbf{R}_+$ given by

$$f(t) = \begin{cases} \exp(\sqrt{t}/(\sqrt{t} - 1)) & \text{for } 0 \leq t < 1; \\ 0 & \text{for } t \geq 1. \end{cases}$$

Let $0 < a < a'$ and

$$S_{a,a'} = \{x \in \Phi_{-p_1} : a \leq \|x\|_{-p_1} \leq a'\}.$$

For any closed subset F of Φ_{-p_1} contained in $S_{a,a'}$ and $k \geq 3$, we define

$$f_k(x) = f(k^2 \rho(x, F)^2 / a^2)$$

where $\rho(x, F)$ is the distance from x to set F in Φ_{-p_1}. Let $\{X^n\}$, $\{\xi^n\}$ and ξ be as defined in the proof of Lemma 6.1.4 and $F^n_{k,t}$ be functionals on $D([0,T], \Phi_{-p_1})$ given by

$$F^n_{k,t}(Z) = \sum_{0<s\leq t} f_k(\Delta Z_s) - \int_0^t \int_U f_k(G^n(s, Z_s, u))\mu(du)ds.$$

Define the functionals $F_{k,t}$ similarly. Then, for fixed k,

$$\begin{aligned}|F^n_{k,t}(\xi^n) - F_{k,t}(\xi)| \;\leq\; & \left|\sum_{0<s\leq t} f_k(\Delta\xi^n_s) - \sum_{0<s\leq t} f_k(\Delta\xi_s)\right| \\ & + \left|\int_0^t \int_U f_k(G^n(s, \xi^n_s, u)) - f_k(G(s, \xi_s, u))\mu(du)ds\right|.\end{aligned}$$

The first term converges to 0 a.s. and, for the second term, let $b^n = \rho(G^n(s, \xi^n_s, u), F)$ and $b = \rho(G(s, \xi_s, u), F)$. Then

$$\begin{aligned}& \left|\int_0^t \int_U f_k(G^n(s, \xi^n_s, u)) - f_k(G(s, \xi_s, u))\mu(du)ds\right| \\ \leq\; & \int_0^t \int_U |f_k(G^n(s, \xi^n_s, u)) - f_k(G(s, \xi_s, u))|\, 1_{b^n\leq\frac{a}{2}, b\leq\frac{a}{2}}\mu(du)ds \\ & + \int_0^t \int_U f_k(G^n(s, \xi^n_s, u)) 1_{b^n\leq\frac{a}{2}, b>\frac{a}{2}}\mu(du)ds \\ & + \int_0^t \int_U f_k(G(s, \xi_s, u)) 1_{b^n>\frac{a}{2}, b\leq\frac{a}{2}}\mu(du)ds \\ \leq\; & \|f'\|_\infty \left(\frac{k}{a}\right)^2 \int_0^t \int_U |\rho(G^n(s, \xi^n_s, u), F)^2 - \rho(G(s, \xi_s, u), F)^2| \\ & \qquad 1_{b^n\leq\frac{a}{2}, b\leq\frac{a}{2}}\mu(du)ds \\ & + \int_0^t \int_U f_k(G^n(s, \xi^n_s, u)) 1_{b^n\leq\frac{a}{k}, b>\frac{a}{2}}\mu(du)ds \\ & + \int_0^t \int_U f_k(G(s, \xi_s, u)) 1_{b^n>\frac{a}{2}, b\leq\frac{a}{k}}\mu(du)ds \\ \leq\; & \|f'\|_\infty \left(\frac{k}{a}\right)^2 \int_0^t \int_U \|G^n(s, \xi^n_s, u) - G(s, \xi_s, u)\|_{-p_1} \\ & \qquad (b^n + b) 1_{b^n\leq\frac{a}{2}, b\leq\frac{a}{2}}\mu(du)ds \\ & + 2\int_0^t \mu\left\{u : |b^n - b| > \left(\frac{1}{2} - \frac{1}{k}\right) a\right\} ds \\ \leq\; & \|f'\|_\infty \left(\frac{k}{a}\right)^2 \int_0^t \int_U \|G^n(s, \xi^n_s, u) - G(s, \xi_s, u)\|_{-p_1} \\ & \qquad (\|G^n(s, \xi^n_s, u)\|_{-p_1} + \|G(s, \xi_s, u)\|_{-p_1})\mu(du)ds \\ & + \frac{8k^2}{(k-2)^2} \int_0^t \int_U \|G^n(s, \xi^n_s, u) - G(s, \xi_s, u)\|^2_{-p_1}\mu(du)ds\end{aligned}$$

which converges to 0 in probability by the same arguments as in the proof of Lemma 6.1.7. It follows as in the proof of Theorem 6.1.2 that, for fixed k and t, $\{F^n_{k,t}(\xi^n)\}$ is uniformly integrable and

$$Eh(\xi^n)(F^n_{k,t}(\xi^n) - F^n_{k,s}(\xi^n)) = 0.$$

Letting n tends to ∞, we get

$$Eh(\xi)(F_{k,t}(\xi) - F_{k,s}(\xi)) = 0$$

Hence, we have

$$E^{\lambda^*}h(Z)\left\{\sum_{s<r\leq t} f_k(\Delta Z_r) - \int_s^t \int_U f_k(G(r, Z_r, u))\mu(du)dr\right\} = 0.$$

Since f_k decreases to 1_F as $k \to \infty$, by the monotone convergence theorem, we have

$$E^{\lambda^*}h(Z)\sum_{s<r\leq t} 1_F(\Delta Z_r) = E^{\lambda^*}h(Z)\int_s^t \int_U 1_F(G(r, Z_r, u))\mu(du)dr \quad (6.1.21)$$

for any closed subset F of $S_{a,a'}$. As both sides of (6.1.21) define two measures on $S_{a,a'}$ and coincide for all closed sets, (6.1.21) holds for any Borel subset of $S_{a,a'}$. Letting $a \to 0$ and $a' \to \infty$, (6.1.21) holds for any Borel subset of Φ_{-p_1}. This proves the lemma. ∎

Theorem 6.1.4 *Under the conditions of Theorem 6.1.2, λ^* is a weak solution on [0,T] of the SDE (6.0.1).*

Proof: From Lemma 6.1.11 we know that the point process $\Delta M_s = \Delta Z_s$ has compensator $\hat{N}_{\Delta M}(dtdv) = q(t, dv, \omega)dt$ while

$$q(t, E, \omega) = \mu\{u : G(t, Z_{t-}, u) \in E\}.$$

Therefore by Theorem 3.4.7, on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ of the stochastic basis

$$(D([0,T], \Phi_{-p_1}), \mathcal{B}(D([0,T], \Phi_{-p_1})), \lambda^*, \mathcal{B}_t),$$

there exists a Poisson random measure N with characteristic measure μ such that

$$M_t = \int_0^t \int_U G(s, Z_{s-}, u)\tilde{N}(dsdu).$$

Hence

$$Z(t) = Z(0) + \int_0^t A(s, Z_s)ds + \int_0^t \int_U G(s, Z_{s-}, u)\tilde{N}(duds).$$ ∎

6.2 Existence of a weak solution

In this section, we use the basic results of last section to derive the existence of a weak solution of the SDE (6.0.1). The idea is as follows: first, we prove the existence of the weak solution on [0,T] of (6.0.1) when the nuclear space Φ is finite dimensional, say $\mathbf{R}^d$. Then, employing the Galerkin method, we project the coefficients of the equation (6.0.1) to a sequence of finite dimensional subspaces and consider the corresponding SDE on these subspaces. We get the desired existence by proving that this sequence of equations satisfies the assumptions (A1) and (A2) of Section 6.1. Applying the results to the intervals [0,T], [2T,3T], $\cdots$, we get a sequence of solutions of (6.0.1) in these intervals and, connecting them, we obtain a solution on the interval $[0, \infty)$.

First of all, let us consider (6.0.1) when $\Phi = \mathbf{R}^d$. In this case, $\Phi_p = \mathbf{R}^d$ for all p. The SDE (6.0.1) can be rewritten as

$$x_t = \xi + \int_0^t a(s, x_s)ds + \int_0^t \int_U c(s, x_{s-}, u)\tilde{N}(duds) \tag{6.2.1}$$

where $a : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}^d$ and $c : \mathbf{R}_+ \times \mathbf{R}^d \times U \to \mathbf{R}^d$ are two measurable mappings, N is a Poisson random measure on $\mathbf{R}_+ \times U$ with respect to a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ and ξ is a $\mathcal{F}_0$-measurable $\mathbf{R}^d$-valued random variable.

In the present setup, we make the following assumptions (F): $\forall T > 0$, there exist constants K_1 and K_2 such that

(F1) (Continuity) $\forall t \in [0, T]$, $a(t, \cdot) : \mathbf{R}^d \to \mathbf{R}^d$ is continuous; $\forall t \in [0, T]$ and $x \in \mathbf{R}^d$, $c(t, x, \cdot) \in L^2(U, \mu; \mathbf{R}^d)$ and, for t fixed, the map $x \to c(t, x, \cdot)$ from $\mathbf{R}^d$ to $L^2(U, \mu; \mathbf{R}^d)$ is continuous.

(F2) (Coercivity) $\forall t \in [0, T]$ and $x \in \mathbf{R}^d$,

$$2 < a(t, x), x > \leq K_1(1 + |x|^2).$$

(F3) (Growth) $\forall t \in [0, T]$ and $x \in \mathbf{R}^d$,

$$|a(t, x)|^2 \leq K_2(1 + |x|^2) \quad \text{and} \quad \int_U |c(t, x, u)|^2 \mu(du) \leq K_1(1 + |x|^2)$$

where $< \cdot, \cdot >$ and $| \cdot |$ are the inner product and norm in $\mathbf{R}^d$ respectively.

Remark 6.2.1 *If we replace K_1 and K_2 by $K = \max(K_1, K_2)$, the assumptions (F) are just re-statements of the assumptions (I) of Section 6.1 in the present setup. We distinguish K_1 and K_2 for technical reasons which will become clear later on (See Remark 6.2.2 below).*

To solve the SDE (6.2.1), we make the following additional assumption (6.2.2) which will be removed later: There exists a constant L such that for any $t \in [0,T]$ and $x, y \in \mathbf{R}^d$,

$$|a(t,x) - a(t,y)|^2 + \int_U |c(t,x,u) - c(t,y,u)|^2 \mu(du) \leq L|x-y|^2. \quad (6.2.2)$$

The estimate (6.2.3) given below is of crucial importance for this chapter.

Lemma 6.2.1 *Under assumptions (F) and (6.2.2), if $E|\xi|^2 < \infty$, then there exists a solution x of (6.2.1) such that*

$$E \sup_{0 \leq t \leq T} |x_t|^2 \leq \tilde{K} \quad (6.2.3)$$

where $\tilde{K} = \tilde{K}(K_1, T, E|\xi|^2)$ is a finite constant.

Proof: Let $x_t^0 = \xi$ and

$$x_t^{n+1} = \xi + \int_0^t a(s, x_s^n) ds + \int_0^t \int_U c(s, x_{s-}^n, u) \tilde{N}(duds), \ n \geq 0.$$

Under the condition (6.2.2), it is easy to see that $\{x^n\}$ converges to a stochastic process x in the following sense:

$$E \sup_{0 \leq t \leq T} |x_t^n - x_t|^2 \to 0.$$

Further, it is clear that x is a solution of (6.2.1). We only need to prove the estimate (6.2.3). Applying Itô's formula to (6.2.1), we get

$$|x_t|^2 = |\xi|^2 + 2\int_0^t < x_s, a(s,x_s) > ds + \int_0^t \int_U |c(s,x_s,u)|^2 \mu(du) ds$$
$$+ \int_0^t \int_U \{|c(s,x_{s-},u)|^2 + 2 < x_{s-}, c(s,x_{s-},u) >\} \tilde{N}(duds). \quad (6.2.4)$$

Let $\tau_m = \inf\{t \leq T : |x_t| > m\}$ be a sequence of increasing stopping times. By (6.2.4), we have

$$\begin{aligned} & |x_{t\wedge\tau_m}|^2 - |\xi|^2 \\ \leq\ & 2K_1 \int_0^{t\wedge\tau_m} (1 + |x_s|^2) ds \\ & + \int_0^{t\wedge\tau_m} \int_U \{|c(s,x_{s-},u)|^2 + 2 < x_{s-}, c(s,x_{s-},u) >\} \tilde{N}(duds). \end{aligned}$$

Let

$$f^m(t) = E \sup_{r \leq t\wedge\tau_m} |x_r|^2$$

and

$$M_t = \int_0^t \int_U < x_{s-}, c(s, x_{s-}, u) > \tilde{N}(duds). \tag{6.2.5}$$

Then

$$\begin{aligned} f^m(t) &\leq E|\xi|^2 + 2K_1 t + 2K_1 \int_0^t f^m(s)ds + 2E \sup_{r \leq t\wedge\tau_m} M_r \\ &\quad + E \sup_{r \leq t\wedge\tau_m} \int_0^r \int_U |c(s, x_{s-}, u)|^2 \tilde{N}(duds). \end{aligned} \tag{6.2.6}$$

Note that

$$\begin{aligned} & E \sup_{r \leq t\wedge\tau_m} \int_0^r \int_U |c(s, x_{s-}, u)|^2 \tilde{N}(duds) \\ \leq\ & E \sup_{r \leq t\wedge\tau_m} \left\{ \int_0^r \int_U |c(s, x_{s-}, u)|^2 N(duds) \right. \\ & \left. + \int_0^r \int_U |c(s, x_{s-}, u)|^2 \mu(du)ds \right\} \\ =\ & 2E \int_0^{t\wedge\tau_m} \int_U |c(s, x_s, u)|^2 \mu(du)ds \\ \leq\ & 2K_1 t + 2K_1 E \int_0^t f^m(s)ds. \end{aligned} \tag{6.2.7}$$

On the other hand, M, defined in (6.2.5) is a martingale with quadratic variation process

$$[M]_t = \int_0^t \int_U < x_{s-}, c(s, x_{s-}, u) >^2 N(duds).$$

It follows from the Burkholder-Davis-Gundy inequality that

$$\begin{aligned} & 2E \sup_{r \leq t\wedge\tau_m} M_r \leq 8E[M]_{t\wedge\tau_m}^{1/2} \\ =\ & 8E \left\{ \int_0^{t\wedge\tau_m} \int_U < x_s, c(s, x_s, u) >^2 N(duds) \right\}^{1/2} \\ \leq\ & 8E \left\{ \int_0^{t\wedge\tau_m} \int_U |x_s|^2 |c(s, x_s, u)|^2 N(duds) \right\}^{1/2} \\ \leq\ & 8E \left(\sup_{r \leq t\wedge\tau_m} |x_r| \left\{ \int_0^{t\wedge\tau_m} \int_U |c(s, x_s, u)|^2 N(duds) \right\}^{1/2} \right) \\ \leq\ & \frac{1}{2} E \sup_{r \leq t\wedge\tau_m} |x_r|^2 + 32E \int_0^{t\wedge\tau_m} \int_U |c(s, x_s, u)|^2 N(duds) \\ \leq\ & \frac{1}{2} f^m(t) + 32K_1 t + 32K_1 \int_0^t f^m(s)ds. \end{aligned} \tag{6.2.8}$$

Hence, by (6.2.6)-(6.2.8), we have

$$f^m(t) \leq 2\left\{E|\xi|^2 + 36K_1t + 36K_1\int_0^t f^m(s)ds\right\},$$

and so

$$\begin{aligned} f^m(t) &\leq 2(E|\xi|^2 + 36K_1T)\int_0^T \exp(36K_1(T-s))ds \\ &\equiv \tilde{K}(K_1, T, E|\xi|^2) < \infty. \end{aligned}$$

Letting $m \to \infty$, we get our estimate. ∎

The following theorem yields the existence of a weak solution on [0,T] of the SDE (6.2.1) without the condition (6.2.2).

Theorem 6.2.1 *Under assumptions (F) and $E|\xi|^2 < \infty$, the SDE (6.2.1) has a weak solution λ on $D([0,T], \mathbf{R}^d)$ such that*

$$E\sup_{0\leq t\leq T}|x_t|^2 \leq \tilde{K}(K_1, T, E|\xi|^2) < \infty \tag{6.2.9}$$

where x is a $\mathbf{R}^d$-valued process on a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ corresponding to the weak solution λ.

Proof: Let J be the Friedrichs mollifier given by

$$J(x) = \begin{cases} k\cdot\exp\{-(1-|x|^2)^{-1}\} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where k is a constant such that $\int J(x)dx = 1$. Let

$$a^n(t,x) = \begin{cases} \int a(t, x - n^{-1}z)J(z)dz & \text{for } |x| \leq n \\ a^n(t, nx/|x|) & \text{for } |x| > n \end{cases}$$

and

$$c^n(t,x,u) = \begin{cases} \int c(t, x - n^{-1}z, u)J(z)dz & \text{for } |x| \leq n \\ c^n(t, nx/|x|, u) & \text{for } |x| > n. \end{cases}$$

It is easy to verify that, for each n, (a^n, c^n, μ) satisfies the assumptions (F) and (6.2.2) with K_1, K_2, L replaced by $3K_1 + 4\sqrt{K_2}$, $3K_2$ and L^n respectively, where L^n is a constant depends on n. Hence, by Lemma 6.2.1, the SDE

$$x_t^n = \xi + \int_0^t a^n(s, x_s^n)ds + \int_0^t\int_U c^n(s, x_{s-}^n, u)\tilde{N}(duds)$$

has a solution x^n such that

$$E \sup_{0\le t\le T} |x_t^n|^2 \le \tilde{K}\left(3K_1 + 4\sqrt{K_2}, T, E|\xi|^2\right) < \infty.$$

This proves that the sequence $\{(a^n, c^n, \mu)\}$ satisfies the assumption (A1) with

$$K = \max(3K_1 + 4\sqrt{K_2}, 3K_2) \text{ and } \tilde{K} = \tilde{K}(3K_1 + 4\sqrt{K_2}, T, E|\xi|^2).$$

The assumption (A2) is easy to check. Hence, by Theorem 6.1.4, the SDE (6.2.1) has a weak solution on [0,T]. (6.2.9) follows from (6.2.3) and Lemma 6.1.4. ∎

Now, we come back to our original problem and project the SDE (6.0.1) onto a sequence of finite dimensional subspaces. Let λ_0 be a probability measure on Φ_{-r_0} such that

$$E^{\lambda_0}\|v\|_{-r_0}^2 < \infty. \qquad (6.2.10)$$

Let $p = \max(p_0, r_0)$ and $\pi : \Phi_{-p} \to \mathbf{R}^d$ be a mapping given by

$$\pi(v)_k = v[\phi_k^p], \quad k = 1, 2, \cdots, d$$

and let $\lambda_0^d \equiv \lambda_0 \circ \pi^{-1}$ be the induced measure on $\mathbf{R}^d$. We define $a^d : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}^d$ and $g^d : \mathbf{R}_+ \times \mathbf{R}^d \times U \to \mathbf{R}^d$ by

$$a^d(s,x)_k = A\left(s, \sum_{j=1}^d x_j \phi_j^{-p}\right)[\phi_k^p]$$

and

$$g^d(s,x,u)_k = G\left(s, \sum_{j=1}^d x_j \phi_j^{-p}, u\right)[\phi_k^p].$$

Lemma 6.2.2 *Under assumptions (I) and (6.2.10), the SDE*

$$x_t^d = x_0^d + \int_0^t a^d(s, x_s^d)ds + \int_0^t \int_U g^d(s, x_{s-}^d, u)\tilde{N}(duds)$$

on $\mathbf{R}^d$ with initial measure λ_0^d has a weak solution λ^d such that

$$E^{P^d} \sup_{0\le t\le T} |x_t^d|^2 \le \tilde{K}\left(K, T, E^{\lambda_0}\|v\|_{-p}^2\right) < \infty$$

where x^d is a $\mathbf{R}^d$-valued process on a stochastic basis $(\Omega^d, \mathcal{F}^d, P^d, (\mathcal{F}_t^d))$ corresponding to the weak solution λ^d.

Proof: For each d, it is easy to see that assumptions (F) are satisfied by (a^d, g^d, μ) with

$$K_1^d = K \quad \text{and} \quad K_2^d = \max\left(\|\phi_k\|_q^2 \|\phi_k\|_p^{-2} : 1 \le k \le d\right) K. \tag{6.2.11}$$

The assertion of the Lemma follows from Theorem 6.2.1. ∎

Remark 6.2.2 *As K_2^d in (6.2.11) depends on d while K_1 does not, we use different notations for them in the assumptions (F) and obtain estimate (6.2.9) depending on K_1 only (cf. Remark 6.2.1).*

For the weak solution x^d, we define the corresponding Φ_{-p}-valued r.c.l.l. process X^d by

$$X_t^d = \sum_{k=1}^d (x_t^d)_k \phi_k^{-p}.$$

Then

$$\sup_d E \sup_{0 \le t \le T} \|X_t^d\|_{-p}^2 \le \tilde{K}\left(K, T, E\|X_0\|_{-p}^2\right).$$

Let $\gamma^d : \Phi' \to \Phi'$ be a mapping given by

$$\gamma^d v = \sum_{k=1}^d v[\phi_k^p]\phi_k^{-p}$$

and let $\lambda_0^d \equiv \lambda_0 \circ (\gamma^d)^{-1}$ be the induced measure on Φ'. Let $A^d : \mathbf{R}_+ \times \Phi' \to \Phi'$ and $G^d : \mathbf{R}_+ \times \Phi' \times U \to \Phi'$ be two sequences of measurable mappings given by

$$A^d(s, v) = \gamma^d A(s, \gamma^d v) \quad \text{and} \quad G^d(s, v, u) = \gamma^d G(s, \gamma^d v, u).$$

Then X^d is a solution of the SDE

$$X_t^d = X_0^d + \int_0^t A^d(s, X_s^d)ds + \int_0^t \int_U G^d(s, X_{s-}^d, u)\tilde{N}(duds)$$

on the stochastic basis $(\Omega^d, \mathcal{F}^d, P^d, (\mathcal{F}_t^d))$ (given in Lemma 6.2.2) with initial measure λ_0^d.

Theorem 6.2.2 *Under assumptions (I) and (6.2.10), the SDE (6.0.1) has a Φ_{-p_1}-valued weak solution λ^* on $D([0,T], \Phi_{-p_1})$ with initial distribution λ_0 and*

$$E \sup_{0 \le t \le T} \|X_t\|_{-p}^2 \le \tilde{K}\left(K, T, E^{\lambda_0}\|v\|_{-p}^2\right)$$

where X_t is the Φ_{-p_1}-valued process on a stochastic basis corresponding to the weak solution λ^*.

Proof: By Theorem 6.1.4 and Corollary 6.1.1, we only need to check that $(A^d, G^d, \mu, \lambda_0^d)$ satisfies the assumptions (A1) and (A2). By the continuity of $A(t,\cdot)$ on Φ_{-p}, $\forall w \in \Phi_{-p}$, $\forall \epsilon > 0$, $\exists \tilde{\delta}(w)$, $\forall w' \in S(w,\tilde{\delta})$, we have $\|A(t,w) - A(t,w')\|_{-q} < \epsilon$, where

$$S(w,\tilde{\delta}) = \{w' \in \Phi_{-p} : \|w - w'\|_{-p} < \tilde{\delta}(w)\}.$$

For fixed $v_0 \in \Phi_{-p}$, let $C = \{\gamma^d v_0 : d \in \mathbf{N}\} \cup \{v_0\}$. As C is a compact subset of Φ_{-p} and $\{S(w,\tilde{\delta}(w)/2) : w \in C\}$ is an open covering of C, there exist $w_1, \cdots, w_n \in C$ such that

$$C \subset \cup_{k=1}^n S(w_k, \tilde{\delta}(w_k)/2).$$

Let $\delta = \min\{\tilde{\delta}(w_k)/2 : k = 1, \cdots, n\}$. For $w \in C$ and $w' \in S(w,\delta)$, we have k such that $w \in S(w_k, \tilde{\delta}(w_k)/2)$ and hence

$$\|w_k - w'\|_{-p} \le \|w - w_k\|_{-p} + \|w - w'\|_{-p} < \tilde{\delta}(w_k),$$

so that

$$\begin{aligned} & \|A(t,w) - A(t,w')\|_{-q} \\ \le \; & \|A(t,w) - A(t,w_k)\|_{-q} + \|A(t,w_k) - A(t,w')\|_{-q} \\ < \; & 2\epsilon. \end{aligned} \tag{6.2.12}$$

Note that

$$v[\phi_j^q]\phi_j^{-q} = v[\phi_j^p]\phi_j^{-p}, \qquad \forall v \in \Phi', p, q \ge 0. \tag{6.2.13}$$

Therefore, for any $v \in S(v_0, \delta)$

$$\begin{aligned} \|A^d(t,v) - A^d(t,v_0)\|_{-q}^2 &= \left\| \sum_{k=1}^d \left(A(t,\gamma^d v) - A(t,\gamma^d v_0) \right) [\phi_k^q] \phi_k^{-q} \right\|_{-q}^2 \\ &\le \|A(t,\gamma^d v) - A(t,\gamma^d v_0)\|_{-q}^2 < 4\epsilon^2 \end{aligned}$$

where the last inequality follows from (6.2.12), $\gamma^d v_0 \in C$ and

$$\|\gamma^d v - \gamma^d v_0\|_{-p} \le \|v - v_0\|_{-p} < \delta.$$

This proves that for $t \in [0,T]$ and $d \in \mathbf{N}$, $A^d(t,\cdot)$ is a continuous map from Φ_{-p} to Φ_{-q} and the continuity is uniform in d. Note that for any $t \in [0,T]$ and $\phi \in \Phi$,

$$\begin{aligned} & 2A^d(t,\phi)[\theta_p \phi] = 2 \sum_{k=1}^d A(s,\gamma^d \phi)[\phi_k^p] \phi_k^{-p} [\theta_p \phi] \\ = \; & 2A(t,\gamma^d \phi)[\theta_p \gamma^d \phi] \le K(1 + \|\gamma^d \phi\|_{-p}^2) \\ \le \; & K(1 + \|\phi\|_{-p}^2). \end{aligned}$$

Further, for any $t \in [0,T]$ and $v \in \Phi_{-p}$, we have

$$\begin{aligned}
\|A^d(t,v)\|_{-q}^2 &= \left\| \sum_{k=1}^d A(s,\gamma^d\phi)[\phi_k^p]\phi_k^{-p} \right\|_{-q}^2 \\
&= \left\| \sum_{k=1}^d A(s,\gamma^d\phi)[\phi_k^q]\phi_k^{-q} \right\|_{-q}^2 \\
&= \sum_{k=1}^d A(s,\gamma^d\phi)[\phi_k^q]^2 \le \|A(s,\gamma^d\phi)\|_{-q}^2 \\
&\le K(1+\|\gamma^d\phi\|_{-p}^2) \le K(1+\|\phi\|_{-p}^2),
\end{aligned}$$

where the second equality follows from (6.2.13).

We can derive the corresponding properties for $\{G^d\}$ in a similar fashion. Therefore the assumption (A1)(1°) holds. The condition (A1)(2°) follows from Lemma 6.2.2. The condition (A2) can be verified easily. Thus the proof of the theorem is complete. ∎

Finally, we construct a weak solution on $[0,\infty)$ for (6.0.1). First of all, let us construct a sequence of measures λ_n on $\mathbf{D}^n \equiv D([0,nT],\Phi_{-p_1(nT)})$ by induction. Taking $\lambda_1 = \lambda^*$ (given by the previous theorem) and assuming that λ_n on $\mathbf{D}^n$ has been constructed, we now construct λ_{n+1} on $\mathbf{D}^{n+1}$.

For $0 \le t \le T$, $v \in \Phi'$ and $u \in U$, let

$$\tilde{A}(t,v) = A(t+nT,v) \quad \text{and} \quad \tilde{G}(t,v,u) = G(t+nT,v,u). \qquad (6.2.14)$$

Then $\tilde{A}$ and $\tilde{G}$ satisfy the assumptions (I) with p_0 and $K(p,q,T)$ replaced by $p_0((n+1)T)$ and $K(p,q,(n+1)T)$ respectively. With initial distribution $\tilde{\lambda}_0 = \lambda_n \circ Z_{nT}^{-1}$, the SDE

$$X_t = X_0 + \int_0^t \tilde{A}(s,X_s)ds + \int_0^t \int_U \tilde{G}(s,X_{s-},u)\tilde{N}(duds)$$

has a $\Phi_{-p_1((n+1)T)}$-valued weak solution $\tilde{\lambda}_n^*$ on [0,T]. As

$$\mathbf{D}^{1,n+1} \equiv D([0,T],\Phi_{-p_1((n+1)T)})$$

is a Polish space, the regular conditional probability measure

$$\hat{\lambda}_{z_0}^*(\cdot) = E^{\tilde{\lambda}_n^*}(Z \in \cdot | Z_0 = z_0)$$

exists. Let

$$\pi : \mathcal{D}(\pi) \subset \mathbf{D}^n \times \mathbf{D}^{1,n+1} \to \mathbf{D}^{n+1}$$

be given by

$$\pi(Z^1, Z^2)_t = \begin{cases} Z^1_t & \text{as } 0 \leq t \leq nT \\ Z^2_{t-nT} & \text{as } nT \leq t \leq (n+1)T \end{cases}$$

where $\mathcal{D}(\pi) = \{(Z^1, Z^2) \in \mathbf{D}^n \times \mathbf{D}^{1,n+1} : Z^1_{nT} = Z^2_0\}$.

Define a measure λ^*_{n+1} on $\mathbf{D}^n \times \mathbf{D}^{1,n+1}$ by

$$\lambda^*_{n+1}(A \times B) = \int_A \hat{\lambda}^*_{Z^1_{nT}}(B)\lambda_n(dZ^1)$$

for $A \subset \mathbf{D}^n$ and $B \subset \mathbf{D}^{1,n+1}$. It is easy to show that $\lambda^*_{n+1}(\mathcal{D}(\pi)) = 1$ and hence, λ^*_{n+1} induces a measure $\lambda_{n+1} = \lambda^*_{n+1} \circ \pi^{-1}$ on $\mathbf{D}^{n+1}$.

The λ_n's can be regarded as probability measures on $D([0,\infty), \Phi')$ and satisfy

$$\lambda_{n+1}|_{\mathcal{B}_{nT}} = \lambda_n$$

where $\mathcal{B}_{nT}$ is the natural σ-algebra on $D([0,\infty), \Phi')$ upto time nT. Hence, the following set function

$$\lambda(B) = \lambda_n(B) \quad \text{for } B \in \mathcal{B}_{nT}.$$

on the field $\cup_n \mathcal{B}_{nT}$ is well-defined and σ-additive. Therefore λ can be extended to a probability measure on the σ-field $\vee_n \mathcal{B}_{nT} = \mathcal{B}$. Denoting this extension also by λ, we have

$$\lambda|_{\mathcal{B}_{nT}} = \lambda_n.$$

Now we proceed to show that λ is a weak solution of the SDE (6.0.1).

Lemma 6.2.3 *λ is a solution of the $\mathcal{L}$-martingale problem.*

Proof: We only need to show that, for any $F \in \mathcal{D}^\infty_0(\Phi')$, $0 \leq s < t < \infty$ and $B \in \mathcal{B}_s$, we have

$$\int_B \left(M^F(Z)_t - M^F(Z)_s\right) \lambda(dZ) = 0. \tag{6.2.15}$$

We prove (6.2.15) by induction. If $t \leq T$, (6.2.15) follows from Theorem 6.1.2. Suppose (6.2.15) holds when $t \leq nT$. We prove it still holds when $t \leq (n+1)T$.

First, assume that $nT \leq s < t \leq (n+1)T$. Let $\tilde{\mathcal{L}}$ and $\tilde{M}^F$ be defined by (6.1.4) with A and G replaced by $\tilde{A}$ and $\tilde{G}$ of (6.2.14). As $B \in \mathcal{B}_s$, $\pi^{-1}(B \cap \mathbf{D}^{n+1}) \in \mathcal{B}^1_{nT} \times \mathcal{B}^2_{s-nT}$, it follows from standard arguments of measure theory that we may assume that $\pi^{-1}(B\cap\mathbf{D}^{n+1}) = C\times D$ with $C\in\mathcal{B}^1_{nT}$ and $D\in\mathcal{B}^2_{s-nT}$ in the following calculations:

$$\begin{aligned}
&\int_B \left(M^F(Z)_t - M^F(Z)_s\right)\lambda(dZ)\\
=\;& \int_{B\cap\mathbf{D}^{n+1}} \left(M^F(Z)_t - M^F(Z)_s\right)\lambda_{n+1}(dZ)\\
=\;& \int_{\pi^{-1}(B\cap\mathbf{D}^{n+1})} \left(\tilde{M}^F(Z^2)_{t-nT} - \tilde{M}^F(Z^2)_{s-nT}\right)\hat{\lambda}^*_{Z^1_{nT}}(dZ^2)\lambda_n(dZ^1)\\
=\;& \int_C \lambda_n(dZ^1)E^{\tilde{\lambda}^*_n}\Big(\left(\tilde{M}^F(Z^2))_{t-nT} - \tilde{M}^F(Z^2)_{s-nT})\right)1_D(Z^2)\Big|Z^2_0 = Z^1_{nT}\Big)\\
=\;& \int_C \lambda_n(dZ^1)E^{\tilde{\lambda}^*_n}\Big(E^{\tilde{\lambda}^*_n}\Big(\left(\tilde{M}^F(Z^2)_{t-nT} - \tilde{M}^F(Z^2)_{s-nT}\right)\\
&1_D(Z^2)|\mathcal{B}^2_{s-nT}\Big)\Big|Z^2_0 = Z^1_{nT}\Big)\\
=\;& 0.
\end{aligned}$$

Finally, if $s\le nT<t\le(n+1)T$, then

$$\begin{aligned}
E^\lambda(M^F(Z)_t|\mathcal{B}_s) &= E^\lambda(E^\lambda(M^F(Z)_t|\mathcal{B}_{nT}|\mathcal{B}_s)\\
&= E^\lambda(M^F(Z)_{nT}|\mathcal{B}_s) = M^F(Z)_s \quad \lambda\text{-a.s.}
\end{aligned}$$ ∎

Similar arguments yield the following Lemma.

Lemma 6.2.4 (1°) *For any $\phi\in\Phi$, $\{M_\phi(t,Z)\}_{t\ge0}$ given by Lemma 6.1.9 is a λ-square integrable purely-discontinuous martingale.*
(2°) *Let*

$$\Gamma = \left\{A\in\mathcal{B}(\Phi'\setminus\{0\}) : E^\lambda\sum_{0<s\le t}1_A(\Delta Z_s)<\infty,\ \forall t>0\right\}.$$

Then, for $A\in\Gamma$, we have

$$\sum_{0<s\le t}1_A(\Delta Z_s) - \int_0^t\int_U 1_A(G(s,Z_s,u))\mu(du)ds$$

is a λ-martingale on $[0,\infty)$.

Theorem 6.2.3 *Suppose that the assumptions (I) hold and $\forall\phi\in\Phi$*

$$E^{\lambda_0}|v[\phi]|^2 \equiv \int_{\Phi'}|v[\phi]|^2\lambda_0(dv)<\infty.$$

Then (6.0.1) has a Φ'-valued weak solution satisfying the following condition: $\forall T>0$, $\exists p_1=p_1(T)$ such that

$$E\sup_{0\le t\le T}\|X_t\|^2_{-p_1}\le\tilde{K}\left(K,T,E^{\lambda_0}\|v\|^2_{-p}\right).$$

Proof: Let

$$V(\phi) = \left(\int_{\Phi'} |v[\phi]|^2 \lambda_0(dv) \right)^{1/2}, \ \forall \phi \in \Phi.$$

Then, it is easy to check the conditions of Lemma 1.3.1 and hence, we have an index r such that, $\forall \phi \in \Phi$, $V(\phi) \leq \theta \|\phi\|_r$. i.e.

$$\int_{\Phi'} |v[\phi]|^2 \lambda_0(dv) \leq \theta^2 \|\phi\|_r^2. \tag{6.2.16}$$

By the definition of nuclear space, there exists an index $r_0 > r$ such that $\sum_k \|\phi_k^{r_0}\|_r^2 < \infty$. Hence, by (6.2.16), we have

$$\int_{\Phi_{-r_0}} \|v\|_{-r_0}^2 \lambda_0(dv) = \sum_k \int_{\Phi'} |v[\phi_k^{r_0}]|^2 \lambda_0(dv) \leq \sum_k \theta^2 \|\phi_k^{r_0}\|_r^2 < \infty.$$

The rest of the the proof follows from exactly the same arguments as in the proof of Theorem 6.1.4. ∎

6.3 Existence and uniqueness of the strong solution

In this section, we shall impose an additional condition to ensure that the SDE (6.0.1) has a unique strong solution. This will be achieved by establishing pathwise uniqueness and extending the Yamada-Watanabe argument to this setup.

To implement the Yamada-Watanabe argument, we need to realize the driving processes (the Poisson random measures in our case) in a common space. This space is to be chosen such that the regular conditional probability measures exist for any probability measures on it. Unfortunately, this property is not enjoyed by the space of all measures on $\mathbf{R}_+ \times U$. Based on these considerations, we shall establish an equivalence relation between the SDE (6.0.1) and another kind of SDE driven by an ℓ^2-valued martingale which will be called a Good process. As the Good processes can be realized on the Polish space $D([0,T], \ell^2)$, the Yamada-Watanabe argument is applicable and we obtain the uniqueness of the solution for the new equation. Hence, by the equivalence, we get the uniqueness of the solution for the SDE (6.0.1).

We first state some basic definitions.

Definition 6.3.1 *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a stochastic basis and $\tilde{N}(duds)$ a compensated Poisson random measure on $[0,T] \times U$. Suppose that X_0 is a Φ_{-p}-valued random variable such that $E\|X_0\|_{-p}^2 < \infty$. Then by an Φ_{-p}-valued* **strong solution** *on Ω to the SDE (6.0.1) we mean a process X_t*

defined on Ω such that
(a) X_t is a Φ_{-p}-valued $\mathcal{F}_t$-measurable random variable;
(b) $X \in D([0,T], \Phi_{-p})$ a.s.;
(c) There exists a sequence (σ_n) of stopping times on Ω increasing to infinity, such that, $\forall$ n

$$E \int_0^{T\wedge\sigma_n} \int_U \|G(s, X_s, u)\|_{-p}^2 \mu(du) ds < \infty. \tag{6.3.1}$$

and

$$E \int_0^{T\wedge\sigma_n} \|A(s, X_s)\|_{-q}^2 ds < \infty;$$

(d) The SDE (6.0.1) is satisfied for all $t \in [0,T]$ and almost all $\omega \in \Omega$.

Definition 6.3.2 (pathwise uniqueness) *We say that the Φ_{-p}-valued solution for the SDE (6.0.1) has the* **pathwise uniqueness** *property if the following is true: Suppose that X and X' are two Φ_{-p}-valued solutions defined on the same probability space $(\Omega, \mathcal{F}, P)$ with respect to the same Poisson random measure N and starting from the same initial point $X_0 \in \Phi_{-p}$, then the paths of X and X' coincide for almost all $\omega \in \Omega$.*

Now, we impose the following monotonicity condition
(M): $\forall t \in [0,T]$, $v_1, v_2 \in \Phi_{-p}$, we have that

$$2 < A(t,v_1) - A(t,v_2), v_1 - v_2 >_{-q} + \int_U \|G(t,v_1,u) - G(t,v_2,u)\|_{-q}^2 \mu(du) \le K\|v_1 - v_2\|_{-q}^2$$

where q is introduced in assumptions (I).

Lemma 6.3.1 *Under assumptions (I) and (M), SDE (6.0.1) satisfies the pathwise uniqueness property.*

Proof: Let X and X' be two Φ_{-p}-valued solutions. Without loss of generality, suppose that the same sequence $\{\sigma_n\}$ of stopping times satisfies (c) of the Definition 6.3.1 for X and X'. For $\phi \in \Phi$, we have

$$\begin{aligned}(X_t - X_t')[\phi] &= \int_0^t (A(s,X_s) - A(s,X_s'))[\phi] ds \\ &\quad + \int_0^t \int_U (G(s,X_{s-},u) - G(s,X_{s-}',u))[\phi]\tilde{N}(duds).\end{aligned}$$

It follows from Itô's formula that

$$Ee^{-K(t\wedge\sigma_n)}[(X_t - X_t')[\phi]]^2$$

$$
\begin{aligned}
&= 2E\int_0^{t\wedge\sigma_n} e^{-Ks}(X_s - X'_s)[\phi](A(s,X_s) - A(s,X'_s))[\phi]ds \\
&\quad -E\int_0^{t\wedge\sigma_n} Ke^{-Ks}((X_s - X'_s)[\phi])^2 ds \\
&\quad +E\int_0^{t\wedge\sigma_n}\int_U e^{-Ks}((G(s,X_s,u) - G(s,X'_s,u))[\phi])^2\mu(du)ds.
\end{aligned}
$$

Letting $\phi = \phi_k^q,\ k \in \mathbf{N}$ and adding, we have

$$
\begin{aligned}
& Ee^{-K(t\wedge\sigma_n)}\|X_t - X'_t\|_{-q}^2 \\
&= 2E\int_0^{t\wedge\sigma_n} e^{-Ks} < X_s - X'_s, A(s,X_s) - A(s,X'_s) >_{-q} ds \\
&\quad -E\int_0^{t\wedge\sigma_n} Ke^{-Ks}\|X_s - X'_s\|_{-q}^2 ds \\
&\quad +E\int_0^{t\wedge\sigma_n}\int_U e^{-Ks}\|G(s,X_s,u) - G(s,X'_s,u)\|_{-q}^2\mu(du)ds \\
&\le 0. \qquad (6.3.2)
\end{aligned}
$$

Hence, by the right continuity of X and X' and (6.3.2), $X = X'$ a.s. ■

Definition 6.3.3 (Uniqueness in law) *We say that uniqueness in law holds for (6.0.1) if, for any two stochastic bases $(\Omega^k, \mathcal{F}^k, P^k, (\mathcal{F}_t^k))$, two Poisson random measures N^k on $\mathbf{R} \times U$ with the same characteristic measure μ and two Φ_{-p}-valued solutions X, X' of (6.0.1) with the same initial distribution on Φ_{-p}, $(k = 1,2)$, we have that X and X' induce the same probability measure on $D([0,T], \Phi_{-p})$.*

The following assumption will be made throughout the rest of the book: $(U, \mathcal{E}, \mu)$ is a separable measure space.

Now, we introduce the Good processes which will play an essential role in the implementation of the Yamada-Watanabe argument.

Definition 6.3.4 *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a stochastic basis. An ℓ^2-valued process H_t on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ is called a* **Good process** *with respect to a CONS $\{f_n\}$ of $L^2(U, \mathcal{E}, \mu)$ if $\exists$ a Poisson random measure $N(duds)$ on $\mathbf{R}_+ \times U$ with characteristic measure μ such that*

$$
H_t = \sum_{n=1}^{\infty} \frac{1}{n}\int_0^t\int_U f_n(u)\tilde{N}(duds)e_n \qquad (6.3.3)
$$

where $e_n = (0, \cdots, 0, 1, 0, \cdots) \in \ell^2$.

It is easy to see that the series in (6.3.3) converges and, with respect to the same CONS $\{f_n\}$ of $L^2(U, \mathcal{E}, \mu)$, all Good processes have the same distribution on $(D([0,T], \ell^2), \mathcal{B}\{D([0,T], \ell^2)\})$ which will be denoted by P_G and called the **Good measure** .

For any $s \in [0,T]$ and $v \in \Phi_{-p_1}$, we define an unbounded linear operator $\psi(s,v)$ from $\mathcal{D}(\psi(s,v)) \subset \ell^2$ to Φ_{-p_1} by

$$\mathcal{D}(\psi(s,v)) = \left\{ a \in \ell^2 : \sum_k k| < a, e_k >_{\ell^2} | < \infty \right\}$$

and

$$\psi(s,v)a = \sum_k k < a, e_k >_{\ell^2} \int_U G(s,v,u) f_k(u) \mu(du).$$

Lemma 6.3.2 *Let X be an Φ_{-p_1}-valued r.c.l.l. process such that (6.3.1) holds. Then $\int_0^t \psi(s, X_{s-}) dH_s$ is well-defined by*

$$\int_0^{t\wedge\sigma_n} \psi(s, X_{s-}) dH_s = \sum_{k=1}^\infty \int_0^{t\wedge\sigma_n} \psi(s, X_{s-}) e_k d < H_s, e_k >_{\ell^2}, \ \forall n \geq 1. \tag{6.3.4}$$

Further, we have

$$\int_0^t \int_U G(s, X_{s-}, u) \tilde{N}(duds) = \int_0^t \psi(s, X_{s-}) dH_s. \tag{6.3.5}$$

Proof: For simplicity of notation, we assume that $\sigma_n = \infty$ in (6.3.1) and (6.3.4). Then

$$\begin{aligned} & E \sup_{0\leq t\leq T} \left\| \sum_{k=1}^m \int_0^t \psi(s, X_{s-}) e_k d < H_s, e_k >_{\ell^2} \right. \\ & \qquad \left. - \int_0^t \int_U G(s, X_{s-}, u) \tilde{N}(duds) \right\|_{-p_1}^2 \\ \leq & \sum_{j=1}^\infty E \sup_{0\leq t\leq T} \left| \sum_{k=1}^m \int_0^t \int_U \left(\int_U G(s, X_{s-}, v)[\phi_j^{p_1}] f_k(v) \mu(dv) \right) \right. \\ & \qquad \left. f_k(u) \tilde{N}(duds) - \int_0^t \int_U G(s, X_{s-}, u)[\phi_j^{p_1}] \tilde{N}(duds) \right|^2 \\ \leq & 4 \sum_{j=1}^\infty E \int_0^T \int_U \left| \sum_{k=1}^m \left(\int_U G(s, X_s, v)[\phi_j^{p_1}] f_k(v) \mu(dv) \right) f_k(u) \right. \\ & \qquad \left. - G(s, X_s, u)[\phi_j^{p_1}] \right|^2 \mu(du) ds \\ = & 4 \sum_{j=1}^\infty E \int_0^T \sum_{k=m+1}^\infty \left(\int_U G(s, X_{s-}, v)[\phi_j^{p_1}] f_k(v) \mu(dv) \right)^2 ds \to 0 \end{aligned}$$

for $m \to \infty$, since

$$\sum_{k=m+1}^{\infty} \left(\int_U G(s, X_{s-}, v)[\phi_j^{p_1}] f_k(v) \mu(dv) \right)^2 \to 0,$$

$$\sum_{k=m+1}^{\infty} \left(\int_U G(s, X_{s-}, v)[\phi_j^{p_1}] f_k(v) \mu(dv) \right)^2 \leq \int_U |G(s, X_{s-}, v)[\phi_j^{p_1}]|^2 \mu(dv)$$

and

$$\begin{aligned} & \sum_{j=1}^{\infty} E \int_0^t \int_U |G(s, X_{s-}, v)[\phi_j^{p_1}]|^2 \mu(dv) ds \\ = \; & E \int_0^t \int_U \|G(s, X_{s-}, v)\|_{-p_1}^2 \mu(dv) ds < \infty. \end{aligned}$$

∎

As a consequence of (6.3.5), the SDE (6.0.1) can be written in a different form

$$X_t = X_0 + \int_0^t A(s, X_s) ds + \int_0^t \psi(s, X_{s-}) dH_s. \tag{6.3.6}$$

Now, we demonstrate how to couple two solutions of (6.0.1) and discuss some properties of the coupled process.

Suppose X' and X'' are two solutions of the SDE (6.0.1) on stochastic bases $(\Omega', \mathcal{F}', P', (\mathcal{F}'_t))$ and $(\Omega'', \mathcal{F}'', P'', (\mathcal{F}''_t))$ with initial random variables X'_0 and X''_0 (having the same distribution λ_0 on Φ_{-p_1}) and Poisson random measures N' and N'' (having the same characteristic measure μ on U) respectively. Let H' and H'' be defined in terms of (6.3.3) with respect to the same CONS $\{f_n\}$ of $L^2(U, \mathcal{E}, \mu)$ with N replaced by N' and N'' respectively. Then (X', H', X'_0) and (X'', H'', X''_0) are two solutions of the SDE (6.3.6) on the stochastic bases $(\Omega', \mathcal{F}', P', (\mathcal{F}'_t))$ and $(\Omega'', \mathcal{F}'', P'', (\mathcal{F}''_t))$ respectively. Let λ' and λ'' be the Borel probability measures on $D([0,T], \Phi_{-p_1}) \times D([0,T], \ell^2) \times \Phi_{-p_1}$ induced by (X', H', X'_0) and (X'', H'', X''_0) respectively. Define a mapping

$$\pi : D([0,T], \Phi_{-p_1}) \times D([0,T], \ell^2) \times \Phi_{-p_1} \to D([0,T], \ell^2) \times \Phi_{-p_1}$$

by $\pi(w_1, w_2, x) = (w_2, x)$. Then, $\lambda' \circ \pi^{-1} = \lambda'' \circ \pi^{-1} = P_G \otimes \lambda_0$.

Let $\lambda'^{w_2,x}(dw_1)$ and $\lambda''^{w_2,x}(dw_1)$ be the regular conditional probability of w_1 given w_2 and x with respect to λ' and λ'' respectively. This is possible since $D([0,T], \Phi_{-p_1})$ is a Polish space. On the space

$$\Omega = D([0,T], \Phi_{-p_1}) \times D([0,T], \Phi_{-p_1}) \times D([0,T], \ell^2) \times \Phi_{-p_1},$$

we define a Borel probability measure λ by

$$\lambda(A) = \int\int\Big(\int\int 1_A(w_1,w_2,w_3,x)\lambda'^{w_3,x}(dw_1)\lambda''^{w_3,x}(dw_2)\Big) P_G(dw_3)\lambda_0(dx) \tag{6.3.7}$$

for $A \in \mathcal{B}(\Omega)$. Then, it is easy to show that (w_1, w_3, x) and (X', H', X'_0) have the same distribution and so do (w_2, w_3, x) and (X'', H'', X''_0).

Lemma 6.3.3 *For any $A \in \mathcal{B}_t(D([0,T],\Phi_{-p_1}))$, we define two functions f_1 and f_2*

$$f_1(w,x) = \lambda'^{w,x}(A) \quad \textit{and} \quad f_2(w,x) = \lambda''^{w,x}(A).$$

Then f_1 and f_2 are measurable with respect to the completion of the σ-field $\mathcal{B}_t(D([0,T],\ell^2)) \times \mathcal{B}(\Phi_{-p_1})$ under the probability measure $P_G \otimes \lambda_0$.

Proof: We only prove the result for f_1. For fixed $t > 0$ and $A \in \mathcal{B}_t(D([0,T], \Phi_{-p_1}))$, let $\lambda_t'^{w,x}(A)$ be defined as $\lambda'^{w,x}(A)$ with λ' replaced by its restriction to the sub-σ-field

$$\mathcal{B}_t(D([0,T],\Phi_{-p_1})) \times \mathcal{B}_t(D([0,T],\ell^2)) \times \mathcal{B}_t(\Phi_{-p_1}).$$

Then $(w,x) \mapsto \lambda_t'^{w,x}(A)$ is measurable with respect to the σ-field $\mathcal{B}_t(D([0,T], \ell^2)) \times \mathcal{B}(\Phi_{-p_1})$. Now, we only need to show that

$$\lambda_t'^{w,x}(A) = f_1(w,x) \quad \text{for } P_G \otimes \lambda_0\text{-a.s } (w,x).$$

i.e. for any $C \in \mathcal{B}(D([0,T],\ell^2)) \times \mathcal{B}(\Phi_{-p})$, we have to show that

$$\int_C \lambda_t'^{w,x}(A) P_G(dw)\lambda_0(dx) = \lambda'(A \times C). \tag{6.3.8}$$

Consider a continuous mapping $\rho : D([0,t],\ell^2) \times D([0,T-t],\ell^2) \to D([0,T],\ell^2)$ given by

$$\rho(w^1,w^2)_s = \begin{cases} w^1_s & \text{if } s < t \\ w^2_{s-t} + w^1_t & \text{if } s \geq t. \end{cases}$$

From the definition of P_G, we have

$$P_G\{w \in D([0,T],\ell^2) : w(t-) \neq w(t)\} = 0$$

and hence, ρ has a continuous inverse ρ^{-1}. So, we only need to prove (6.3.8) for C of the form

$$C = \{w \in D([0,T],\ell^2) : \rho^{-1}w \in A_1 \times A_2\} \times D,$$

where $A_1 \in \mathcal{B}(D([0,t],\ell^2))$, $A_2 \in \mathcal{B}(D([0,T-t],\ell^2))$ and $D \in \mathcal{B}(\Phi_{-p_1})$. As Good processes are of independent increments, $P_G \circ \rho = P_1 \otimes P_2$, where P_1 and P_2 are probability measures on $D([0,t],\ell^2)$ and $D([0,T-t],\ell^2)$ respectively. Furthermore, as $\lambda_t'^{w,x}(A)$ is $\mathcal{B}_t(D([0,T],\ell^2)) \times \mathcal{B}(\Phi_{-p_1})$-measurable, we can find a measurable function g in $D([0,t],\ell^2) \times \Phi_{-p_1}$ such that

$$\lambda_t'^{w,x}(A) = g(\rho^{-1}(w)^1, x)$$

where $\rho^{-1}(w)^1 \in D([0,t],\ell^2)$ is the first component of $\rho^{-1}(w)$ in the product space $D([0,t],\ell^2) \times D([0,T-t],\ell^2)$. Hence

$$\begin{aligned}
& \int_C \lambda_t'^{w,x}(A) P_G(dw)\lambda_0(dx) \\
= & \int_{A_1 \times A_2 \times D} g(w^1, x) P_1(dw^1) P_2(dw^2) \lambda_0(dx) \\
= & \int_{A_1 \times D} g(w^1, x) P_1(dw^1) \lambda_0(dx) P_2(A_2) \\
= & \int \lambda_t'^{w,x}(A) 1_{\rho^{-1}(w)^1 \in A_1} 1_D(x) P_G(dw) \lambda_0(dx) P_2(A_2) \\
= & \lambda'(A \times \{(\rho^{-1}w)^1 \in A_1\} \times D) P_2(A_2) \\
= & P'\left\{X' \in A, H'|_{[0,t]} \in A_1, X_0' \in D\right\} P'\left\{H'(t+\cdot) - H'(t) \in A_2\right\} \\
= & P'\left\{X' \in A, H'|_{[0,t]} \in A_1, X_0' \in D, H'(t+\cdot) - H'(t) \in A_2\right\} \\
= & P'\left\{X' \in A, (H', X_0') \in C\right\} = \lambda'(A \times C).
\end{aligned}$$

■

Lemma 6.3.4 *Let $\mathcal{B}_t'$ be the completion of*

$$\mathcal{B}_t(D([0,T],\Phi_{-p_1})) \times \mathcal{B}_t(D([0,T],\Phi_{-p_1})) \times \mathcal{B}_t(D([0,T],\ell^2)) \times \mathcal{B}(\Phi_{-p_1}).$$

Then w_3 is a Good process on an extension $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{\lambda}, \tilde{\mathcal{B}}_t)$ of $(\Omega, \mathcal{B}', \lambda, \mathcal{B}_t')$.

Proof: By the definition of P_G, there exists a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ and a Good process H on it such that P_G is the distribution of H. We prove our lemma in four steps.

Step 1. w_3 is an ℓ^2-valued λ-square-integrable martingale.

Let $A_1, A_2 \in \mathcal{B}_s(D([0,T],\Phi_{-p_1}))$, $A_3 \in \mathcal{B}_s(D([0,T],\ell^2))$, $A_4 \in \mathcal{B}(\Phi_{-p_1})$ and $a \in \ell^2$. Then we have

$$\begin{aligned}
& E^\lambda\{\exp(i < a, w_3(t) - w_3(s) >_{\ell^2}) 1_{A_1 \times A_2 \times A_3 \times A_4}\} \\
= & \int_{A_3 \times A_4} \exp(i < a, w_3(t) - w_3(s) >_{\ell^2}) \\
& \qquad \lambda'^{w_3,x}(A_1) \lambda''^{w_3,x}(A_2) P_G(dw_3) \lambda_0(dx)
\end{aligned}$$

$$\begin{aligned}
&= \int_{A_3\times A_4} \exp(i < a, w_3(t) - w_3(s) >_{\ell^2}) f_1(w_3, x) \\
&\qquad f_2(w_3, x) P_G(dw_3)\lambda_0(dx) \\
&= E^\lambda \exp(i < a, w_3(t) - w_3(s) >_{\ell^2})\lambda(A_1 \times A_2 \times A_3 \times A_4)
\end{aligned}$$

where f_1, f_2 are defined in Lemma 6.3.3. Hence, w_3 is of independent increments. Since

$$E^\lambda (w_3)_t = E^P H_t = 0,$$

and

$$E^\lambda \|(w_3)_t\|^2_{\ell^2} = E^P \|H_t\|^2_{\ell^2} = \sum_{n=1}^\infty \frac{t}{n^2} < \infty,$$

w_3 is an ℓ^2-valued λ-square-integrable martingale.

Step 2. $\forall a \in \ell^2$, the quadratic variation of the square-integrable martingales $< w_3, a >_{\ell^2}$ is given by

$$< w_3 >_t (a, a) = t \sum_n \frac{a_n^2}{n^2}.$$

We only need to prove that

$$R_t =< (w_3)_t, a >^2_{\ell^2} - t \sum_n \frac{a_n^2}{n^2}$$

is a λ-martingale. In fact,

$$\begin{aligned}
& E^\lambda (R_t - R_s | \mathcal{B}'_s) \\
&= E^\lambda (< (w_3)_t - (w_3)_s, a >^2_{\ell^2} + 2 < (w_3)_t - (w_3)_s, a >_{\ell^2} \\
&\qquad < (w_3)_s, a >_{\ell^2} | \mathcal{B}'_s) - (t - s) \sum_n \frac{a_n^2}{n^2} \\
&= E^\lambda < (w_3)_t - (w_3)_s, a >^2_{\ell^2} - (t - s) \sum_n \frac{a_n^2}{n^2} \\
&= E^P < H_t - H_s, a >^2_{\ell^2} - (t - s) \sum_n \frac{a_n^2}{n^2} = 0.
\end{aligned}$$

Step 3. $< w_3, a >_{\ell^2}$ is purely-discontinuous.

It is easy to see that the mapping

$$w_3 \to \sum_{s \le t} |\Delta < (w_3)_s, a >_{\ell^2} |^2$$

from $D([0, T], \ell^2)$ into $\mathbf{R}$ is measurable. Hence

$$E^\lambda \sum_{s\le t} |\Delta < (w_3)_s, a >_{\ell^2} |^2 = E^P \sum_{s \le t} |\Delta < H_s, a >_{\ell^2} |^2$$

$$= E^P \sum_{n,m=1}^{\infty} \frac{a_n a_m}{nm} \int_0^t \int_U f_n(u) f_m(u) N(duds)$$

$$= \sum_{n=1}^{\infty} \frac{a_n^2}{n^2} t = E^\lambda < w_3 >_t (a, a).$$

It then follows from the same argument as in the proof of Theorem 6.1.3 that $< w_3, a >_{\ell^2}$ is purely-discontinuous.

Step 4. As w_3 and H have the same distribution, the point process $\Delta w_3(s)$ has the same compensator as the point process ΔH_s which is $\hat{N}_{\Delta H}(dtdv) = q(t, dv, \omega)dt$ while

$$q(t, E, \omega \qquad \left\{ u : \sum_{n=1}^{\infty} \frac{1}{n} f_n(u) e_n \in) E \right\} \mu \quad \forall E \in \mathcal{B}(\ell^2).$$

It follows from the same arguments as in the proof of the Theorem 6.1.4 that there exists a Poisson random measure M with characteristic measure μ on an extension of $(\Omega, \mathcal{B}', \lambda, \mathcal{B}'_t)$ such that

$$(w_3)_t = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^t \int_U f_n(u) \tilde{M}(duds) e_n.$$

Hence, w_3 is a Good process on an extension of $(\Omega, \mathcal{B}', \lambda, \mathcal{B}'_t)$. ∎

Lemma 6.3.5 *Let P_1 and P_2 be two probability measures on a Polish space X with metric ρ. If $(P_1 \times P_2)\{(x_1, x_2) : x_1 = x_2\} = 1$, there exists a unique $x \in X$ such that $P_1 = P_2 = \delta_{\{x\}}$.*

Proof: As

$$1 = \int P_1(dx) \int 1_{x=y} P_2(dy) = \sum_x P_1(\{x\}) P_2(\{x\}) \leq \sum_x P_2(\{x\}) \leq 1, \tag{6.3.9}$$

we have

$$(P_1(\{x\}) - 1) P_2(\{x\}) = 0, \quad \forall x \in X.$$

If $P_1(\{x\}) < 1$, $\forall x \in X$, then $P_2(\{x\}) = 0$, $\forall x \in X$ and hence,

$$\sum_x P_1(\{x\}) P_2(\{x\}) = 0 \neq 1,$$

which contradicts (6.3.9) and hence, there exists $x \in X$ such that $P_1 = \delta_{\{x\}}$. By (6.3.9) again, $P_1(\{x\}) P_2(\{x\}) = 1$ and hence, $P_2 = \delta_{\{x\}}$. ∎

Theorem 6.3.1 *Under assumptions (I) and (M), uniqueness in law holds and the SDE (6.0.1) has a unique strong solution.*

Proof: Let X' and X'' be two solutions of the SDE (6.0.1). From the arguments above, we see that (w_1, w_3, x) and (w_2, w_3, x) are two solutions of (6.3.6) on the same stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{\lambda}, \tilde{\mathcal{B}}_t)$. Let M be the Poisson random measure on this stochastic basis corresponding to the Good process w_3. Then (w_1, M, x) and (w_2, M, x) are solutions of (6.0.1) on the same stochastic basis, where M is given in the proof of Lemma 6.3.4. By the pathwise uniqueness proved in Lemma 6.3.1, we have that $\tilde{\lambda}(w_2 = w_1) = 1$. Coming back to the original probability space, we have $\lambda(w_2 = w_1) = 1$. But, by (6.3.7),

$$\lambda(w_2 = w_1) = \int\int \lambda'^{w,x} \otimes \lambda''^{w,x}(w_2 = w_1) P_G(dw)\lambda_0(dx),$$

so, for $P_G \otimes \lambda_0$-a.s. (w, x), we have

$$\lambda'^{w,x} \otimes \lambda''^{w,x}(w_1 = w_2) = 1. \tag{6.3.10}$$

By Lemma 6.3.5 and (6.3.10), we have a mapping

$$F : D([0,T], \ell^2) \times \Phi_{-p_1} \to D([0,T], \Phi_{-p_1})$$

such that

$$\lambda'^{w,x} = \lambda''^{w,x} = \delta_{F(w,x)}. \tag{6.3.11}$$

For any $A \in \mathcal{B}_t(D([0,T], \Phi_{-p_1}))$, by (6.3.11), Lemma 6.3.3 and

$$1_{F^{-1}(A)}(w, x) = \lambda'^{w,x}(A),$$

it follows that $F^{-1}(A)$ is in the completion of $\mathcal{B}_t(D([0,T], \ell^2)) \times \mathcal{B}(\Phi_{-p_1})$ under $P_G \otimes \lambda_0$, and hence, $F(w, x)$ is adapted. Then, for any Poisson random measure N and initial Φ_{-p_1}-valued random variable X_0, corresponding to a Good process H with respect to a fixed CONS $\{f_n\}$ of $L^2(U, \mathcal{E}, \mu)$, $F(H, X_0)$ is a strong solution of the SDE (6.0.1).

The uniqueness of the strong solution follows directly from the pathwise uniqueness of the SDE (6.0.1). The uniqueness in law follows from (6.3.11). ∎

Finally, we consider the strong solution of (6.0.1) on $[0, \infty)$.

Definition 6.3.5 *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a stochastic basis, $\tilde{N}(duds)$ a compensated Poisson random measure on $\mathbf{R}_+ \times U$ and X_0 a Φ'-valued random variable. Then by a Φ'-valued strong solution on Ω to the SDE (6.0.1) we mean a process X_t defined on Ω such that*

(a) X_t *is* Φ'*-valued,* $\mathcal{F}_t$*-measurable;*
(b) $X \in D([0,\infty),\Phi')$*;*
(c) There exists a sequence (σ_n) *of stopping times on* Ω *increasing to infinity and independent of* ϕ *such that,* $\forall n \in \mathbf{N}$ *and* $\forall \phi \in \Phi$

$$\begin{aligned} E|X_0[\phi]|^2 &+ E\int_0^{\sigma_n} |A(s,X_s)[\phi]|^2 ds \\ &+ E\int_0^{\sigma_n}\int_U |G(s,X_s,u)[\phi]|^2 \mu(du)ds < \infty. \end{aligned}$$

(d) For each $t > 0$*,*

$$\begin{aligned} X_t[\phi] &= X_0[\phi] + \int_0^t A(s,X_s)[\phi]ds \\ &\quad + \int_0^t \int_U G(s,X_{s-},u)[\phi]\tilde{N}(duds), \text{ a.s.} \end{aligned}$$

Theorem 6.3.2 *Under assumptions (I) and (M), if* $E|X_0[\phi]|^2 < \infty\ \forall \phi \in \Phi$*, SDE (6.0.1) has a unique* Φ'*-valued solution on* $[0,\infty)$*.*

Proof: 1° (existence) By the proof of Theorem 6.2.3, we have r_0 such that X_0 lies in Φ_{-r_0} and $E\|X_0\|^2_{-r_0} < \infty$. For every $n \in \mathbf{N}$, by Theorem 6.3.1, there exists a $\Phi_{-p_1(n)}$-valued solution X^n for the SDE (6.0.1) in $[0,n]$. As $p_1(n) \le p_1(n+1), X^{n+1}$ and X^n are two $\Phi_{-p_1(n+1)}$-valued solutions for the SDE (6.0.1) in $[0,n]$ and hence, by Theorem 6.3.1, $X_t^n = X_t^{n+1}$ for $t \le n$. Let $\xi_t = X_t^n$ for $n-1 \le t < n$, $n \in \mathbf{N}$, then it is easy to see that ξ is a Φ'-valued solution of the SDE (6.0.1) on $[0,\infty)$.
2° (uniqueness) Let X be another Φ'-valued solution of SDE (6.0.1). By (c) of Definition 6.3.5 we have

$$E \sup_{0\le t\le n\wedge\sigma_n} (X_t[\phi])^2 < \infty.$$

It follows from the same arguments as in the proof of Theorem 6.2.3 that there exists an index p_n such that X_t lies in Φ_{-p_n} when $t \le n \wedge \sigma_n$. By the proof of 1°, we may assume without restricting the generality that ξ_t also lies in Φ_{-p_n} when $t \le n \wedge \sigma_n$. By the same arguments as in the proof of Lemma 6.3.1 we get our uniqueness. ∎