Chapter 1

Topological vector spaces

In the first two sections of this chapter we collect some necessary facts from functional analysis about topological vector spaces and their operator theory to make this book as self-contained as possible. Although we have provided all the proofs, the only exception being the proof of the spectral decomposition theorem, these two sections are not intended as an introduction to functional analysis for the beginner. We refer the reader who is interested in a more detailed treatment to standard textbooks on this topic such as Conway [5], Reed and Simon [47], Yosida [61].

In Section 3 we treat a special class of topological vector spaces: countable Hilbertian nuclear space and their dual spaces. As we shall see in later chapters, these spaces are very convenient for some practical problems and will play a major role in the course of this book. Most of the material in this section is taken from Kallianpur [23].

1.1 Topological vector spaces.

In this section we introduce the definition of a topological vector space (TVS) and state some basic properties of special classes of topological vector spaces such as Frèchet, Banach and Hilbert spaces for later use.

Definition 1.1.1 *A non-empty set X is called a* **topological vector space** *if it is a vector space with a topology compatible with the space structure, i.e., the following two maps*

$$(x, y) \in X \times X \mapsto x + y \in X \tag{1.1.1}$$

$$(\alpha, x) \in \mathbf{R} \times X \mapsto \alpha x \in X \tag{1.1.2}$$

are continuous.

Next we introduce seminorms on vector spaces and the topology determined by them.

Definition 1.1.2 *A real valued function p on a linear space X is called a* **semi-norm** *if*
a) $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$,
b) $p(\alpha x) = |\alpha| p(x) \quad \forall x \in X$ *and* $\alpha \in \mathbf{R}$.
Further, p is called a **norm** *on X if, in addition to a) and b), we have*
c) $p(x) = 0$ *implies* $x = 0$.

Theorem 1.1.1 *If p is a semi-norm on X, then*
i) $p(0) = 0$,
ii) $p(x-y) \geq |p(x) - p(y)|$, $\forall x, y \in X$. *In particular,* $p(x) \geq 0$, $\forall x \in X$.

Proof: i) It follows from b) that

$$p(0) = p(0 \cdot x) = 0 \cdot p(x) = 0.$$

ii) Without loss of generality, we assume that $p(x) \geq p(y)$. By a), we have

$$p(x) = p(y + (x-y)) \leq p(y) + p(x-y).$$

■

Definition 1.1.3 *a) Let Γ be an index set and let $\mathcal{G} = \{p_v : v \in \Gamma\}$ be a family of semi-norms on X. A set $U \subset X$ is said to be a* **neighborhood of** *$x_0 \in X$ if there exist $n \in \mathbf{N}, v_j \in \Gamma$ and $\epsilon_j \geq 0, j = 1, 2, \cdots, n$ such that*

$$U = \left\{ x \in X : p_{v_j}(x - x_0) < \epsilon_j,\ j = 1, 2, \cdots, n \right\}. \tag{1.1.3}$$

b) A set $G \subset X$ is said to be **open** *if, for any $x_0 \in G$, there exists a neighborhood U of x_0 such that $U \subset G$. Let τ be the collection of all open subsets of X. τ is called* **the topology of X determined by** $\mathcal{G}$.

Theorem 1.1.2 *i) Let $\mathcal{G} = \{p_v : v \in \Gamma\}$ be a family of semi-norms on X and let τ be given as above. Then (X, τ) is a topological space.*
ii) (X, τ) is a Hausdorff topological space if $\mathcal{G}$ satisfies the following separating condition: For any $x_0 \neq 0$, there exists $v_0 \in \Gamma$ such that $p_{v_0}(x_0) > 0$.

Proof: i) It is easy to see that

a) $\emptyset \in \tau$ and $X \in \tau$, where $\emptyset$ is the empty set.

b) For any family $\{G_\alpha : \alpha \in A\}$ of open sets, we have $\cup_{\alpha \in A} G_\alpha \in \tau$.

c) For any finite family $\{G_j : j = 1, 2, \cdots, n\}$ of open sets, we have $\cap_{j=1}^n G_j \in \tau$.

Hence, (X, τ) is a topological space.
ii) We only need to show that, for any $x_1 \neq x_2$, there exist two disjoint open sets G_1 and G_2 such that $x_1 \in G_1$ and $x_2 \in G_2$.

Without loss of generality, we assume that $x_1 = 0$ and $x_2 \neq 0$. It follows from the separating condition that there exists a seminorm p_v such that $p_v(x_2) = \alpha > 0$. Let

$$G_1 = \left\{x \in X : p_v(x) < \frac{\alpha}{2}\right\} \quad \text{and} \quad G_2 = x_2 + G_1.$$

For any $y \in G_2$, there exists $y_1 \in G_1$ such that $y = x_2 + y_1$. Hence

$$p_v(y) = p_v(x_2 + y_1) \geq p_v(x_2) - p_v(-y_1) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}$$

and so G_1 and G_2 are disjoint. ∎

Definition 1.1.4 *A topological vector space (X, τ) is called a* **pre-Frèchet space** *if τ is given by a countable family of seminorms which satisfies the separating condition given in Theorem 1.1.2. It is called a* **pre-Banach space** *if τ is given by a norm. It is called a* **pre-Hilbert space** *if τ is given by a Hilbertian norm $\|\cdot\|$ in the following sense: For any $x, y \in X$*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{1.1.4}$$

We shall show in Theorem 1.1.8 that a Hilbertian norm is uniquely determined by an **inner product** *$< \cdot, \cdot >$, i.e., a continuous symmetric bilinear form on $X \times X$ such that $< x, x > \geq 0$ and $< x, x >= 0$ iff $x = 0$.*

Definition 1.1.5 *a) A sequence $\{x_n\}$ in the topological vector space (X, τ) is called a* **Cauchy sequence** *if $x_n - x_m \to 0$ as $n, m \to \infty$. A topological vector space (X, τ) is said to be* **sequentially complete** *if every Cauchy sequence converges in X.*
b) A complete pre-Frèchet (resp. Banach, Hilbert) space is called a **Frèchet** *(resp.* **Banach, Hilbert***)* **space**.

Theorem 1.1.3 *Under the conditions of Theorem 1.1.2 we have the following:*
a) (X, τ) is a topological vector space.
b) A sequence $\{x_n\}$ converges to an element x_∞ in X if and only if

$$p_v(x_n - x_\infty) \to 0 \qquad \text{as} \quad n \to \infty \qquad \text{for any} \quad v \in \Gamma;$$

$\{x_n\}$ is a Cauchy sequence in X iff

$$p_v(x_n - x_m) \to 0 \qquad \text{as} \quad n, m \to \infty \qquad \text{for any} \quad v \in \Gamma.$$

c) If Γ is countable, then (X, τ) is a metric space, i.e., there exists d : $X \times X \to \mathbf{R}$ such that *i)* $d(x_1, x_2) \geq 0$ *and* $d(x_1, x_2) = 0$ *iff* $x_1 = x_2$,

ii) $d(x_1, x_2) = d(x_2, x_1)$,

iii) $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$, *for any* $x_1, x_2, x_3 \in X$,

and $x_n \to x_\infty$ *in* τ*-topology iff* $d(x_n, x_\infty) \to 0$. *The map d satisfying i)-iii) above is called a metric on X and (X,d) is called a metric space.*

Proof: a) Let $(x_0, y_0) \in X \times X$. For any neighborhood U of $x_0 + y_0 \in X$ we have
$$U = \left\{x \in X : p_{v_j}(x - x_0 - y_0) < \epsilon_j, j = 1, 2, \cdots, n\right\}.$$
Let
$$U' = \left\{x \in X : p_{v_j}(x) < \frac{\epsilon_j}{2}, j = 1, 2, \cdots, n\right\}.$$
Then $x_0 + U'$ (resp. $y_0 + U'$) is a neighborhood of x_0 (resp. y_0). For any $x \in x_0 + U'$ and $y \in y_0 + U'$, we have

$$x + y = x_0 + y_0 + x' + y'$$

where $x', y' \in U'$. Then

$$p_{v_j}(x' + y') \leq p_{v_j}(x') + p_{v_j}(y') < \epsilon_j.$$

i.e. $x + y \in U$ and hence (1.1.1) holds. Similarly we can prove (1.1.2) and therefore (X, τ) is a topological vector space.
b) If $x_n \to x_\infty$ in X and $v \in \Gamma$, then for a neighborhood

$$U = \{x \in X : p_v(x - x_\infty) < \epsilon\}$$

of x_∞, there exists N such that $n \geq N$ implies $x_n \in U$. Hence $p_v(x_n - x_\infty) \to 0$.

On the other hand, if $p_v(x_n - x_\infty) \to 0$ for any $v \in \Gamma$ then for any positive ϵ there exists $N(\epsilon, v)$ such that $n \geq N(\epsilon, v)$ implies that $p_v(x_n - x_\infty) < \epsilon$. For any neighborhood

$$U = \left\{x \in X : p_{v_j}(x - x_\infty) < \epsilon_j, j = 1, 2, \cdots, m\right\}$$

of x_∞, letting $N = max\{N(v_j, \epsilon_j) : j = 1, 2, \cdots, m\}$, we have $x_n \in U$ for $n \geq N$, i.e. $x_n \to x_\infty$ in X. The second statement can be proved by similar arguments.
c) Let $\mathcal{G} = \{p_j : j = 1, 2, \cdots\}$ and

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j}(p_j(x - y) \wedge 1), \ \forall x, y \in X. \tag{1.1.5}$$

Then $d(x,y) = d(y,x) \geq 0$. By the separating condition, $x = y$ iff $p_j(x-y) = 0$ for all $j \geq 1$ and hence, it is equivalent to $d(x,y) = 0$. The condition iii) can be verified easily.

It follows from b) that $x_n \to x_\infty$ in X if and only if $p_j(x_n - x_\infty) \to 0$ for any $j \geq 1$. This is equivalent to $d(x_n, x_\infty) \to 0$ by the definition of d. ∎

Theorem 1.1.4 *Suppose that $(X, \|\cdot\|)$ is a pre-Banach space. There exists a unique (in the sense of isometric isomorphism) Banach space $(\tilde{X}, \|\cdot\|\tilde{})$ such that X is isometrically isomorphic to a dense subspace $\tilde{X}_0$ of $\tilde{X}$. Further, if X is a pre-Hilbert space, then $\tilde{X}$ is a Hilbert space. $\tilde{X}$ is called the* **completion** *of X.*

Proof: (Uniqueness) If we have $\tilde{\tilde{X}}_0$ and $\tilde{\tilde{X}}$ with the same property, then $\tilde{\tilde{X}}_0$ is isometrically isomorphic to $\tilde{X}_0$. Hence, by denseness of $\tilde{\tilde{X}}_0$ and $\tilde{X}_0$, it is easy to show that $\tilde{\tilde{X}}$ is isometrically isomorphic to $\tilde{X}$.

(Existence) Let Y be the collection of all Cauchy sequences in X. For $\{x_n\}, \{y_n\} \in Y$, we say that $\{x_n\} \sim \{y_n\}$ if $x_n - y_n \to 0$ as $n \to \infty$. It is easy to show that Y is a vector space and " $\sim$ " is an equivalence relationship in Y. Let $\tilde{X}$ be the quotient space $Y/\sim$, i.e. the collection of all equivalence classes.

For each $\tilde{x} \in \tilde{X}$, let

$$\|\tilde{x}\|\tilde{} = \lim_{n\to\infty} \|x_n\|.$$

It can be shown that $\|\cdot\|\tilde{}$ is a well-defined norm on $\tilde{X}$.

Let $\tilde{X}_0 = \{\{x, x, \cdots\} \in \tilde{X} : x \in X\}$ and $\iota : X \to \tilde{X}$ be given by $\iota x = \{x, x, \cdots\}$. For any $\tilde{x} = \{x_n\} \in \tilde{X}$, let $\tilde{x}_m = \{x_m, x_m, \cdots\} \in \tilde{X}_0$. Then

$$\lim_{m\to\infty} \|\tilde{x}_m - \tilde{x}\|\tilde{} = \lim_{m\to\infty} \lim_{n\to\infty} \|x_m - x_n\| = 0.$$

Hence $\tilde{X}_0$ is dense in $\tilde{X}$. It follows directly from the construction that X is isometrically isomorphic to $\tilde{X}_0$.

Finally, we prove that $\tilde{X}$ is complete. Let $\{\tilde{x}_k\}_{k\geq 1} = \{\{x_n^{(k)}\}_{n\geq 1}\}_{k\geq 1}$ be a Cauchy sequence in $\tilde{X}$. For each k, there exists n_k such that for any $m \geq n_k$

$$\|x_m^{(k)} - x_{n_k}^{(k)}\| < k^{-1}.$$

Let $\tilde{x} = \{x_{n_k}^{(k)}\}$. Then

$$\begin{aligned}
&\|x_{n_k}^{(k)} - x_{n_m}^{(m)}\| \\
= \; & \|\{x_{n_k}^{(k)}, x_{n_k}^{(k)}, \cdots\} - \{x_{n_m}^{(m)}, x_{n_m}^{(m)}, \cdots\}\|\tilde{} \\
\leq \; & \|\{x_{n_k}^{(k)}, x_{n_k}^{(k)}, \cdots\} - \tilde{x}_k\|\tilde{} + \|\tilde{x}_k - \tilde{x}_m\|\tilde{} + \|\tilde{x}_m - \{x_{n_m}^{(m)}, x_{n_m}^{(m)}, \cdots\}\|\tilde{} \\
\leq \; & k^{-1} + \|\tilde{x}_k - \tilde{x}_m\|\tilde{} + m^{-1}.
\end{aligned}$$

Hence $\{x_{n_k}^{(k)}\}$ is a Cauchy sequence and therefore, $\tilde{x} \in \tilde{X}$. Note that

$$\begin{aligned}
\|\tilde{x}_k - \tilde{x}\|^{\sim} &\le \|\tilde{x}_k - \{x_{n_k}^{(k)}, x_{n_k}^{(k)}, \cdots\}\|^{\sim} + \|\{x_{n_k}^{(k)}, x_{n_k}^{(k)}, \cdots\} - \tilde{x}\|^{\sim} \\
&\le k^{-1} + \lim_{p\to\infty} \|x_{n_k}^{(k)} - x_{n_p}^{(p)}\| \\
&\le k^{-1} + \lim_{p\to\infty} (k^{-1} + \|\tilde{x}_k - \tilde{x}_p\|^{\sim} + p^{-1}) \\
&= 2k^{-1} + \lim_{p\to\infty} \|\tilde{x}_k - \tilde{x}_p\|^{\sim}.
\end{aligned}$$

As $\{\tilde{x}_k\}$ is a Cauchy sequence, $\tilde{x}_k \to \tilde{x}$ in $\tilde{X}$.

The second statement follows from the definition directly. ∎

Definition 1.1.6 *Let M be a subset of the topological space X. If the closure of M does not contain any non-empty open set, we say M is a* **nowhere dense set**. *If M can be represented as the union of countable many nowhere dense sets, we say it is in* **Baire's first category**; *otherwise it is in* **Baire's second category**.

Theorem 1.1.5 *A complete metric space is in Baire's second category.*

Proof: Suppose that X is a complete metric space and is in Baire's first category, i.e., there exists a sequence of nowhere dense closed sets M_n such that $X = \cup_n M_n$.

As M_1^c is a non-empty open set, there exists a closed sphere $S_1 = \{x \in X : d(x, x_1) \le r_1\}$ such that $0 < r_1 < \frac{1}{2}$ and $S_1 \subset M_1^c$, where M_1^c is the complement of M_1. As M_2 is nowhere dense, $M_2^c \cap S_1^0$ is a non-empty open set, where S_1^0 is the interior of S_1. Hence there exists a closed sphere $S_2 = \{x \in X : d(x, x_2) \le r_2\}$ such that $0 < r_2 < \frac{1}{2^2}$ and $S_2 \subset M_2^c \cap S_1^0$. By induction, we can find a sequence of closed spheres $S_n = \{x \in X : d(x, x_n) \le r_n\}$ such that

$$0 < r_n < \frac{1}{2^n} \qquad \text{and} \qquad S_n \subset M_n^c \cap S_{n-1}^0, \;\; \forall n \ge 1,$$

where, by convention, $S_0 = X$. Note that for any $n \le m$, $d(x_n, x_m) \le r_n \to 0$ so that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists $x_\infty \in X$ such that $d(x_n, x_\infty) \to 0$. As

$$d(x_n, x_\infty) \le d(x_n, x_m) + d(x_m, x_\infty) \le r_n + d(x_m, x_\infty),$$

we have that $d(x_n, x_\infty) \le r_n$ by taking $m \to \infty$, i.e., $x_\infty \in S_n \subset M_n^c$ for any $n \ge 1$. Therefore $x_\infty \notin \cup_n M_n$. This contradicts the fact that $X = \cup_n M_n$. ∎

Corollary 1.1.1 *If X is a Frèchet space, then X is in Baire's second category.*

Next we define the dual space of a topological vector space X whose topology is given by a family of seminorms and introduce the strong topology on this space.

Definition 1.1.7 *a) A subset B of X is said to be* **bounded** *if it can be absorbed by any neighborhood of $0 \in X$, i.e., for any neighborhood U of $0 \in X$ there exists a constant $\alpha > 0$ such that $\alpha^{-1}B \subset U$.*
b) Let X' be the collection of all continuous linear maps from X to **R**. *Then X' is called the* **dual space** *of X.*
c) For any bounded subset B of X, let

$$q_B(f) = \sup_{x \in B} |f[x]|, \qquad f \in X'.$$

Then $\{q_B\}$ is a family of seminorms on X' and gives X' a **strong topology** *τ'. (X, τ') is called the* **strong dual of X**.

Theorem 1.1.6 *a) If $f \in X'$, then f is a* **bounded functional** *in the sense that f maps bounded subsets of X to bounded subsets of* **R**.
b) For any bounded subset B of X, q_B is a seminorm on X'.
c) If X is a pre-Banach space and f is a bounded linear functional on X, then $f \in X'$. Further, X' is a Banach space with norm

$$\|f\|_{X'} = \sup_{\|x\| \leq 1} |f[x]|.$$

Proof: a) As f is continuous at $0 \in X$, there exists a neighborhood U of $0 \in X$ such that $x \in U$ implies $|f[x]| < 1$. For any bounded set B, let $\alpha > 0$ such that $\alpha^{-1}B \subset U$. Hence for any $x \in B$

$$|f[x]| = \alpha|f[\alpha^{-1}x]| \leq \alpha.$$

b) It follows from a) that $q_B(f) < \infty$ for any $f \in X'$. Note that for any $f, g \in X'$ and $\alpha \in \mathbf{R}$

$$q_B(f+g) = \sup_{x \in B} |f[x] + g[x]| \leq \sup_{x \in B} |f[x]| + \sup_{x \in B} |g[x]| = q_B(f) + q_B(g)$$

and

$$q_B(\alpha f) = \sup_{x \in B} |\alpha f[x]| = |\alpha| q_B(f),$$

i.e., q_B is a seminorm.
c) As $\{x \in X : \|x\| \leq 1\}$ is a bounded set and f is a bounded functional, there exists M such that $\|x\| \leq 1$ implies $|f[x]| \leq M$. Hence for any $x \in X$

$$\left| f\left[\frac{x}{\|x\|}\right] \right| \leq M, \text{ i.e. } |f[x]| \leq M\|x\|.$$

The continuity of f follows directly.

It is easy to show that $\|\cdot\|_{X'}$ is a norm in X'. Let $\tilde{\tau}$ be the topology of X' given by $\|\cdot\|_{X'}$. Let B be a bounded subset of X. Then there exists $\alpha > 0$ such that $\alpha^{-1}B \subset \{x \in X : \|x\| < 1\}$. Hence for any $f \in X'$

$$q_B(f) \leq \sup_{\|x\|<\alpha} |f[x]| = \sup_{\|y\|\leq 1} |f[\alpha y]| = \alpha \|f\|_{X'}.$$

On the other hand, as $S = \{x \in X : \|x\| \leq 1\}$ is a bounded subset of X, we see that $\|\cdot\|_{X'} = q_S$. Hence two topologies $\tilde{\tau}$ and τ' are equivalent and therefore, X' is a pre-Banach space.

Further, if $\{f_n\}$ is a Cauchy sequence in X', $\lim_{n\to\infty} f_n[x]$ exists for any $x \in X$ since

$$|f_n[x] - f_m[x]| \leq \|f_n - f_m\|_{X'}\|x\| \to 0, \qquad \text{as} \quad n, m \to \infty. \tag{1.1.6}$$

Denoting it by f(x), it is obvious that f is a linear functional. Further as the limit exists uniformly for $x \in S$, we see that $f \in X'$. As $\{f_n\}$ is a Cauchy sequence, $\forall \epsilon > 0$, $\exists N$, s.t. $\forall n, m \geq N$, $\|x\| \leq 1$,

$$|f_n[x] - f_m[x]| \leq \epsilon.$$

Taking $m \to \infty$, we have

$$|f_n[x] - f[x]| \leq \epsilon, \;\; \forall n \geq N, \;\; \|x\| \leq 1.$$

Therefore, $\|f_n - f\|_{X'} \to 0$ as $n \to \infty$. Hence X' is a Banach space. ∎

Theorem 1.1.7 (Hahn-Banach) *If X_0 is a subspace of the pre-Banach space X and $f \in X_0'$, then there exists $\tilde{f} \in X'$ such that $\tilde{f}|_{X_0} = f$ and $\|\tilde{f}\|_{X'} = \|f\|_{X_0'}$. In particular, for any $x_0 \in X, x_0 \neq 0$, there exists $f \in X'$ such that $f[x_0] \neq 0$.*

Proof: Let $\mathcal{M}$ be the collection of all subspaces X_α containing X_0 such that there exists $f_\alpha \in X_\alpha'$ with

$$f_\alpha|_{X_0} = f \qquad \text{and} \qquad \|f_\alpha\|_{X_\alpha'} = \|f\|_{X_0'}. \tag{1.1.7}$$

Then $\mathcal{M}$ is a set with the partial order: $X_\alpha \prec X_\beta$ if $X_\alpha \subset X_\beta$ and f_β is an extension of f_α, i.e. $f_\beta|_{X_\alpha} = f_\alpha$. For any ordered subset $\mathcal{M}_0 = \{X_\alpha : \alpha \in A\}$ of $\mathcal{M}$, let

$$X_{\mathcal{M}_0} = \cup_{\alpha\in A} X_\alpha.$$

Then $X_{\mathcal{M}_0}$ is a subspace containing X_0. Define

$$f_{\mathcal{M}_0}[x] = f_\alpha[x], \qquad \text{if} \quad x \in X_\alpha.$$

It is easy to see that $f_{\mathcal{M}_0}$ is well-defined on $X_{\mathcal{M}_0}$ and $f_{\mathcal{M}_0}|_{X_0} = f$. Note that

$$\begin{aligned} \|f\|_{X_0'} &\le \|f_{\mathcal{M}_0}\|_{X'_{\mathcal{M}_0}} \\ &= \sup\{|f_{\mathcal{M}_0}[x]| : \|x\| \le 1 \text{ and } x \in \mathcal{M}_0\} \\ &\le \sup\{|f_\alpha[x]| : \|x\| \le 1, x \in X_\alpha \text{ and } \alpha \in A\} \\ &\le \sup\left\{\|f_\alpha\|_{X_\alpha'} : \alpha \in A\right\} = \|f\|_{X_0'}. \end{aligned}$$

Hence

$$f_{\mathcal{M}_0} \in X'_{\mathcal{M}_0} \qquad \text{with} \qquad \|f_{\mathcal{M}_0}\|_{X'_{\mathcal{M}_0}} = \|f\|_{X_0'}.$$

It is easy to see that $f_{\mathcal{M}_0}$ is an extension of f_α for all $\alpha \in A$. Therefore $X_{\mathcal{M}_0}$ is a maximum element of $\mathcal{M}_0$. By Zorn's lemma, there exists a local maximum X_1 (with linear functional f_1) of $\mathcal{M}$.

If $X_1 \neq X$, there exists $m \in X \setminus X_1$. Let

$$X_2 = \{x = x_1 + \lambda m : x_1 \in X_1 \quad \text{and} \quad \lambda \in \mathbf{R}\}$$

and

$$f_2[x] = f_1[x_1] + \lambda c$$

where c is a real constant to be determined later such that

$$\|f_2\|_{X_2'} \le \|f\|_{X_0'}. \tag{1.1.8}$$

Then X_2 is a subspace containing X_1 and f_2 is an extension of f_1 such that (1.1.7) holds. This contradicts the fact that X_1 is a local maximum of $\mathcal{M}$. Therefore $X_1 = X$ and taking $\tilde{f} = f_1$ we have the first assertion of the theorem. The second part of the theorem follows directly by taking $X_0 = \{\lambda x_0 : \lambda \in \mathbf{R}\}$ and $f[\lambda x_0] = \lambda$.

To finish the proof we have to find c. (1.1.8) is equivalent to

$$f_1[x_1] + \lambda c \le \|f\|_{X_0'}\|x_1 + \lambda m\| \qquad \text{for any} \quad x_1 \in X_1 \quad \text{and} \quad \lambda \in \mathbf{R},$$

i.e.

$$f_1\left[\frac{x_1}{\lambda}\right] + c \le \|f\|_{X_0'}\left\|m + \frac{x_1}{\lambda}\right\| \qquad \text{for any} \quad \lambda > 0$$

and

$$f_1\left[\frac{x_1}{-\lambda}\right] - c \le \|f\|_{X_0'}\left\|-m + \frac{x_1}{-\lambda}\right\| \qquad \text{for any} \quad \lambda < 0.$$

We only need to choose c to lie between

$$\sup\left\{f_1[x] - \|f\|_{X_0'}\|x - m\| : x \in X_1\right\}$$

and

$$\inf\left\{\|f\|_{X_0'}\|x+m\| - f_1[x] : x \in X_1\right\}.$$

This is possible as for any $x, y \in X_1$,

$$\begin{aligned} f_1[x] + f_1[y] &= f_1[x+y] \le \|f_1\|_{X_1'}\|x+y\| \\ &= \|f\|_{X_0'}\|(x-m)+(y+m)\| \\ &\le \|f\|_{X_0'}\|x-m\| + \|f\|_{X_0'}\|y+m\|. \end{aligned}$$

∎

Corollary 1.1.2 *Let X be a Banach space. Then there exists an isometric isomorphism ι from X onto a closed subspace X_0'' of X''. If $X_0'' = X''$, we call X a* **reflexive space***.*

Proof: For any $x_0 \in X$, let $\iota x_0 \in X''$ be such that $(\iota x_0)[f] = f[x_0]$ for all $f \in X'$. Let $X_0'' = \iota X$. Note that

$$|f[x_0]| \le \|f\|_{X'}\|x_0\|_X,$$

Hence $\|\iota x_0\|_{X''} \le \|x_0\|_X$.

Define a continuous linear functional f on a one-dimensional subspace

$$X_0 = \{\alpha x_0 : \alpha \in \mathbf{R}\}$$

of X by

$$f[\alpha x_0] = \alpha\|x_0\|_X.$$

Then $\|f\|_{X_0'} = 1$. It follows from the Hahn-Banach theorem that there exist $\tilde{f} \in X'$ such that $\|\tilde{f}\|_{X'} = 1$ and $\tilde{f}[x_0] = \|x_0\|_X, i.e., (\iota x_0)[\tilde{f}] = \|x_0\|_X$. Therefore $\|x_0\|_X \le \|\iota x_0\|_{X''}$. The linearity of ι follows from the definition directly. ∎

In the rest of this section, it will be assumed that $(X, \|\cdot\|)$ is a Hilbert space.

Theorem 1.1.8 *The relations*

$$< x, y >= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2),\ \forall x, y \in X \tag{1.1.9}$$

and

$$\|x\|^2 =< x, x >,\quad \forall x \in X, \tag{1.1.10}$$

define $< \cdot, \cdot >$ as an inner product in X.

Proof: First suppose we have a Hilbertian norm $\|\cdot\|$ and define $<\cdot,\cdot>$ in terms of (1.1.9). It is easy to see that $<\cdot,\cdot>$ is symmetric and continuous with respect to the topology given by $\|\cdot\|$. It follows from (1.1.4) that

$$\begin{aligned} & < x, z > + < y, z > \\ = & \ \frac{1}{4}(\|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2) \\ = & \ \frac{1}{8}((\|x+y+2z\|^2 + \|x-y\|^2) - (\|x+y-2z\|^2 + \|x-y\|^2)) \\ = & \ \frac{1}{8}(\|x+y+2z\|^2 - \|x+y-2z\|^2) = \frac{1}{2} < x+y, 2z > . \end{aligned} \tag{1.1.11}$$

Letting $x = 0$ in (1.1.9), we see that $< 0, y >= 0$, $\forall y \in X$. Let $y = 0$ in (1.1.11). Then

$$< x, z >= \frac{1}{2} < x, 2z > . \tag{1.1.12}$$

Hence, by (1.1.11) and (1.1.12), we have

$$< x, z > + < y, z >=< x+y, z >, \quad \forall x, y, z \in X. \tag{1.1.13}$$

Now we prove that for any $\alpha \in \mathbf{R}$ and $x, y \in X$, we have

$$< \alpha x, y >= \alpha < x, y > . \tag{1.1.14}$$

It follows from (1.1.13) that (1.1.14) holds for $\alpha = n \in \mathbf{N}$. By (1.1.12),

$$\left\langle \frac{x}{2}, y \right\rangle = \left\langle y, \frac{x}{2} \right\rangle = \frac{1}{2} < y, x >= \frac{1}{2} < x, y > .$$

Therefore we see that (1.1.14) holds for $\alpha = \frac{n}{2^m}, n, m \in \mathbf{N}$. By the continuity of $<\cdot,\cdot>$ we have that (1.1.14) holds for any $\alpha \in \mathbf{R}$. Hence $<\cdot,\cdot>$ is a continuous symmetric bilinear form on $X \times X$. The other conditions follow from the definition and the properties of the norm.

Now we assume we have a continuous symmetric bilinear form $<\cdot,\cdot>$ on $X \times X$ such that $< x, x >\geq 0$ and $< x, x >= 0$ iff $x = 0$. Define $\|\cdot\|$ by (1.1.10).

For any $t \in \mathbf{R}$

$$0 \leq \|x+ty\|^2 =< x+ty, x+ty >= \|x\|^2 + 2t < x, y > +t^2\|y\|^2,$$

therefore we have the following **Schwartz inequality**

$$| < x, y > | \leq \|x\| \cdot \|y\|, \qquad \text{for any} \quad x, y \in X.$$

Hence

$$\begin{aligned} \|x+y\|^2 & = < x+y, x+y > \\ & = \|x\|^2 + 2 < x, y > +\|y\|^2 \\ & \leq (\|x\| + \|y\|)^2, \end{aligned}$$

this prove the condition (a) of the Definition 1.1.2. The conditions (b) and (c) of that definition are immediate. Therefore $\|\cdot\|$ is a norm on X.

Finally, we note that

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= <x+y,x+y>+<x-y,x-y> \\ &= 2(\|x\|^2+\|y\|^2),\end{aligned}$$

i.e., $\|\cdot\|$ is a Hilbertian norm on X. ∎

Definition 1.1.8 *a) Let $x,y\in X$. If $<x,y>=0$, we say x is* **orthogonal** *to y. For a subset M of X, let*

$$M^{\perp}=\{x\in X:<x,m>=0,\quad \forall m\in M\}.$$

b) A subset S of X is called an **orthogonal system** *if for any $x,y\in S$ we have $x\neq 0$, $y\neq 0$ and $<x,y>=0$. S is called a* **complete orthogonal system** *(COS) if there exists no other orthogonal system which strictly contains S. S is called a* **complete orthonormal system** *(CONS) if S is a COS and, for any $x\in S, \|x\|=1$.*

Theorem 1.1.9 *If M is a closed subspace of X, then $M^{\perp}$ is a closed subspace of X and is called the* **orthogonal complement** *of M. For any $x\in X$, there exists a unique decomposition*

$$x=m+n,\ m\in M\quad \text{and}\quad n\in M^{\perp}.$$

We denote m by P_Mx and call it the **orthogonal projection** *of x. The operator P_M from X to X is called the* **orthogonal projection operator with range M.**

Proof: From the properties of the inner product we see that $M^{\perp}$ is a closed subspace. Suppose we have two decompositions, i.e.,

$$x=m+n=m'+n'.$$

Then

$$m-m'=n'-n\in M\cap M^{\perp}.$$

Hence

$$<m-m',m-m'>=0,$$

i.e., $m=m'$ and therefore, $n=n'$. This proves the uniqueness.

For the existence, we may assume that $M\neq X$ and $x\notin M$. As M is closed

$$d=\inf\{\|x-m\|:m\in M\}>0.$$

Let $m_n \in M$ be such that $\|x - m_n\| \to d$. Then, by (1.1.4),

$$\begin{aligned} &\|m_n - m_k\|^2 = \|(m_n - x) - (m_k - x)\|^2 \\ = &\ 2\|m_n - x\|^2 + 2\|m_k - x\|^2 - 4\left\|\frac{m_n + m_k}{2} - x\right\|^2 \\ \le &\ 2\|m_n - x\|^2 + 2\|m_k - x\|^2 - 4d^2 \to 0, \quad \text{as } n,\ k \to \infty. \end{aligned}$$

Therefore $\{m_n\}$ is a Cauchy sequence and hence, there exists $m \in M$ such that $m_n \to m$ as $n \to \infty$. Further, we have $\|x - m\| = d$.

Let $n = x - m$. Then for any $m' \in M$ and $\alpha \in \mathbf{R}$

$$d^2 \le \|x - m - \alpha m'\|^2 = \|n\|^2 - 2\alpha < n, m' > +\alpha^2\|m'\|^2.$$

Hence

$$2\alpha < n, m' > \le \alpha^2\|m'\|^2, \quad \forall \alpha \in \mathbf{R}.$$

This implies $< n, m' >= 0$, i.e., $n \in M^\perp$. ∎

Theorem 1.1.10 *If X is a separable (i.e. X has a countable dense subset) Hilbert space, then there exists a CONS of X which contains only countably many elements. Further, for any CONS $\{e_n\}$, we have*

a)

$$x = \sum_{j=1}^{\infty} < x, e_j > e_j, \quad \forall x \in X, \tag{1.1.15}$$

*b) (***Parseval equation***)*

$$\|x\|^2 = \sum_n < x, e_n >^2, \quad \forall x \in X,$$

c)

$$< x, y >= \sum_n < x, e_n >< y, e_n >, \ \forall x, y \in X. \tag{1.1.16}$$

Proof: Let $S = \{x_n\}$ be a countable dense subset of X. Without loss of generality, assume that $0 \notin S$. We define a sequence $\{y_n\}$ inductively by

$$y_n = x_n - \sum_{j=1}^{n-1} < x_n, u_j > u_j, \ n = 1, 2, \cdots \tag{1.1.17}$$

where $u_j = y_j/\|y_j\|$ if $y_j \neq 0$ and $u_j = 0$ otherwise.

Let S' be the collection of all non-zero elements of u_j. Now we show that S' is a CONS by induction. Note that

$$\begin{aligned} < y_2, u_1 > &= \langle x_2 - < x_2, u_1 > u_1, u_1 \rangle \\ &= < x_2, u_1 > - < x_2, u_1 >< u_1, u_1 > \\ &= 0. \end{aligned}$$

We assume that for any $j < k \leq n, < y_k, u_j >= 0$. Then for $k < n+1$,

$$\begin{aligned} < y_{n+1}, u_k > &= \left\langle x_{n+1} - \sum_{j=1}^{n} < x_{n+1}, u_j > u_j, u_k \right\rangle \\ &= < x_{n+1}, u_k > - \sum_{j=1}^{n} < x_{n+1}, u_j >< u_j, u_k > \\ &= < x_{n+1}, u_k > - < x_{n+1}, u_k >< u_k, u_k > \\ &= 0. \end{aligned}$$

Hence for any $j \neq k$ we have $< u_j, u_k >= 0$. Therefore S' is an orthogonal system.

If S' is not a CONS, there exists an $0 \neq x_0 \in (S')^{\perp}$. Hence $< x_0, y_n >= 0$, $\forall n \geq 1$. It follows from (1.1.17) that $< x_0, x_n >= 0$, $\forall n \geq 1$. As $\{x_n\}$ is dense in X, there exists a sequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$. Therefore $< x_0, x_0 >= 0$ which contradicts $x_0 \neq 0$ and hence S' is a CONS.

Let $\{e_n\}$ be a countable CONS of X. For any $x \in X$, let

$$x^{(n)} = \sum_{j=1}^{n} < x, e_j > e_j.$$

Note that

$$\begin{aligned} \|x - x^{(n)}\|^2 &= \left\langle x - \sum_{j=1}^{n} < x, e_j > e_j, x - \sum_{j=1}^{n} < x, e_j > e_j \right\rangle \\ &= \|x\|^2 - \sum_{j=1}^{n} < x, e_j >^2 . \end{aligned} \tag{1.1.18}$$

Hence

$$\sum_{j=1}^{n} < x, e_j >^2 \leq \|x\|^2.$$

Letting $n \to \infty$, we have the following **Bessel inequality**:

$$\sum_{j=1}^{\infty} < x, e_j >^2 \leq \|x\|^2 < \infty. \tag{1.1.19}$$

Hence for any $m \leq n$

$$\|x^{(n)} - x^{(m)}\|^2 = \sum_{j=m+1}^{n} < x, e_j >^2 \to 0 \qquad \text{as} \quad n, m \to \infty.$$

As X is a Hilbert space, there exists $x' \in X$ such that $x^{(n)} \to x'$. But $\forall k \geq 1$

$$< x - x', e_k >= \lim_{n\to\infty} \left\langle x - \sum_{j=1}^{n} < x, e_j > e_j, e_k \right\rangle = 0.$$

It then follows from the completeness of $\{e_k\}$ that $x = x'$. This proves (1.1.15). By (1.1.18) we see that the Parseval equation holds. (1.1.16) follows from (1.1.9) and the Parseval equation. ∎

Remark 1.1.1 *The procedure of constructing an orthogonal system $\{y_n\}$ (resp. $\{u_n\}$) in the proof of the last Theorem is called* **Hilbert-Schmidt orthogonalization** *(resp.* **orthonormalization***). Further, the denseness of $\{x_n\}$ guarantees the completeness of the system. More generally, if the collection of all finite linear combinations of x_n's is dense in X, then $\{u_n\}$ is a CONS of X.*

Finally we give the Riesz representation theorem which allows a Hilbert space to be identified with its dual space.

Theorem 1.1.11 (Riesz representation theorem) *Let X be a Hilbert space. Then there exists an isometric isomorphism ι from X onto X'. In particular, X' is a Hilbert space.*

Proof: For any $y \in X$, let

$$(\iota y)[x] = < x, y >, \ \forall x \in X.$$

It follows from the Schwartz inequality that $\iota y \in X'$ and $\|\iota y\|_{X'} \leq \|y\|$.

For any $f \in X'$, we search for $y \in X$ such that $f = \iota y$. Let $N = \{x \in X : f[x] = 0\}$. If $N = X$, then $f = \iota 0$. If $N \neq X$, it follows from the continuity of f that N is a closed subspace of X. Let $0 \neq y_0 \in N^{\perp}$. Defining

$$y = (f[y_0]/\|y_0\|^2)y_0,$$

we prove that

$$< x, y >= f[x], \ \forall x \in X. \tag{1.1.20}$$

If $x \in N$, both sides of (1.1.19) are equal to 0. If $x = \alpha y_0$, then

$$< x, y >=< \alpha y_0, (f[y_0]/\|y_0\|^2)y_0 >= \alpha f[y_0] = f[x].$$

For any $x \in X$, since

$$x = \left(x - \frac{f[x]}{f[y]}y\right) + \frac{f[x]}{f[y]}y$$

and

$$f\left[x - \frac{f[x]}{f[y]}y\right] = 0,$$

(1.1.19) holds. Note that

$$\|\iota y\|_{X'} = \sup_{\|x\| \leq 1} |(\iota y)[x]| \geq (\iota y)\left[\frac{y}{\|y\|}\right] = \|y\|.$$

Then ι is an isometry from X onto X'. The linearity of ι is immediate. ∎

1.2 Linear operators on topological vector spaces

First we study linear maps between topological vector spaces.

Definition 1.2.1 *Let X, Y be two topological vector spaces. T is called a* **linear operator from X to Y** *if T is a linear map defined on a subspace $\mathcal{D}(T)$ of X with values in Y. $\mathcal{D}(T)$ is called the* **domain** *of T. If $\mathcal{D}(T)$ is dense in X, we define the* **dual operator** *T' of T from Y' to X' as follows:*

$$\mathcal{D}(T') = \{y' \in Y' : \exists x' \in X' \ s.t.\ y'[Tx] = x'[x],\ \forall x \in \mathcal{D}(T)\}$$

and

$$T'y' = x',\ \forall y' \in \mathcal{D}(T').$$

Remark 1.2.1 *We will mostly be considering operators with dense domains. If $\mathcal{D}(T)$ is dense, $T'y'$ is well-defined for $y' \in \mathcal{D}(T')$. It is obvious that T' is a linear operator.*

Definition 1.2.2 *Let X, Y be two Hilbert spaces and let T be a linear operator from X to Y.*
a) A linear operator T^* from $\mathcal{D}(T^*) \subset Y$ to X is said to be the **adjoint operator** *of T if*

$$< T^*y, x >_X = < y, Tx >_Y,\ \forall x \in \mathcal{D}(T) \text{ and } y \in \mathcal{D}(T^*).$$

b) If $X = Y$, $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $T^*|_{\mathcal{D}(T)} = T$, then we call T a **symmetric operator on X**. *If in addition, $\mathcal{D}(T) = \mathcal{D}(T^*)$, then T is called a* **self-adjoint operator on X**.

Definition 1.2.3 *Let X, Y be two topological vector spaces. We denote the collection of all continuous linear maps T from X to Y with $\mathcal{D}(T) = X$ by $L(X,Y)$.*

It is easy to see that $L(X,Y)$ is a linear space. If both X and Y are Frèchet spaces, then similar to Definition 1.1.7 and Theorem 1.1.6 we have the following theorem.

Theorem 1.2.1 *Suppose that both X and Y are Frèchet spaces whose topologies are given by seminorms $\{p_v : v \in \Gamma\}$ and $\{\tilde{p}_{v'} : v' \in \Gamma'\}$ respectively.*
a) If $T \in L(X,Y)$, then T is a **bounded operator** *in the sense that the image of any bounded subset of X is bounded in Y.*
b) For any bounded subset B of X and $v' \in \Gamma'$, let

$$q_{B,v'}(T) = \sup_{x \in B} \tilde{p}_{v'}(Tx), \qquad \forall T \in L(X,Y).$$

Then $q_{B,v'}$ is a family of seminorms on $L(X,Y)$. The topology of $L(X,Y)$ given by this family of seminorms is called the **strong topology** *of $L(X,Y)$.*
c) If X, Y are pre-Banach spaces and T is a bounded linear operator from X to Y, then $T \in L(X,Y)$. Further $L(X,Y)$ is a Banach space with norm

$$\|T\|_{L(X,Y)} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y.$$

Now we introduce three classes of linear operators: compact operator, nuclear operator and Hilbert-Schmidt operator. These important classes of operators possess many interesting properties which will be used in this book frequently.

Definition 1.2.4 *$T \in L(X,Y)$ is said to be a* **compact operator** *if the image of any bounded subset of X is pre-compact in Y. We denote the class of all compact operators by $L_c(X,Y)$.*

Theorem 1.2.2 *Suppose that X, Y, Z are three Banach spaces.*
a) $L_c(X,Y)$ is a closed subspace of the Banach space $L(X,Y)$.
b) If $T \in L_c(X,Y), S \in L(Y,Z)$ or $T \in L(X,Y), S \in L_c(Y,Z)$, then the composition $ST \in L_c(X,Z)$.

Proof: Part b) follows directly from the definition and Theorem 1.2.1.
a) It is easy to see that $L_c(X,Y)$ is a subspace. We only need to prove that it is closed. Let $T_n \in L_c(X,Y)$ and $T_n \to T$ with respect to the norm given by Theorem 1.2.1 (c). Let $\{x_n\}$ be a sequence such that $\|x_n\|_X \leq M$, $\forall n \geq 1$ where M is a finite constant.

By the compactness of each T_n, we can choose a subsequence $\{\tilde{x}_k\}$ of $\{x_n\}$ by making use of the diagonalization argument such that, for each n fixed, $\{T_n\tilde{x}_k\}$ converges in Y as $k \to \infty$. Hence

$$\begin{aligned}
& \|T\tilde{x}_k - T\tilde{x}_m\|_Y \\
\leq\ & \|T\tilde{x}_k - T_n\tilde{x}_k\|_Y + \|T_n\tilde{x}_k - T_n\tilde{x}_m\|_Y + \|T_n\tilde{x}_m - T\tilde{x}_m\|_Y \\
\leq\ & 2M\|T - T_n\|_{L(X,Y)} + \|T_n\tilde{x}_k - T_n\tilde{x}_m\|_Y.
\end{aligned}$$

Therefore

$$\limsup_{k,m\to\infty} \|T\tilde{x}_k - T\tilde{x}_m\|_Y \leq 2M\|T - T_n\|_{L(X,Y)}.$$

Letting $n \to \infty$, we see that

$$\lim_{k,m\to\infty} \|T\tilde{x}_k - T\tilde{x}_m\|_Y = 0,$$

i.e., $\{T\tilde{x}_k\}$ is a Cauchy sequence in Y. By the completeness of Y, $\{T\tilde{x}_k\}$ converges. Hence the image of $\{x \in X : \|x\|_X \leq M\}$ is pre-compact and then, $T \in L_c(X,Y)$. ∎

Let X, Y be two separable Hilbert spaces and $T \in L(X,Y)$. For any CONS $\{e_n\}$ of X and CONS $\{f_m\}$ of Y, note that

$$\sum_{n=1}^{\infty} \|Te_n\|_Y^2 = \sum_{m=1}^{\infty} \|T^* f_m\|_X^2.$$

Therefore

$$\|T\|_{(2)} \equiv \left(\sum_{n=1}^{\infty} \|Te_n\|_Y^2 \right)^{1/2}.$$

does not depend on the choice of the CONS $\{e_n\}$ of X and $\|T\|_{(2)}^2 = \|T^*\|_{(2)}^2$.

Definition 1.2.5 *Let X, Y be two separable Hilbert spaces. $T \in L(X,Y)$ is said to be* **Hilbert-Schmidt** *if $\|T\|_{(2)} < \infty$. $\|\cdot\|_{(2)}$ is called the* **Hilbert-Schmidt norm** *of T. We denote the class all of Hilbert-Schmidt operators from X to Y by $L_{(2)}(X,Y)$.*

Theorem 1.2.3 *Let X, Y be two separable Hilbert spaces.*
a) $(L_{(2)}(X,Y), \|T\|_{(2)}^2)$ is a separable Hilbert space with inner product given by

$$< T, S >_{(2)} = \sum_{n=1}^{\infty} < Te_n, Se_n >_Y, \quad \forall T, S \in L_{(2)}(X,Y). \tag{1.2.1}$$

b) $L_{(2)}(X,Y) \subset L_c(X,Y)$.
c) Let Z be another separable Hilbert space. Suppose that $T \in L_{(2)}(X,Y)$, $S \in L(Y,Z)$ or $T \in L(X,Y)$, $S \in L_{(2)}(Y,Z)$, then the composition $ST \in L_{(2)}(X,Z)$.

Proof: a) It is easy to see that $L_{(2)}(X,Y)$ is a linear space and $< \cdot, \cdot >_{(2)}$ is an inner product on $L_{(2)}(X,Y)$.

Now we prove the completeness of $(L_{(2)}(X,Y), \|T\|_{(2)}^2)$. For any $x \in X$ and $T \in L_{(2)}(X,Y)$, we have

$$\begin{aligned}
\|Tx\|_Y^2 &= \left\| \sum_{n=1}^{\infty} < x, e_n >_X Te_n \right\|_Y^2 \\
&\le \left(\sum_{n=1}^{\infty} | < x, e_n >_X | \|Te_n\|_Y \right)^2 \\
&\le \left(\sum_{n=1}^{\infty} < x, e_n >_X^2 \right) \left(\sum_{n=1}^{\infty} \|Te_n\|_Y^2 \right) \\
&= \|x\|_X^2 \|T\|_{(2)}^2.
\end{aligned}$$

Hence

$$\|T\|_{L(X,Y)} \leq \|T\|_{(2)}, \quad \forall T \in L(X,Y). \tag{1.2.2}$$

Let $\{T_n\} \subset L_{(2)}(X,Y)$ be a Cauchy sequence. By (1.2.2), $\{T_n\}$ is a Cauchy sequence in the Banach space L(X,Y) and hence, there exists $T \in L(X,Y)$ such that $\|T_n - T\|_{L(X,Y)} \to 0$. Making use of Fatou's lemma, we have

$$\sum_{j=1}^{\infty} \|Te_j\|_Y^2 \leq \liminf_{n\to\infty} \sum_{j=1}^{\infty} \|T_n e_j\|_Y^2 \leq \sup_n \|T_n\|_{(2)}^2 < \infty. \tag{1.2.3}$$

Therefore $T \in L_{(2)}(X,Y)$. As $\{T_n\}$ is a Cauchy sequence in $L_{(2)}(X,Y)$, $\forall \epsilon > 0$, there exists N such that

$$\sum_{j=1}^{J} \|(T_n - T_m)e_j\|_Y^2 \leq \epsilon, \qquad \forall\ n,\ m \geq N \quad \text{and} \quad J \geq 1.$$

Letting $m \to \infty$ and then $J \to \infty$, we have

$$\sum_{j=1}^{\infty} \|(T_n - T)e_j\|_Y^2 \leq \epsilon,\ \forall n \geq N,$$

i.e., $T_n \to T$ in $L_{(2)}(X,Y)$ and hence, $L_{(2)}(X,Y)$ is a Hilbert space.

Finally, for a CONS $\{e_j\}$ in X and a CONS $\{f_j\}$ in Y, it is easy to see that

$$T = \sum_{i,j} < T, T_{ij} >_{(2)} T_{ij}$$

where the summation converges in $L_{(2)}(X,Y)$ and $T_{ij} \in L_{(2)}(X,Y), i,j \geq 1$, is given by

$$T_{ij}x =< e_i, x >_X f_j, \quad \forall x \in X.$$

Hence $L_{(2)}(X,Y)$ is separable.

b) Let $T \in L_{(2)}(X,Y)$ and let $\{x_n\}$ be a bounded sequence in X. As

$$| < Tx_n, f_j >_Y | \leq \|T^* f_j\|_X \sup_n \|x_n\|_X, \tag{1.2.4}$$

it follows from the diagonalization arguments that there exists a subsequence $\{\tilde{x}_n\}$ of $\{x_n\}$ such that $< Tx_n, f_j >_Y \to \alpha_j$ as $n \to \infty$ for each j. Similar to (1.2.3), we have $\sum_j \alpha_j^2 < \infty$. By (1.2.4) it is easy to see that

$$T\tilde{x}_n \to \sum_j \alpha_j f_j.$$

Hence T is a compact operator.

c) Suppose that $T \in L_{(2)}(X,Y)$ and $S \in L(Y,Z)$, then

$$\sum_{n=1}^{\infty} \|STe_n\|_Z^2 \leq \|S\|_{L(Y,Z)}^2 \sum_{n=1}^{\infty} \|Te_n\|_Y^2 < \infty.$$

Therefore $ST \in L_{(2)}(X,Y)$. ∎

Definition 1.2.6 *Let X, Y be two separable Hilbert spaces. $T \in L(X,Y)$ is said to be a* **nuclear operator** *if there exists a CONS $\{e_n\}$ of X such that*

$$\sum_{n=1}^{\infty} \|Te_n\|_Y < \infty.$$

We denote by $L_{(1)}(X,Y)$ the class of all nuclear operators from X to Y.

For $T \in L_{(1)}(X,Y)$, we define

$$\|T\|_{(1)} \equiv \inf \left\{ \sum_{n=1}^{\infty} \|Te_n\|_Y : \quad \{e_n\} \quad \text{is a CONS of X} \right\}. \tag{1.2.5}$$

Then $(L_{(1)}(X,Y), \|\cdot\|_{(1)})$ is a Banach space.

If $X = Y$ is a separable Hilbert space and $T \in L_{(1)}(X,X)$, we define the **trace** of T as follows:

$$Trace(T) = \sum_j < Te_j, e_j >$$

where $\{e_j\}$ is a CONS of X. It is easy to verify that the definition does not depend on the choice of the CONS of X.

The following theorem can be proved by similar arguments as in the previous theorem, we omit its proof.

Theorem 1.2.4 *Let X, Y, Z be three Hilbert spaces. Then*
a) $L_{(1)}(X,Y) \subset L_{(2)}(X,Y)$.
b) If $T \in L_{(2)}(X,Y), S \in L_{(2)}(Y,Z)$, *then* $ST \in L_{(1)}(X,Z)$.
c) If $T \in L_{(1)}(X,Y), S \in L(Y,Z)$ *or* $T \in L(X,Y), S \in L_{(1)}(Y,Z)$, *then the composition* $ST \in L_{(1)}(X,Z)$.

Now we study self-adjoint operators in more details. The easiest self-adjoint operators are the projection operators introduced in Theorem 1.1.9. In fact, if P is a projection operator on X, it is easy to show that P is a bounded self-adjoint operator and $P^2 = P$. We will show that each self-adjoint operator corresponds to a family of projection operators.

Definition 1.2.7 *A family of projection operators $\{E_\lambda : \lambda \in \mathbf{R}\}$ on a Hilbert space X is called a* **spectral family** *if*
a) $E_\lambda E_\mu = E_{\lambda \wedge \mu}$, *where* $\lambda \wedge \mu = \min(\lambda, \mu)$.
b) $E_{-\infty} = 0$ *and* $E_\infty = I$, *where I is the identity operator on X,*

$$E_{-\infty}x = \lim_{\lambda \to \infty} E_\lambda x \qquad \text{and} \qquad E_\infty x = \lim_{\lambda \to \infty} E_\lambda x, \ \forall x \in X.$$

c) $E_{\lambda+} = E_\lambda$, *where* $E_{\lambda+}x = \lim_{\lambda < \mu \to \lambda} E_\lambda x$.

Let $\{E_\lambda\}$ be a spectral family and $x \in X$. Then $F_x(\lambda) \equiv< E_\lambda x, x >_X$ is a bounded nondecreasing function on $\mathbf{R}$. In fact, for $\lambda < \mu$, we have

$$\begin{aligned} 0 &\leq < (E_\mu - E_\lambda)x, (E_\mu - E_\lambda)x >_X \\ &= < (E_\mu - E_\lambda)^2 x, x >_X \\ &= < (E_\mu - 2E_\lambda + E_\lambda)x, x >_X \\ &= < E_\mu x, x >_X - < E_\lambda, x >_X, \end{aligned} \tag{1.2.6}$$

and

$$< E_\lambda x, x >\leq< E_\infty x, x >_X= \|x\|_X^2.$$

Let f be a simple function on $\mathbf{R}$ given by

$$f(\lambda) = \sum_{j=1}^{n-1} a_j 1_{(\lambda_j,\lambda_{j+1}]}(\lambda),$$

where $-\infty < \lambda_1 < \cdots < \lambda_n < \infty$, $a_j \in \mathbf{R}$. We define $I(f) \equiv \int f(\lambda)dE_\lambda x \in X$ as follows:

$$I(f) = \sum_{j=1}^{n-1} a_j(E_{\lambda_{j+1}}x - E_{\lambda_j}x). \tag{1.2.7}$$

Then

$$\begin{aligned} \|I(f)\|_X^2 &= \left\langle \sum_{j=1}^{n-1} a_j(E_{\lambda_{j+1}} - E_{\lambda_j})x, \sum_{k=1}^{n-1} a_k(E_{\lambda_{k+1}} - E_{\lambda_k})x \right\rangle_X \\ &= 2 \sum_{1\leq j<k\leq n-1} a_j a_k < (E_{\lambda_{k+1}} - E_{\lambda_k})(E_{\lambda_{j+1}} - E_{\lambda_j})x, x >_X \\ &\quad + \sum_{j=1}^{n-1} a_j^2 < (E_{\lambda_{j+1}} - E_{\lambda_j})^2 x, x >_X \\ &= \sum_{j=1}^{n-1} a_j^2(< E_{\lambda_{j+1}}x, , x >_X - < E_{\lambda_j}x, x >_X) \\ &= \int f(\lambda)^2 dF_x(\lambda). \end{aligned} \tag{1.2.8}$$

Therefore I is an isometrical mapping from the collection of all simple functions (as a subspace of $L^2(\mathbf{R}, F_x)$) to X. Hence for continuous function $f \in L^2(\mathbf{R}, F_x)$, $\int f(\lambda)dE_\lambda x$ is well-defined.

Theorem 1.2.5 *Let f be a continuous function on* $\mathbf{R}$ *and* $\{E_\lambda\}$ *be a spectral family. Define*

$$\mathcal{D}(T) = \left\{ x \in X : \int f(\lambda)^2 d < E_\lambda x, x >_X < \infty \right\}$$

and

$$Tx = \int f(\lambda) dE_\lambda x, \quad \forall x \in \mathcal{D}(T).$$

Then T is a self-adjoint operator on X.

Proof: For any $x \in X$, by Definition 1.2.7 we have

$$E_{(\alpha,\beta]}x \equiv E_\beta x - E_\alpha x \to x \qquad \text{as} \quad \alpha \to -\infty, \ \beta \to \infty.$$

Further, we have

$$\int f(\lambda)^2 d < E_\lambda E_{(\alpha,\beta]}x, x >_X = \int_\alpha^\beta f(\lambda)^2 d < E_\lambda x, x >_X < \infty$$

and hence $E_{(\alpha,\beta]}x \in \mathcal{D}(T)$ and $\mathcal{D}(T)$ is dense in X.

Similar to the proof of (1.2.8) we can show that

$$\left\langle \int f(\lambda) dE_\lambda x, y \right\rangle_X = \int f(\lambda) d < E_\lambda x, y >_X, \tag{1.2.9}$$

$\forall x \in \mathcal{D}(T)$ and $y \in X$. If $y \in \mathcal{D}(T)$, then

$$< Tx, y >_X = \int f(\lambda) d < x, E_\lambda y >_X = < x, Ty >_X, \ \ \forall \ x \in \mathcal{D}(T)$$

and hence $y \in \mathcal{D}(T^*)$ and $T^*y = Ty$, i.e. T is a symmetric operator.

On the other hand, for any $y \in \mathcal{D}(T^*)$, we have

$$\begin{aligned} < E_{(\alpha,\beta]}T^*y, x >_X &= < y, TE_{(\alpha,\beta]}x >_X \\ &= \int_\alpha^\beta f(\lambda) d < E_\lambda x, y >_X \\ &= < TE_{(\alpha,\beta]}y, x >_X, \end{aligned}$$

i.e. $E_{(\alpha,\beta]}T^*y = TE_{(\alpha,\beta]}y$. As $E_{(\alpha,\beta]}T^*y \to T^*y$, we have

$$\infty > \lim_{\alpha \to -\infty, \beta \to \infty} ||E_{(\alpha,\beta]}T^*y||_X^2 = \int f(\lambda)^2 d < E_\lambda y, y >_X .$$

Therefore $y \in \mathcal{D}(T)$ and hence, T is a self-adjoint operator. ∎

Theorem 1.2.6 (Spectral decomposition theorem) *Let T be a self-adjoint operator on X. Then there exists a unique spectral family $\{E_\lambda\}$ such that*

$$\mathcal{D}(T) = \left\{ x \in X : \int \lambda^2 d < E_\lambda x, x >_X < \infty \right\}$$

and

$$Tx = \int \lambda dE_\lambda x, \ \forall x \in \mathcal{D}(T).$$

As a consequence, for a self-adjoint operator T on X and a continuous function f on $\mathbf{R}$, we can define a self-adjoint operator $f(T)$ as follows

$$\mathcal{D}(f(T)) = \left\{ x \in X : \int f(\lambda)^2 d < E_\lambda x, x >_X < \infty \right\}$$

and

$$f(T)x = \int f(\lambda) dE_\lambda x, \ \forall x \in \mathcal{D}(f(T)).$$

For self-adjoint compact operators, the corresponding spectral families can be given by a simpler form.

Theorem 1.2.7 *If T is a self-adjoint compact operator, then there exists a CONS $\{e_{mn} : m = 0, 1, 2, \cdots, m_\infty; n = 1, 2, \cdots, n_m\}$ of X and a sequence $\{\lambda_m : m = 1, 2, \cdots, m_\infty\}$ of real numbers such that*
i) $\lambda_m \neq \lambda_{m'}$ for any $m \neq m'$,
ii) $|\lambda_m|$ decreasing and $\lambda_m \to 0$ if $m_\infty = \infty$,
iii) $n_m < \infty$ if $m \neq 0$,
iv)

$$Tx = \sum_{m=1}^{m_\infty} \lambda_m \sum_{n=1}^{n_m} < x, e_{mn} > e_{mn}, \ \forall x \in X. \tag{1.2.10}$$

Proof: Let $\{E_\lambda\}$ be the spectral family of T. For $\alpha < \beta$ such that $\alpha\beta > 0$, if the range of $E_{(\alpha,\beta]}$ is of infinite dimension, then there exists a CONS $\{x_j\}$ with infinite many elements. It follows from the Bessel inequality (1.1.19) that $< x_j, x > \to 0$, $\forall x \in X$ as $j \to \infty$.

As T is a compact operator and $\{x_j\}$ is bounded, there exists a subsequence $\{\tilde{x}_j\}$ such that $T\tilde{x}_j \to x_\infty$. Hence for any $x \in X$

$$< x_\infty, x > = \lim_j < T\tilde{x}_j, x > = \lim_j < \tilde{x}_j, T'x > = 0,$$

i.e. $x_\infty = 0$. But for any x in the range of $E_{(\alpha,\beta]}$, we have

$$\begin{aligned} \|Tx\|^2 &= \int \lambda^2 d\|E_\lambda x\|^2 = \int_\alpha^\beta \lambda^2 d\|E_\lambda x\|^2 \\ &\geq \min(\alpha^2, \beta^2)\|E_{(\alpha,\beta]}x\|^2 = \min(\alpha^2, \beta^2)\|x\|^2. \end{aligned} \tag{1.2.11}$$

Then

$$0 < \min(\alpha^2, \beta^2) \leq \|T\tilde{x}_j\|^2 \to 0,$$

a contradiction. Hence the range of $E_{(\alpha,\beta]}$ is of finite dimension whenever $0 \notin [\alpha, \beta]$.

Further, by (1.2.11), if $\min(\alpha^2, \beta^2) > ||T||^2_{L(X,Y)}$, then $E_{(\alpha,\beta]} = 0$. Therefore there exist

$$\lambda_1^- < \lambda_2^- < \cdots < 0 < \cdots < \lambda_2^+ < \lambda_1^+$$

such that 0 is the only possible limit point for either sequence $\{\lambda_j^-\}$ or sequence $\{\lambda_j^+\}$ and

$$M(\lambda) = \begin{cases} \phi & \text{if } \lambda < \lambda_1^- \\ M(\lambda_j^-) & \text{if } \lambda \in [\lambda_j^-, \lambda_{j+1}^-) \\ M(\lambda_{j+1}^+) & \text{if } \lambda \in [\lambda_{j+1}^+, \lambda_j^+) j = 1, 2, \cdots; \\ X & \text{if } \lambda \geq \lambda_1^+ \end{cases}$$

where $M(\lambda)$ is the range of E_λ. Rearrange the two sequences into λ_j such that $|\lambda_j|$ decreasing. Let $M_j = M(\lambda_j) \ominus M(\lambda_j-)$, $j \geq 1$ and $M_0 = M(0) \ominus M(0-)$. Then

$$X = \oplus_{j=0}^\infty M_j \qquad \text{and} \qquad Tx = \sum_{j=1}^\infty \lambda_j P_{M_j} x, \quad \forall x \in X.$$

The conclusion of the theorem then follows easily. ∎

Corollary 1.2.1 *a) If T is a self-adjoint nuclear operator from X to X, then (1.2.10) holds with*

$$||T||_{(1)} = \sum_{m=1}^{m_\infty} |\lambda_m| n_m < \infty$$

and

$$Trace(T) = \sum_{m=1}^{m_\infty} \lambda_m n_m.$$

b) If T is a self-adjoint Hilbert-Schmidt operator from X to X, then (1.2.10) holds with

$$||T||^2_{(2)} = \sum_{m=1}^{m_\infty} |\lambda_m|^2 n_m < \infty.$$

Finally we study the semigroup theory of linear operators which will be useful in solving some stochastic evolution equations.

Definition 1.2.8 *Let X be a Banach space. A family $\{T_t : t \geq 0\} \subset L(X, X)$ is said to be a* **strongly continuous semigroup on X** *if*
a) $T_{t+s} = T_t T_s$, $\forall s, t \geq 0$,
b) $T_0 = I$
c) $||T_t x - T_{t_0} x|| \to 0$, $\forall x \in X$ *as* $t \to t_0$.

Theorem 1.2.8 *For a strongly continuous semigroup $\{T_t\}$ on a Banach space X, there exist $M \geq 1$ and $\beta \in \mathbf{R}$ such that*

$$\|T_t\|_{L(X)} \leq Me^{\beta t}, \ \forall t \geq 0. \tag{1.2.12}$$

Proof: Let

$$M = \sup_{0 \leq t \leq 1} \|T_t\|_{L(X)} \qquad \text{and} \qquad \beta = \ln(\|T_1\|_{L(X)}).$$

Then

$$\|T_t\|_{L(X)} = \|T_{[t]+t-[t]}\|_{L(X)} = \|(T_1)^{[t]}T_{t-[t]}\|_{L(X)} \leq Me^{\beta[t]} \leq Me^{\beta t}. \qquad \blacksquare$$

Definition 1.2.9 *Let X be a Banach space. Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup on X. Define*

$$\mathcal{D}(A) = \left\{ x \in X : \quad \text{the limit} \quad \frac{T_h x - x}{h} \quad \text{in } X \text{ exists as} \quad h \to 0+ \right\}$$

and

$$Ax = \lim_{h \to 0+} \frac{T_h x - x}{h}, \qquad \forall x \in \mathcal{D}(A).$$

A is called the **generator of the semigroup**.

Theorem 1.2.9 *Let A be the generator of a strongly continuous semigroup $\{T_t\}$ on a Banach space X. Then*
a) For any $t > 0$ and $x \in X$,

$$\int_0^t T_s x ds \in \mathcal{D}(A) \quad \text{and} \quad T_t x - x = A \int_0^t T_s x ds. \tag{1.2.13}$$

b) If $x \in \mathcal{D}(A)$, then $T_t x \in \mathcal{D}(A)$ and $AT_t x = T_t Ax$. Further

$$T_t x - x = \int_0^t AT_s x ds = \int_0^t T_s Ax ds.$$

Proof: a) It follows from the definition that

$$\begin{aligned}
\lim_{h \to 0+} h^{-1}(T_h - I) \int_0^t T_s x ds &= \lim_{h \to 0+} h^{-1} \int_0^t (T_{s+h} - T_s) x ds \\
&= \lim_{h \to 0+} h^{-1} \left(\int_t^{t+h} T_s x ds - \int_0^h T_s x ds \right) \\
&= T_t x - x.
\end{aligned}$$

b) As

$$\lim_{h \to 0+} h^{-1}(T_h - I)T_t x = \lim_{h \to 0+} T_t(h^{-1}(T_h - I)x) = T_t Ax,$$

$T_tx \in \mathcal{D}(A)$ and $(\frac{d}{dt})^+T_tx = AT_tx = T_tAx$.

On the other hand, for $t > 0$

$$\begin{aligned}
& \left\| \frac{1}{-h}(T_{t-h}x - T_tx) - T_tAx \right\|_X \\
= \; & \left\| T_{t-h}\left(\frac{1}{h}(T_hx - x) - Ax\right) + T_{t-h}Ax - T_tAx \right\|_X \\
\leq \; & \|T_{t-h}\|_{L(X)} \left\| \frac{1}{h}(T_hx - x) - Ax \right\|_X + \|T_{t-h}Ax - T_tAx\|_X \to 0.
\end{aligned}$$

Hence $\frac{d}{dt}T_tx = AT_tx = T_tAx$. This proves (1.2.13). ∎

Corollary 1.2.2 *If A is the generator of a strongly continuous semigroup $\{T_t\}$ on a Banach space X, then A is a linear operator from the dense subspace $\mathcal{D}(A) \subset X$ to X.*

Proof: It is easy to see that $\mathcal{D}(A)$ is a subspace of X and A is a linear operator. For any $x \in X$ and $t > 0$, note that

$$t^{-1}\int_0^t T_sxds \in \mathcal{D}(A) \qquad \text{and} \qquad t^{-1}\int_0^t T_sxds \to x, \qquad \text{as} \quad t \to 0+.$$

Hence $\mathcal{D}(A)$ is a dense subspace of X. ∎

Definition 1.2.10 *A is called a* **closed operator** *if for $\{x_n\} \subset \mathcal{D}(A)$ such that $x_n \to x, Ax_n \to y$ in X we have $x \in \mathcal{D}(A)$ and $y = Ax$.*

Corollary 1.2.3 *If A is the generator of a strongly continuous semigroup $\{T_t\}$ on a Banach space X, then A is a closed operator.*

Proof: By (1.2.13), we have

$$T_tx_n - x_n = \int_0^t T_sAx_nds.$$

Hence

$$T_tx - x = \int_0^t T_syds.$$

This proves that $x \in \mathcal{D}(A)$ and $y = Ax$. ∎

Definition 1.2.11 *Let A be a closed operator on X. The* **resolvent set** *$\rho(A)$ of A is the collection of all $\lambda \in \mathbf{R}$ such that $\lambda - A$ is invertible, $\mathcal{R}(\lambda - A) = X$ and $R_\lambda = (\lambda - A)^{-1} \in L(X)$. For each $\lambda \in \rho(A)$, R_λ is called the* **resolvent of A at** *λ.*

Theorem 1.2.10 *Let A be the generator of a strongly continuous semigroup $\{T_t\}$ on a Banach space X. Let M and β be given by Theorem 1.2.8. Then $(\beta, \infty) \subset \rho(A)$,*

$$R_\lambda = \int_0^\infty e^{-\lambda t} T_t dt \tag{1.2.14}$$

and

$$\|(R_\lambda)^n\| \leq M(\lambda - \beta)^{-n}, \qquad n = 1, 2, \cdots, \ \lambda > \beta. \tag{1.2.15}$$

Proof: From the proof of Corollary 1.1.2 we see that, $\forall x \in X$ there exists $x' \in X'$ such that $\|x'\|_{X'} = 1$ and $x'[x] = \|x\|_X$. Note that for any $x \in \mathcal{D}(A)$ and $\lambda > \beta$,

$$\frac{d}{dt} x'[T_t x] = x'[T_t A x] = x'[T_t(A - \lambda)x] + \lambda x'[T_t x]. \tag{1.2.16}$$

Hence

$$x'[T_t x] = e^{\lambda t} x'[x] + \int_0^t e^{\lambda(t-s)} x'[T_s(A - \lambda)x] ds.$$

Therefore

$$\begin{aligned} \|x\|_X &= x'[x] = \left| e^{-\lambda t} x'[T_t x] - \int_0^t e^{-\lambda s} x'[T_s(A - \lambda)x] ds \right| \\ &\leq M e^{(\beta - \lambda)t} \|x\|_X + M(\lambda - \beta)^{-1} \|(\lambda - A)x\|_X. \end{aligned}$$

Letting $t \to \infty$ we have

$$M^{-1}(\lambda - \beta)\|x\|_X \leq \|(\lambda - A)x\|_X. \tag{1.2.17}$$

Hence $\lambda - A$ is invertible and $\mathcal{R}(\lambda - \mathcal{A})$ is a closed subspace of X.

If $\mathcal{R}(\lambda - A) \neq X$, it follows from the proof of Corollary 1.1.2 that there exists $x' \in X'$ such that $\|x'\|_{X'} = 1$ and $x'[(\lambda - A)x] = 0$, $\forall x \in \mathcal{D}(A)$. By Theorem 1.2.9 (b) we have $x'[T_s(A - \lambda)x] = 0$, $\forall s \geq 0$. It follows from (1.2.16) that

$$\frac{d}{dt} x'[T_t x] = \lambda x'[T_t x].$$

Hence $x'[T_t x] = x'[x] e^{\lambda t}$ and

$$|x'[x]| \leq M\|x\|_X e^{(\beta - \lambda)t} \to 0 \qquad \text{as } t \to \infty,$$

i.e., $x'[x] = 0$, $\forall x \in \mathcal{D}(A)$. This contradicts the denseness of $\mathcal{D}(A)$. Therefore $\mathcal{R}(\lambda - A) = X$. It follows from (1.2.17) that $R_\lambda \in L(X)$. Hence $\lambda \in \rho(A)$, $\forall \lambda > \beta$.

Let $y = R_\lambda x$. Then $y \in \mathcal{D}(A)$ and $x = \lambda y - Ay$. Hence

$$\begin{aligned}\int_0^\infty e^{-\lambda t} T_t x dt &= \int_0^\infty e^{-\lambda t} T_t(\lambda y - Ay) dt \\ &= \lambda \int_0^\infty e^{-\lambda t} T_t y dt - \int_0^\infty e^{-\lambda t} \frac{d}{dt} T_t y dt \\ &= \lambda \int_0^\infty e^{-\lambda t} T_t y dt - \int_0^\infty e^{-\lambda t} dT_t y \\ &= \lambda \int_0^\infty e^{-\lambda t} T_t y dt + y - \lambda \int_0^\infty e^{-\lambda t} T_t y dt = y.\end{aligned}$$

This proves (1.2.14). Making use of (1.2.14) repeatedly, we have

$$(R_\lambda)^n = \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1+\cdots+t_n)} T_{t_1+\cdots+t_n} dt_1 \cdots dt_n.$$

Hence

$$\begin{aligned}\|(R_\lambda)^n\|_{L(X)} &\le \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1+\cdots+t_n)} M e^{\beta(t_1+\cdots+t_n)} dt_1 \cdots dt_n \\ &= M(\lambda - \beta)^{-n}.\end{aligned}$$

∎

Theorem 1.2.11 *Let A be a densely-defined closed linear operator on a Banach space X such that $(\beta, \infty) \subset \rho(A)$ and*

$$\|(\lambda - A)^{-n}\| \le M(\lambda - \beta)^{-n}, n = 1, 2, \cdots, \lambda > \beta.$$

Then there exists a unique strongly continuous semigroup $\{T_t\}$ with generator A such that (1.2.12) holds.

Proof: First we assume that $\beta = 0$. Let $I_n = n(n - A)^{-1}$. Then for any $x \in \mathcal{D}(A)$

$$\begin{aligned}\|x - I_n x\|_X &= \|x - n(n - A)^{-1} x\|_X = \|(n - A)^{-1} Ax\|_X \\ &\le Mn^{-1} \|Ax\|_X \to 0, \qquad (1.2.18)\end{aligned}$$

as $n \to \infty$. As $\|I_n\|_{L(X)} \le M$ and $\mathcal{D}(A)$ is dense in X, we see that for any $x \in X$

$$\|x - I_n x\|_X \to 0, \qquad \text{as } n \to \infty.$$

Let $A_n = n(I_n - I)$. Then $A_n = AI_n = I_n A$. Hence for any $x \in \mathcal{D}(A)$ we have $A_n x \to Ax$ in X. Let

$$T_t^{(n)} x = e^{-nt} e^{tnI_n} = e^{-nt} \sum_j \frac{(tn)^j}{j!} (I_n)^j x, \ \forall x \in X. \qquad (1.2.19)$$

It is easy to show that (1.2.19) is well-defined, $\|T_t^{(n)}\|_{L(X)} \leq M$ and, $\forall n \geq 1, \{T_t^{(n)}\} \subset L(X)$ is a strongly continuous semigroup with generator A_n.

For any $x \in \mathcal{D}(A)$ we obtain

$$\begin{aligned} \|T_t^{(n)}x - T_t^{(m)}x\|_X &= \left\| - \int_0^t \frac{\partial}{\partial s}(T_{t-s}^{(n)}T_s^{(m)}x)ds \right\|_X \\ &= \left\| \int_0^t T_{t-s}^{(n)}T_s^{(m)}(A_n - A_m)xds \right\|_X \\ &\leq M^2\|(A_n - A_m)x\|_X t \to 0. \end{aligned} \qquad (1.2.20)$$

By the uniform boundedness of $\|T_t^{(n)}\|_{L(X)}$, (1.2.20) holds for any $x \in X$ uniformly for t in any bounded intervals. Therefore $\{T_t\}$ is a strongly continuous semigroup if we define $T_t x$ as the limit of $T_t^{(n)}x$.

Let $\tilde{A}$ be the generator of $\{T_t\}$. By Theorem 1.2.9 (b) we have

$$T_t^{(n)}x - x = \int_0^t T_s^{(n)}A_n x ds, \ \forall x \in X.$$

Then for $x \in \mathcal{D}(A)$

$$T_t x - x = \int_0^t T_s A x ds,$$

and hence $x \in \mathcal{D}(\tilde{A})$ and $\tilde{A}x = Ax$. On the other hand, for any $x \in \mathcal{D}(\tilde{A})$, set $y = (1 - \tilde{A})x$. As $1 \in \rho(A)$, there exists $z \in \mathcal{D}(A)$ such that $y = (1 - A)z = (1 - \tilde{A})x$. Hence $x = z \in \mathcal{D}(A)$. Therefore $\tilde{A} = A$. As $\|T_t^{(n)}\|_{L(X)} \leq M$, (1.2.12) holds with $\beta = 0$.

For general case, let $A_1 = A - \beta$. We obtain a strongly continuous semigroup $\{S_t\}$ with generator A_1 such that $\|S_t\|_{L(X)} \leq M$. Let $T_t = e^{\beta t}S_t$. Then $\{T_t\}$ satisfies the condition of the theorem.

Let $\{U_t\}$ be another strongly continuous semigroup with generator A. Then for any $x \in \mathcal{D}(A)$

$$\frac{\partial}{\partial s}(T_{t-s}U_s x) = -AT_{t-s}U_s x + T_{t-s}AU_s x = 0.$$

Hence $T_t x = U_t x$. This proves the uniqueness. ∎

1.3 Countably Hilbertian nuclear spaces.

In this section we introduce countably Hilbertian nuclear spaces (CHNS) and give some typical examples.

Definition 1.3.1 *Let X be a vector space. A family of norms $\{\|\cdot\|_v : v \in \Gamma\}$ on X is called* **compatible** *if $\forall p, q \in \Gamma$, $\{x_n\}$ in X is a Cauchy sequence with respect to both norms and tends to 0 with respect to one norm, then $\{x_n\}$ tends to 0 with respect to another norm.*

Remark 1.3.1 *Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two compatible norms on X such that $\|x\|_1 \leq \|x\|_2$ for any $x \in X$. Let X_j be the completion of X with respect to $\|\cdot\|_j, j = 1, 2$. For $\tilde{x} \in X_2$, let $\iota\tilde{x} = \tilde{x}$. Then ι is a well-defined map from X_2 to X_1. We call it the* **canonical injection** *from X_2 to X_1.*

Proof: Let $\tilde{x} \in X_2$ with a representation $\{x_n\} \subset X$ which is a Cauchy sequence with respect to $\|\cdot\|_2$ (cf. Theorem 1.1.4). As $\|\cdot\|_1 \leq \|\cdot\|_2, \{x_n\}$ is a Cauchy sequence with respect to $\|\cdot\|_1$. Further if $\{y_n\}$ is another representation of $\tilde{x}$ *in* X_2, i.e., $\|x_n - y_n\|_2 \to 0$. Then $\|x_n - y_n\|_1 \to 0$ and hence, $\{x_n\}$ and $\{y_n\}$ are equivalent in X_1. Therefore $\tilde{x} \in X_1$.

Now we only need to prove that, for any $\tilde{x} = \{x_n\}$ and $\tilde{y} = \{y_n\}$ in X_2, if $\tilde{x} = \tilde{y}$ in X_1 then $\tilde{x} = \tilde{y}$ in X_2. In fact, $\tilde{x} = \tilde{y}$ in X_1 implies that $\|x_n - y_n\|_1 \to 0$. It is obvious that $\{x_n - y_n\}$ is Cauchy with respect to both norms. Therefore, by the compatibility of the norms we have $\|x_n - y_n\|_2 \to 0$ and hence $\tilde{x} = \tilde{y}$ in X_2. ∎

Definition 1.3.2 *A separable Frèchet space Φ is called a* **countably Hilbertian space** *if its topology τ is given by an increasing sequence $\|\cdot\|_n, n \geq 0$, of compatible Hilbertian norms. A countably Hilbertian space Φ is called* **nuclear** *if for each $n \geq 0$ there exists $m > n$ such that the canonical injection from Φ_m into Φ_n is Hilbert-Schmidt, where Φ_n is the completion of Φ with respect to $\|\cdot\|_n$.*

The following Baire category argument will be used frequently in this book.

Lemma 1.3.1 *Let $V(\cdot) : \Phi \to [0, \infty)$ satisfy the following conditions:*
(1) V is lower semicontinuous, i.e.

$$\phi_n \to \phi \qquad \textit{implies} \qquad V(\phi) \leq \liminf_{n\to\infty} V(\phi_n).$$

(2) $V(\phi + \psi) \leq V(\phi) + V(\psi)$, $\forall \phi, \psi \in \Phi$.
(3) $V(\phi) = V(-\phi)$ and $\lim_{n\to\infty} V(\frac{\phi}{n}) = 0$, $\forall \phi \in \Phi$.
Then V is continuous. Further, if (3) is replaced by the following stronger condition
(3)′ $V(a\phi) = |a|V(\phi), \quad \forall a \in \mathbf{R}, \phi \in \Phi,$
then $V(\phi)$ is a continuous function in ϕ and there exist $\theta > 0$ and $r \geq 0$ such that

$$V(\phi) \leq \theta\|\phi\|_r, \qquad \forall \phi \in \Phi.$$

Proof: For any $\epsilon > 0$, let $E_\epsilon = \{\phi \in \Phi : V(\phi) \leq \epsilon\}$. By (1), E_ϵ is a closed subset of Φ. It follows from (3) that

$$\Phi = \cup_{n=1}^{\infty} n E_\epsilon.$$

Then by Theorem 1.1.5 and Theorem 1.1.3 (c) E_ϵ is not a nowhere dense set. Therefore there exists a nonempty open set $U \subset E_\epsilon$. Let $V = \{\phi - \psi : \phi, \psi \in U\}$. For any $\phi_0 \in \Phi, \phi_0 + V$ is a neighborhood of ϕ_0 and for any $\phi_0 + \phi - \psi \in \phi_0 + V$ we have

$$|V(\phi_0 + \phi - \psi) - V(\phi_0)| \leq V(\phi - \psi) \leq V(\phi) + V(-\psi) \leq 2\epsilon.$$

Hence V is continuous.

By the continuity of V at 0, there exists a neighborhood $\mathcal{U}_0$ of 0 such that

$$\mathcal{U}_0 = \left\{\phi \in \Phi : \|\phi\|_{r_j} < \delta_j, j = 1, 2, \cdots, m\right\} \subset E_{\epsilon_0}.$$

Let $r = \max\{r_j, j = 1, 2, \cdots, m\}$ and $\delta_0 = \min\{\delta_j, j = 1, 2, \cdots, m\}$. Then we may assume that

$$\mathcal{U}_0 = \{\phi \in \Phi : \|\phi\|_r < \delta_0\}.$$

For any $\phi \in \Phi, \phi \neq 0$, we have $\frac{\delta_0 \phi}{2\|\phi\|_r} \in \mathcal{U}_0$ and hence, $V(\frac{\delta_0 \phi}{2\|\phi\|_r}) \leq \epsilon_0$. If $(3)'$ holds, then

$$V(\phi) \leq \theta \|\phi\|_r, \qquad \forall \phi \in \Phi$$

by taking $\theta = 2\epsilon_0/\delta_0$. ∎

Theorem 1.3.1 *a)* $\{\Phi_r\}_{r\geq 0}$ *is a sequence of decreasing Hilbert spaces and*

$$\Phi = \cap_{r=0}^{\infty} \Phi_r. \tag{1.3.1}$$

b) Identifying Φ_0' *with* Φ_0 *by Riesz's representation theorem, we denote* Φ_r' *by* Φ_{-r} *with norm* $\|\cdot\|_{-r}, r \geq 0$. *Then* $\{\Phi_{-r}\}_{r\geq 0}$ *is a sequence of increasing Hilbert spaces,* Φ' *is sequentially complete and*

$$\Phi' = \cup_{r=0}^{\infty} \Phi_{-r}. \tag{1.3.2}$$

Proof: a) It follows from Remark 1.3.1 that $\{\Phi_r\}_{r\geq 0}$ is a sequence of decreasing Hilbert spaces. It is obvious that $\Phi \subset \cap_{r=0}^{\infty} \Phi_r$.

Let $\phi \in \cap_{r=0}^{\infty} \Phi_r$. For any $r \geq 0$, there exists $\{\phi_n^{(r)}\} \subset \Phi$ such that $\phi_n^{(r)} \to \phi$ in Φ_r as $n \to \infty$. Without loss of generality, we assume that $\|\phi_r^{(r)} - \phi\|_r < r^{-1}$. Then for any $n, m \geq r$

$$\begin{aligned} \|\phi_n^{(n)} - \phi_m^{(m)}\|_r &\leq \|\phi_n^{(n)} - \phi\|_r + \|\phi_m^{(m)} - \phi\|_r \\ &\leq \|\phi_n^{(n)} - \phi\|_n + \|\phi_m^{(m)} - \phi\|_m \\ &< n^{-1} + m^{-1}. \end{aligned} \tag{1.3.3}$$

Hence $\{\phi_n^{(n)}\}$ is a Cauchy sequence in Φ_r with limit ϕ. It follows from Theorem 1.1.3 (b) that $\{\phi_n^{(n)}\}$ is a Cauchy sequence in Φ. By the completeness of Φ, there exists $\psi \in \Phi$ such that $\phi_n^{(n)} \to \psi$ in Φ. By Theorem 1.1.3 (b) again, $\|\phi_n^{(n)} - \psi\|_r \to 0$. Therefore $\phi = \psi \in \Phi$ and hence (1.3.1) holds.
b) It follows from Theorem 1.1.11 that $\{\Phi_{-r}\}_{r\geq 0}$ is a sequence of Hilbert spaces. Let $0 \leq r < r'$ and $f \in \Phi_{-r}$. Then for any $\phi \in \Phi_{r'} \subset \Phi_r$

$$|f[\phi]| \leq \|f\|_{-r}\|\phi\|_r \leq \|f\|_{-r}\|\phi\|_{r'}.$$

Hence $f \in \Phi_{-r'}$, i.e., $\Phi_{-r} \subset \Phi_{-r'}$. Similarly we have $\cup_{r=0}^{\infty}\Phi_{-r} \subset \Phi'$.

For any $f \in \Phi'$, we define a map $V : \Phi \to [0,\infty)$ by $V(\phi) = |f[\phi]|$, $\forall \phi \in \Phi$. It is easy to verify the conditions of Lemma 1.3.1 for V and hence, there exist $r \geq 0$ and $\theta > 0$ such that

$$|f[\phi]| \leq \theta\|\phi\|_r, \qquad \forall \phi \in \Phi.$$

Therefore f can be regarded as a bounded linear functional in Φ_r. This proves (1.3.2).

Finally we prove that Φ' is sequentially complete. Let $\{f_n\}$ be a Cauchy sequence in Φ'. Then by Theorem 1.1.3 (b) and Definition 1.1.7 (c) that for any bounded subset B of Φ

$$\sup\{|f_n[\phi] - f_m[\phi]| : \phi \in B\} \to 0 \quad \text{as } n,\ m \to \infty. \tag{1.3.4}$$

For any $\phi \in \Phi$, as $\{\phi\}$ is a bounded subset of Φ, we see that the limit of $f_n[\phi]$ exists in $\mathbf{R}$ and we denote it by $f(\phi)$.

It is easy to see that f is a linear functional on Φ. If $f \notin \Phi'$, then there exists $\epsilon_0 > 0$ such that, for any neighborhood U of 0 in Φ, $\exists \phi_U \in U$ such that $|f[\phi_U]| \geq \epsilon_0$. Let d be the metric on Φ given by (1.1.5). Then for each $k \geq 1$,

$$U_k = \{\phi \in \Phi : d(\phi, 0) < k^{-2}\}$$

is a neighborhood of 0 in Φ and therefore, there exists $\phi_k \in U_k$ such that $|f[\phi_k]| \geq \epsilon_0$. Note that

$$\begin{aligned} d(k\phi_k, 0) &= \sum_{j=1}^{\infty} 2^{-j}(\|k\phi_k\|_j \wedge 1) \\ &= k\sum_{j=1}^{\infty} 2^{-j}\|\phi_k\|_j 1_{\|\phi_k\|_j < k^{-1}} + \sum_{j=1}^{\infty} 2^{-j} 1_{\|\phi_k\|_j \geq k^{-1}} \\ &\leq k\sum_{j=1}^{\infty} 2^{-j}(\|\phi_k\|_j \wedge 1)1_{\|\phi_k\|_j < k^{-1}} \\ &\quad + k\sum_{j=1}^{\infty} 2^{-j}(\|\phi_k\|_j \wedge 1)1_{1\geq\|\phi_k\|_j \geq k^{-1}} + \sum_{j=1}^{\infty} 2^{-j} 1_{\|\phi_k\|_j > 1} \\ &\leq kd(\phi_k, 0) + d(\phi_k, 0) \leq k^{-2}(k+1) \to 0. \end{aligned}$$

Hence $k\phi_k \to 0$ in Φ. Therefore $B = \{k\phi_k : k \geq 1\}$ is a bounded subset of Φ. But

$$\infty > \sup_n q_B(f_n) \geq q_B(f) = \sup_k |f[k\phi_k] \geq \sup_k k\epsilon_0 = \infty.$$

This contradiction implies that $f \in \Phi'$. ∎

Definition 1.3.3 *Suppose there is an inner product $< \cdot, \cdot >_H$ on Φ which is continuous in the τ-topology of Φ. Let H be the Hilbert space completion of Φ with respect to $< \cdot, \cdot >_H$. Then the triplet*

$$\Phi \hookrightarrow H \hookrightarrow \Phi'$$

is called a **rigged Hilbert space** *or a* **Gel'fand triplet**.

Remark 1.3.2 *The Hilbert space H may be one of the Hilbert space Φ_r defining the topology of Φ but this is not always the case as we shall illustrate later on.*

Example 1.3.1 *Schwartz space*

Let

$$\mathcal{S}(\mathbf{R}) = \{\phi \in C^\infty(\mathbf{R}) : ||\phi||_{\alpha,\beta} < \infty, \quad \forall \alpha, \beta \in \mathbf{N}\}$$

where

$$||\phi||_{\alpha,\beta} = \sup_{x \in \mathbf{R}} |x^\alpha \phi^{(\beta)}(x)|$$

and $\phi^{(\beta)}(x)$ is the β-order derivative of ϕ.

Lemma 1.3.2 *$\mathcal{S}(\mathbf{R})$ is a Frèchet space whose topology is given by the family $\{|| \cdot ||_{\alpha,\beta} : \alpha, \beta \in \mathbf{N}\}$ of seminorms.*

Proof: It is easy to see that for any α, $\beta \in \mathbf{N}$, $|| \cdot ||_{\alpha,\beta}$ is a seminorm and $\mathcal{S}(\mathbf{R})$ is a vector space. We only need to prove the completeness of $\mathcal{S}(\mathbf{R})$.

Let $\{\phi_n\}$ be a Cauchy sequence in $\mathcal{S}(\mathbf{R})$. Then for any $\alpha, \beta \in \mathbf{N}$

$$||\phi_n - \phi_m||_{\alpha,\beta} \to 0 \qquad \text{as} \quad n, m \to \infty.$$

As $C_b(\mathbf{R})$ is a Banach space with supremum norm, there exists $\psi_{\alpha,\beta} \in C_b(\mathbf{R})$, $\forall \alpha, \beta \in \mathbf{N}$ such that

$$\sup_x |x^\alpha \phi_n^{(\beta)}(x) - \psi_{\alpha,\beta}(x)| \to 0 \qquad \text{as} \quad n \to \infty. \tag{1.3.5}$$

Let $\phi(x) = \psi_{0,0}(x)$. Then $\phi \in C^\infty(\mathbf{R})$ and

$$\psi_{\alpha,\beta}(x) = x^\alpha \phi^{(\beta)}(x). \tag{1.3.6}$$

In fact it follows from the definition of ϕ that (1.3.6) is true for $\alpha = 0$ and $\beta = 0$. We assume that (1.3.6) is true for $\alpha = 0$ and $\beta \leq k$. Note that

$$\phi_n^{(k)}(x) - \phi_n^{(k)}(0) = \int_0^x \phi_n^{(k+1)}(y)dy$$

and letting $n \to \infty$

$$\psi_{0,k}(x) - \phi_{0,k}(0) = \int_0^x \psi_{0,k+1}(y)dy,$$

i.e.

$$\phi^{(k)}(x) - \phi^{(k)}(0) = \int_0^x \psi_{0,k+1}(y)dy.$$

Hence $\phi \in C^{k+1}(\mathbf{R})$ and $\phi^{(k+1)}(x) = \psi_{0,k+1}(x)$. This proves that (1.3.6) holds for $\alpha = 0$ and $\beta \in \mathbf{N}$.

For any $\alpha \in \mathbf{N}$ and $x \in \mathbf{R}$

$$x^\alpha \phi_n^{(\beta)}(x) \to x^\alpha \psi_{0,\beta}(x) = x^\alpha \phi^{(\beta)}(x).$$

Hence (1.3.6) holds for any $\alpha, \beta \in \mathbf{N}$. Therefore $\phi \in \mathcal{S}(\mathbf{R})$ and $\phi_n \to \phi$ in $\mathcal{S}(\mathbf{R})$. Hence $\mathcal{S}(\mathbf{R})$ is a Frèchet space. ∎

The space $\mathcal{S}(\mathbf{R})$ can also be defined using a sequence of Hilbertian norms. Let

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and

$$h_n(x) = \frac{(-1)^n}{\sqrt{n!}} g(x)^{-1} \left(\frac{d}{dx}\right)^n g(x),\ n = 0, 1, \cdots. \tag{1.3.7}$$

Then the Hermite polynomials $\{h_n(x) : n \geq 0\}$ forms a CONS of the Hilbert space $L^2(\mathbf{R}, g(x)dx)$ and

$$\sqrt{n} h_n(x) = x h_{n-1}(x) - h_{n-1}^{'}(x) \tag{1.3.8}$$

and

$$h_n^{'}(x) = \sqrt{n} h_{n-1}(x),\ n \geq 1. \tag{1.3.9}$$

Now we define the sequence of Hermite functions $\{\phi_n\}_{n \geq 1}$:

$$\phi_{n+1}(x) = \sqrt{g(x)} h_n(x),\ n \geq 0. \tag{1.3.10}$$

Then $\{\phi_n\}$ is a CONS of $L^2(\mathbf{R})$.

For $p \in \mathbf{R}$ and $\phi \in \Phi$, define

$$||\phi||_p^2 = \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right)^{2p} < \phi, \phi_n >_{L^2(\mathbf{R})}^2 .$$

Lemma 1.3.3 *For any $p \in \mathbf{R}$ there exist $k \in \mathbf{N}$ and $C > 0$ such that*

$$\|\phi\|_p \le C \max_{0\le\alpha,\beta\le k+1} \|\phi\|_{\alpha,\beta}. \tag{1.3.11}$$

Proof: It follows from (1.3.9) and integration by part that

$$\begin{aligned} <\phi, \phi_n>_{L^2(\mathbf{R})} &= \int \phi(x)\sqrt{g(x)} h_{n-1}(x)dx \\ &= n^{-1/2}\int \phi(x)\sqrt{g(x)}dh_n(x) \\ &= -n^{-1/2}\int \left(\phi(x)\sqrt{g(x)}\right)' h_n(x)dx. \end{aligned}$$

Repeating the argument above, we have

$$<\phi, \phi_n>_{L^2(\mathbf{R})} = \frac{(-1)^k \int \left(\phi(x)\sqrt{g(x)}\right)^{(k)} h_{n+k-1}dx}{\sqrt{n(n+1)\cdots(n+k-1)}}.$$

Note that by (1.3.7)

$$\begin{aligned} \left(\phi(x)\sqrt{g(x)}\right)^{(k)} &= \sum_{j=0}^{k}\binom{k}{j}\phi^{(k-j)}(x)\left(\sqrt{g(x)}\right)^{(j)} \\ &= \sum_{j=0}^{k}\binom{k}{j}\phi^{(k-j)}(x)(-1)^j 2^{-\frac{j}{2}}\sqrt{j!}h_j\left(\frac{x}{\sqrt{2}}\right)\sqrt{g(x)}. \end{aligned}$$

It is easy to see that

$$C_j \equiv 2^{-\frac{j}{2}}\sqrt{j!}\sup_x \left|h_j\left(\frac{x}{\sqrt{2}}\right)\right| \frac{1+|x|}{1+|x|^{j+1}} < \infty, \ \forall j \ge 0.$$

Hence

$$\begin{aligned} &\left|\left(\phi(x)\sqrt{g(x)}\right)^{(k)}\right| \\ \le\ & \sum_{j=0}^{k}\binom{k}{j}C_j\frac{1+|x|^{j+1}}{1+|x|}\sqrt{g(x)}|\phi^{(k-j)}(x)| \\ \le\ & \sum_{j=0}^{k}\binom{k}{j}C_j\sqrt{g(x)}(1+|x|)^{-1}\left(\|\phi\|_{0,k-j}+\|\phi\|_{j+1,k-j}\right) \\ \le\ & 2\left\{\sum_{j=0}^{k}\binom{k}{j}C_j\right\}\left(\max_{0\le\alpha,\beta\le k+1}\|\phi\|_{\alpha,\beta}\right)(1+|x|)^{-1}\sqrt{g(x)}. \end{aligned}$$

Therefore

$$\begin{aligned}
<\phi,\phi_n>^2_{L^2(\mathbf{R})} &\le \left(n+\frac{1}{2}\right)^{-k} 4\left\{\sum_{j=0}^{k}\binom{k}{j}C_j\right\}^2 \left(\max_{0\le\alpha,\beta\le k+1}\|\phi\|^2_{\alpha,\beta}\right) \\
&\quad \int h_{n+k-1}(x)^2 g(x)dx \int (1+|x|)^{-2}dx \\
&\le 8\left\{\sum_{j=0}^{k}\binom{k}{j}C_j\right\}^2 \left(n+\frac{1}{2}\right)^{-k} \max_{0\le\alpha,\beta\le k+1}\|\phi\|^2_{\alpha,\beta}.
\end{aligned}$$

Taking k such that $2p-k<-1$ we have

$$\|\phi\|_p^2 \le 8\left\{\sum_{j=0}^{k}\binom{k}{j}C_j\right\}^2 \sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)^{2p-k} \max_{0\le\alpha,\beta\le k+1}\|\phi\|^2_{\alpha,\beta}. \qquad \blacksquare$$

Lemma 1.3.4 *For any $\alpha,\beta\in\mathbf{N}$ there exist $M>0$ and $p\in\mathbf{N}$ such that*

$$\|\phi\|_{\alpha,\beta}\le M\|\phi\|_p \tag{1.3.12}$$

for any ϕ which can be written as a linear combination of finite many ϕ_n's.

Proof: By (1.3.8) and (1.3.9), we have

$$\begin{aligned}
\phi_n'(x) &= h_{n-1}'(x)\sqrt{g(x)}-\frac{x}{2}h_{n-1}(x)\sqrt{g(x)} \\
&= \frac{\sqrt{n-1}}{2}\phi_{n-1}(x)-\frac{\sqrt{n}}{2}\phi_{n+1}(x)
\end{aligned}$$

and

$$x\phi_n(x)=\sqrt{n-1}\phi_{n-1}(x)+\sqrt{n}\phi_{n+1}(x).$$

By induction on α and β it is easy to show that

$$x^\alpha\phi_n^{(\beta)}(x)=\sum_{j=-\alpha-\beta}^{\alpha+\beta}C_{j,n,\alpha,\beta}\phi_{n+j}(x)$$

where

$$|C_{j,n,\alpha,\beta}|\le (n+\alpha+\beta)^{(\alpha+\beta)/2}$$

and we define $\phi_k(x)\equiv 0\ \forall k<1$. Let

$$\phi\equiv\sum_{n=1}^{N}<\phi,\phi_n>_{L^2(\mathbf{R})}\phi_n.$$

Then

$$
\begin{aligned}
&||\phi||^2_{\alpha,\beta,2} \equiv \int |x^\alpha \phi^{(\beta)}(x)|^2 dx \\
= &\int \left| \sum_{n=1}^N < \phi, \phi_n >_{L^2(\mathbf{R})} x^\alpha \phi_n^{(\beta)}(x) \right|^2 dx \\
= &\int \left| \sum_{n=1}^N < \phi, \phi_n >_{L^2(\mathbf{R})} \sum_{j=-\alpha-\beta}^{\alpha+\beta} C_{j,n,\alpha,\beta}\phi_{n+j}(x) \right|^2 dx \\
= &\sum_{n,m=1}^N \sum_{j,k=-\alpha-\beta}^{\alpha+\beta} < \phi, \phi_n >< \phi, \phi_m > C_{j,n,\alpha,\beta} C_{k,m,\alpha,\beta} 1_{n+j=m+k} \\
\leq &\sum_{n,m=1}^N \sum_{j,k=-\alpha-\beta}^{\alpha+\beta} (n \vee m + \alpha + \beta)^{\alpha+\beta} | < \phi, \phi_n >< \phi, \phi_m > | 1_{n+j=m+k} \\
\leq &\sum_{n,m=1}^N (2\alpha + 2\beta + 1)^2 (n \vee m + \alpha + \beta)^{\alpha+\beta} \\
&| < \phi, \phi_n >< \phi, \phi_m > | 1_{|n-m| \leq 2\alpha+2\beta} \\
\leq &\ 2(\alpha + \beta + 1)^2 \sum_{n,m=1}^N (n \vee m + \alpha + \beta)^{\alpha+\beta} \\
&(< \phi, \phi_n >^2 + < \phi, \phi_m >^2) 1_{|n-m| \leq 2\alpha+2\beta} \\
\leq &\ 16(\alpha + \beta + 1)^3 \sum_{n=1}^N (n + 3\alpha + 3\beta)^{\alpha+\beta} < \phi, \phi_n >^2 \\
\leq &\ K^2 ||\phi||^2_{(\alpha+\beta)/2}
\end{aligned}
\tag{1.3.13}
$$

where $K = 2^{\alpha+\beta+4}(\alpha + \beta + 1)^{\alpha+\beta+3}$.

Finally, as

$$
\left(x^\alpha \phi^{(\beta)}(x)\right)' = \alpha x^{\alpha-1} \phi^{(\beta)}(x) + x^\alpha \phi^{(\beta+1)}(x),
$$

we have

$$
\begin{aligned}
||\phi||_{\alpha,\beta} \leq &\int |\alpha x^{\alpha-1} \phi^{(\beta)}(x) + x^\alpha \phi^{(\beta+1)}(x)| dx \\
\leq &\ \alpha \sqrt{\int (1 + x^2) |x^{\alpha-1} \phi^{(\beta)}(x)|^2 dx} \sqrt{\int (1 + x^2)^{-1} dx} \\
&+ \sqrt{\int (1 + x^2) |x^\alpha \phi^{(\beta+1)}(x)|^2 dx} \sqrt{\int (1 + x^2)^{-1} dx} \\
\leq &\ \alpha(||\phi||_{\alpha-1,\beta,2} + ||\phi||_{\alpha,\beta,2}) + ||\phi||_{\alpha,\beta+1,2} + ||\phi||_{\alpha+1,\beta+1,2} \\
\leq &\ 2K(\alpha + 1) ||\phi||_{(\alpha+\beta+2)/2}.
\end{aligned}
\tag{1.3.14}
$$

(1.3.12) then follows by taking $p = \frac{\alpha+\beta+2}{2}$ and $M = 2K(\alpha+1)$. ∎

Theorem 1.3.2 *a)*

$$\mathcal{S}(\mathbf{R}) = \{\phi \in L^2(\mathbf{R}) : \|\phi\|_p < \infty, \ \forall p \in \mathbf{R}\}. \tag{1.3.15}$$

Further the topology of $\mathcal{S}(\mathbf{R})$ is given by the family $\{\|\cdot\|_p : p \in \mathbf{R}\}$ or equivalently, by a sequence $\{\|\cdot\|_p : p \in \mathbf{N}\}$ of increasing Hilbertian norms.
b) $\mathcal{S}(\mathbf{R})$ is a CHNS.
c) $\mathcal{S}_p' = \mathcal{S}_{-p}$.
d) If $H = L^2(\mathbf{R})$, then

$$\mathcal{S}(\mathbf{R}) \hookrightarrow H \hookrightarrow \mathcal{S}(\mathbf{R})'$$

is a rigged Hilbert space.

Proof: a) It follows from Lemma 1.3.3 that

$$\mathcal{S}(\mathbf{R}) \subset \{\phi \in L^2(\mathbf{R}) : \|\phi\|_p < \infty, \ \forall p \in \mathbf{R}\},$$

Let $\phi \in L^2(\mathbf{R})$ be such that $\|\phi\|_p < \infty, \ \forall p \in \mathbf{R}$. Let

$$\phi^{(n)}(x) = \sum_{j=1}^{n} < \phi, \phi_j >_{L^2(\mathbf{R})} \phi_j.$$

Then $\phi^{(n)} \to \phi$ with respect to any $\|\cdot\|_p$. Hence, by Lemma 1.3.4, $\{\phi^{(n)}\}$ is a Cauchy sequence with respect to each $\|\cdot\|_{\alpha,\beta}$. Therefore it follows from Theorem 1.1.3 (b) and the completeness of $\mathcal{S}(\mathbf{R})$ proved in Lemma 1.3.2 that there exists $\psi \in \mathcal{S}(\mathbf{R})$ such that $\phi^{(n)} \to \psi$ in $\mathcal{S}(\mathbf{R})$. By Lemma 1.3.3 again, $\phi^{(n)} \to \psi$ with respect to any $\|\cdot\|_p$. Hence $\phi = \psi \in \mathcal{S}(\mathbf{R})$ i.e.

$$\{\phi \in L^2(\mathbf{R}) : \|\phi\|_p < \infty, \ \forall p \in \mathbf{R}\} \subset \mathcal{S}(\mathbf{R}).$$

This proves (1.3.15). The equivalence of the topologies follows similarly from the above arguments.
b) We first prove that $\{\|\cdot\|_p : p \in \mathbf{R}\}$ is a family of compatible norms. In fact, let $p < q$ and let $\{f_n\} \subset \Phi$ be a Cauchy sequence with respect to both norms and $\|f_n\|_p \to 0$. Then

$$\left(k + \frac{1}{2}\right)^{2p} < f_n, \phi_k >^2_{L^2(\mathbf{R})} \leq \|f_n\|_p^2 \to 0 \qquad \text{for each} \quad k \geq 1.$$

i.e.

$$< f_n, \phi_k >_{L^2(\mathbf{R})} \to 0 \qquad \text{for each} \quad k \geq 1 \quad \text{as} \quad n \to \infty. \tag{1.3.16}$$

As $\{f_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_q$, $\forall\epsilon > 0$, $\exists N$, s.t. $\forall n, m \geq N$ we have

$$\sum_{k=1}^{K}\left(k+\frac{1}{2}\right)^{2q} < f_n - f_m, \phi_k >^2_{L^2(\mathbf{R})} < \epsilon, \qquad \text{for any} \quad K \geq 1.$$

Taking $m \to \infty$, it follows from (1.3.16) that

$$\sum_{k=1}^{K}\left(k+\frac{1}{2}\right)^{2q} < f_n, \phi_k >^2_{L^2(\mathbf{R})} \leq \epsilon, \qquad \text{for any} \quad K \geq 1 \quad \text{and} \quad n \geq N.$$

Letting $K \to \infty$, we have

$$\sum_{k=1}^{\infty}\left(k+\frac{1}{2}\right)^{2q} < f_n, \phi_k >^2_{L^2(\mathbf{R})} \leq \epsilon, \ \forall n \geq N.$$

This proves $\|f_n\|_q \to 0$ and hence the norms are compatible.

For any $p \in \mathbf{R}$, $\{(n+\frac{1}{2})^{-p}\phi_n\}$ is a CONS for $\mathcal{S}_p$. If $p > q + \frac{1}{2}$, then

$$\sum_{n=1}^{\infty}\left\|\left(n+\frac{1}{2}\right)^{-p}\phi_n\right\|_q^2 = \sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)^{-2(p-q)} < \infty.$$

Hence the canonical injection from $\mathcal{S}_p$ into $\mathcal{S}_q$ is Hilbert-Schmidt for $p > q+\frac{1}{2}$. Therefore $\mathcal{S}(\mathbf{R})$ is a CHNS.

c) Let $f, g \in \mathcal{S}$, then for all $p \in \mathbf{R}$

$$\begin{aligned} |f[g]|^2 &= \left|\sum_{n=1}^{\infty} < f, \phi_n >_0 < g, \phi_n >_0\right|^2 \\ &\leq \left(\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)^{-2p} < f, \phi_n >_0^2\right)\left(\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)^{2p} < g, \phi_n >_0^2\right) \\ &= \|f\|_{-p}^2\|g\|_p^2 \end{aligned}$$

which shows that $\mathcal{S}_{-p}$ is the dual of $\mathcal{S}_p$.

d) As the inner product $< \cdot, \cdot >_H = < \cdot, \cdot >_0$ is continuous in $\mathcal{S}(\mathbf{R}) \times \mathcal{S}(\mathbf{R})$, (d) follows directly from the Definition 1.3.3. ∎

Definition 1.3.4 $\mathcal{S}(\mathbf{R})$ *is called* **the space of rapidly decreasing functions on R.** $\mathcal{S}(\mathbf{R})'$ *is called* **the space of tempered distributions.**

The following model is useful in solving some evolution equations.

Definition 1.3.5 *Given a rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$, a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on H is said to be* **compatible with** (Φ, H, Φ') *or equivalently we will refer to (Φ, H, T_t) as a* **compatible family** *if*
a) $T_t\Phi \subset \Phi$ $\forall t \geq 0$.
b) The restriction $T_t|_\Phi : \Phi \to \Phi$ is τ-continuous for any $t \geq 0$.
c) $t \to T_t\phi$ is continuous for any $\phi \in \Phi$.
d) Let A be the generator of T_t on H. Then $A|_\Phi : \Phi \to \Phi$ is continuous.

Remark 1.3.3 *If Φ, H, Φ' are already given, (a)-(d) is a restriction on the type of $\{T_t\}$ that can be considered. However, it is important to observe that in practical problems, physical considerations usually give no idea of the rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$ to be used and only the Hilbert space H and the semigroup $\{T_t\}$ are naturally given in the problem, so that the Schwartz space cannot be chosen in advance.*

The following example gives a method of choosing Φ and Φ' when T_t is given and satisfies certain conditions.

Example 1.3.2 *A class of examples of $(\Phi, H, \Phi', \{T_t\})$*

Let H be a real separable Hilbert space and $A = -L$ be a closed densely defined self-adjoint operator on H such that $\langle -L\phi, \phi\rangle_H \leq 0$ for $\phi \in Dom(L)$, the domain of L. Let $\{T_t\}$ be the semigroup on H determined by A. Further assume that some power of the resolvent of L is a Hilbert-Schmidt operator i.e.

$$\exists\, r_1 \quad \text{such that} \quad (\lambda I + L)^{-r_1} \quad \text{is Hilbert-Schmidt.} \tag{1.3.17}$$

This condition enables us to find an appropriate CHNS Φ for the model, as we shall now indicate: it follows from Corollary 1.2.1 that there exist $\{\mu_j\} \subset \mathbf{R}$ and $\{\phi_j\}_{j\geq 1} \subset H$ such that

$$(\lambda I + L)^{-r_1}\phi_j = \mu_j\phi_j, \qquad \text{for any} \quad j \geq 1.$$

As L is a nonnegative-definite self-adjoint operator, we see that $\{\mu_j\}$ is a decreasing sequence with lower bound $\lambda > 0$ and hence, $\{\phi_j\}_{j\geq 1}$ is a CONS of H. Let λ_j be such that $(\lambda + \lambda_j)^{-r_1} = \mu_j, j \geq 1$. Then $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and, by Theorem 1.2.6

$$L\phi_j = \lambda_j\phi_j, \qquad \text{for any} \quad j \geq 1.$$

Letting $\lambda = 1$, define

$$\begin{aligned} \Phi &= \left\{\phi \in H : \|(I+L)^r\phi\|_H^2 < \infty, \ \forall r \in \mathbf{R}\right\} \qquad (1.3.18) \\ &= \left\{\phi \in H : \sum_{j=1}^{\infty}(1+\lambda_j)^{2r} < \phi, \phi_j >_H^2 < \infty, \ \forall r \in \mathbf{R}\right\}. \end{aligned}$$

Define the inner product $< \cdot, \cdot >_r$ on Φ by

$$< \phi, \psi >_r = \sum_{j=1}^{\infty} (1+\lambda_j)^{2r} < \phi, \phi_j >_H < \psi, \phi_j >_H$$

and

$$\|\phi\|_r^2 = < \phi, \phi >_r .$$

Let Φ_r be the $\|\cdot\|_r$-completion of Φ. We then have

$$\Phi = \cap_r \Phi_r, \quad \Phi' = \cup_r \Phi_r$$

and for $r \leq s, \phi \in \Phi, \|\phi\|_r \leq \|\phi\|_s$ and so $\Phi_s \subset \Phi_r$ with $\Phi_0 = H$. Condition (1.3.17) implies that the canonical injection from Φ_p into Φ_q is Hilbert-Schmidt for $p \geq q + r_1$ and therefore Φ is a CHNS.

For each $r > 0$, Φ_{-r} and Φ_r are in duality under the pairing

$$\eta[\phi] = \sum_{j=1}^{\infty} < \eta, \phi_j >_{-r} < \phi, \phi_j >_r, \quad \eta \in \Phi_{-r}, \ \phi \in \Phi_r \tag{1.3.19}$$

and therefore $\Phi_{-r} = \Phi_r'$. We also have that $\{\phi_j\}_{j\geq 1}$ is a COS (not normal) in Φ_r for all $r \in \mathbf{R}$.

From now on we will write $< \cdot, \cdot >_0 = < \cdot, \cdot >_H$. For $\phi \in H$ and $t \geq 0$, let

$$T_t\phi = \sum_{j=1}^{\infty} e^{-t\lambda_j} < \phi, \phi_j >_0 \phi_j \in H.$$

Then $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup on H with generator $-L$. Now we shall prove that the semigroup $\{T_t\}$ satisfies conditions (a)-(d) in Definition 1.3.5:

If $\phi \in \Phi$, then $\forall r \in \mathbf{R}$

$$\begin{aligned} \sum_{j=1}^{\infty} < T_t\phi, \phi_j >_0^2 (1+\lambda_j)^{2r} &= \sum_{j=1}^{\infty} < \phi, \phi_j >_0^2 (1+\lambda_j)^{2r} e^{-2t\lambda_j} \\ &\leq \sum_{j=1}^{\infty} < \phi, \phi_j >_0^2 (1+\lambda_j)^{2r} < \infty, \end{aligned}$$

i.e. $T_t\phi \in \Phi$ which implies (a). Next for $\phi \in \Phi$ and $t \geq 0$

$$\begin{aligned} \|T_t\phi\|_r^2 &= \sum_{j=1}^{\infty} e^{-2t\lambda_j} (1+\lambda_j)^{2r} < \phi, \phi_j >_0^2 \\ &\leq \sum_{j=1}^{\infty} (1+\lambda_j)^{2r} < \phi, \phi_j >_0^2 \\ &= \|\phi\|_r^2. \end{aligned}$$

It follows from Theorem 1.1.3 (b) that $T_t : \Phi \to \Phi$ is continuous and therefore condition (b) is satisfied.

Now for $s, t \geq 0$ and $\phi \in \Phi$

$$||T_t\phi - T_s\phi||_r^2 = \sum_{j=1}^{\infty}(e^{-t\lambda_j} - e^{-s\lambda_j})^2(1+\lambda_j)^{2r} < \phi, \phi_j >_0^2$$

and for each $j \geq 1$

$$(e^{-t\lambda_j} - e^{-s\lambda_j})^2(1+\lambda_j)^{2r} < \phi, \phi_j >_0^2 \leq 4(1+\lambda_j)^{2r} < \phi, \phi_j >_0^2 .$$

Since $e^{-t\lambda_j}$ is continuous in t for all $j \geq 1$, then by the dominated convergence theorem,

$$\lim_{t \to s} ||T_t\phi - T_s\phi||_r = 0 \quad \forall \phi \in \Phi, r \in \mathbf{R}$$

which implies (c).

Now to prove (d) let $\phi \in \Phi$ and define $\psi_n = \sum_{j=1}^{n} < \phi, \phi_j >_0 \phi_j$. Then $\psi_n \to \phi$ on Φ,

$$L\psi_n = \sum_{j=1}^{n} < \phi, \phi_j >_0 L\phi_j = \sum_{j=1}^{n} \lambda_j < \phi, \phi_j >_0 \phi_j$$

and for $m > n$ and $r \in \mathbf{R}$

$$\left\| \sum_{j=n+1}^{m} \lambda_j < \phi, \phi_j >_0 \phi_j \right\|_r^2 = \sum_{j=n+1}^{m} \lambda_j^2 (1+\lambda_j)^{2r} < \phi, \phi_j >_0^2 \to 0$$

as $n, m \to \infty$. Hence

$$L\psi_n \to \psi = \sum_{j=1}^{\infty} \lambda_j < \phi, \phi_j >_0 \phi_j \qquad \text{in } \Phi.$$

But since L is closed in H and $|| \cdot ||_H$ is Φ-continuous, then $\phi \in Dom(L)$ and $\psi = L\phi$, *i.e.* $L|_\Phi : \Phi \to \Phi$ is given by

$$L\phi = \sum_{j=1}^{\infty} \lambda_j < \phi, \phi_j >_0 \phi_j.$$

It is obvious that L is linear and

$$||L\phi||_r^2 \leq ||\phi||_{r+1}^2 \; \forall \phi \in \Phi, r \in \mathbf{R}. \tag{1.3.20}$$

Hence L is continuous in Φ which implies (d).

Remark 1.3.4 *A compatible family* (Φ, H, T_t) *or* (Φ, H, L) *is called a* **special compatible family** *if the generator* L *satisfies condition (1.3.17) and* Φ *is constructed as in Example 1.3.2, i.e.* Φ *is given by (1.3.18).*

Remark 1.3.5 *The Schwartz space* $\mathcal{S}(\mathbf{R})$ *of Example 1.3.1 may be obtained in the framework of the last example by taking*

$$-L = \frac{d^2}{dx^2} - \frac{x^2}{4}, \quad H = L^2(\mathbf{R}), \quad \lambda_j = j - \frac{1}{2}, \ \forall j \geq 1$$

and $\{\phi_j\}$ *the Hermite functions given by (1.3.10). In fact for* $r > \frac{1}{2}$

$$\sum_{j=1}^{\infty} \|(I+L)^{-r}\phi_j\|_H^2 = \sum_{j=1}^{\infty}(1+\lambda_j)^{-2r} = \sum_{j=1}^{\infty}\left(j+\frac{1}{2}\right)^{-2r} < \infty.$$

Hence L *satisfies condition (1.3.17) for* $r_1 > \frac{1}{2}$ *and* $(\mathcal{S}(\mathbf{R}), L^2(\mathbf{R}), L)$ *is a special compatible family.*