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ADAPTIVE BAYESIAN DESIGNS FOR ACCELERATED LIFE TESTING

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Abstract

In this paper, we present a Bayesian decision theoretic framework for the design of accelerated life tests. In our development, we assume that quality of inference at the "use stress" is the only concern to the designer and use a quadratic loss function as the design criterion. We derive optimal designs for exponential life models under a given form of an "acceleration function" using a complete test. Linear Bayes methods play an important role in our making inference. Sequential processing of information and the ability to obtain one-point designs make the approach attractive for developing adaptive design strategies.

1. Introduction. In accelerated life testing (ALT), items are subjected to an environment that is more severe than the *use environment* (i.e., the normal operating environment) in order to induce early failures. The accelerated environment is achieved by increasing the levels of one or more of the stress variables that constitute the environment. For instance, typical stresses associated with mechanical and electronic devices include temperature, wind, pressure, amplitude, and voltage. Test data collected in the accelerated environment are then used for

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inference about the failure characteristics of the items in the use environment. An important assumption that facilitates inference is the assumed form of the *time transformation function* or *acceleration function* [see Mann, Schafer and Singpurwalla (1974), p. 421] that describes the relationship between the failure characteristic of interest and the applied stress level. This relationship is specified based on engineering judgement and the physics of failure for the given situation.

The design problem in accelerated life testing is concerned with specifying the number and magnitude of the accelerated stress levels, and the number of items to be tested at these stress levels. To date, the majority of the literature on accelerated life testing has focused on inference about the failure behavior in the use environment given the data collected in the accelerated environment. A review of the sample theoretic literature is given in Nelson (1990), and Mazzuchi and Singpurwalla (1988) provide an overview of the Bayesian methods for inference from ALT's.

The majority of the work published regarding the design of ALT's relied on sample theoretic methods [see, for example, Nelson (1990)]. Recently, some Bayesian approaches have been presented by Verdinelli, Polson and Singpurwalla (1993), Menzenfricke (1991) and Chaloner and Larntz (1992). Most of these approaches are based on the theory of optimal Bayesian designs for linear models [see, for example, Chaloner (1984)]. Consequently, the results are applicable to ALT designs when the life model is normal or lognormal. In this paper, we present a Bayesian approach for obtaining optimal ALT designs when the underlying life model is exponential. The extension of our approach to the normal, lognormal, and Weibull models is straightforward. Our approach accommodates complete sample tests, as well as Type I and Type II censored tests. In addition, the methodology can be used for a wide variety of specified time transformation functions including the Power Law, the Arrhenius and Eyring Rules, and their stress dependent (dynamic) equivalents.

2. Formulation of the optimal design problem. Let m denote the number of distinct stress levels used for ALT, and let S_i denote the value of the ith accelerated stress level for $i = 1, 2, \ldots, m$. The subscripts are used to indicate *distinct* stress levels and do not imply any specific ordering in terms of the magnitude of the stresses. It is assumed, however, that each of the accelerated stress levels yields an environment at least as severe as S_u, the stress level in the use environment, that is, $S_i > S_u$ for $i = 1, 2, \ldots, m$. Let n_i denote the number of items tested at the ith stress level and $n = \sum_{i=1}^{m} n_i$ is the *predetermined* number of items to be used in the ALT.

Finally, let Y_{ij} represent the lifelength of the jth item on test at the ith stress level and y_{ij} its realization for $j = 1, 2, \ldots, n_i$. The number of failures observed at the ith stress level is denoted by r_i. Using the notation above, define the information I_i from the ith stress level by

$$I_i = \{S_i, n_i, r_i, y_{ij}, \text{ for } j = 1, 2, \ \ldots, n_i\} \text{ for } i = 1, 2, \ \ldots, m,$$

and assuming that testing proceeds from stress S_1 to S_m, define the available information D_i after testing at the ith stress level by

$$D_i = \{I_i, D_{i-1}\} \text{ for } i = 1, 2, \ \ldots, m.$$

The information available prior to testing is denoted by D_0.

The main purpose of ALT is to provide a prediction of a failure characteristic, such as the mean life or the failure rate, at the stress level in the use environment. We call this level the *use stress*. Assuming that quality of inference is the only concern to the designer and denoting the failure characteristic of interest at the use stress by η_u, we assume that the designer's loss function is quadratic

(2.1)
$$L(\eta_u, \hat{\eta}_u) = (\eta_u - \hat{\eta}_u)^2 .$$

Having selected the optimality criterion, the design problem consists of the following decisions:

- what is the form of the estimator for the failure characteristic at the use stress?
- how many stress levels should be used?
- what levels of stress should be used?
- how many items should be allocated to each stress level?

The optimal ALT design problem can be viewed from a decision theoretic perspective and the corresponding decision tree can be presented as shown in Figure 1.

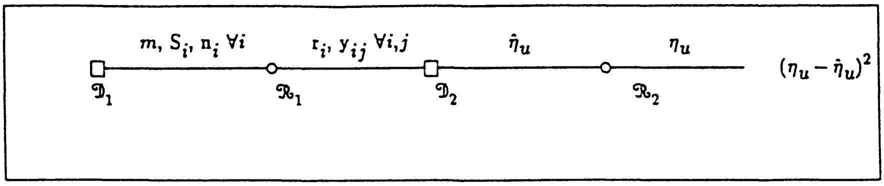

Figure 1. Decision Tree Representation of the Optimal ALT Design Problem.

In Figure 1, the number m, the values of the stress levels $S_1, \ldots, S_m$ and the numbers of items $n_1, \ldots, n_m$ to be allocated to each stress level are specified at decision node $\mathcal{D}_1$. The node $\Re_1$ is random and represents the results of ALT (i.e., the observable quantities

$$\{r_i, y_{ij}, i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n_i\}).$$

The selection of the form of the estimator $\hat{\eta}_u$ given the test information is represented by the node $\mathcal{D}_2$. Finally, the random node $\Re_2$ represents the true but unknown value of the failure characteristic η_u, and $(\eta_u - \hat{\eta}_u)^2$ denotes the realized loss.

The solution of the design problem is obtained in the conventional manner by folding back the decision tree [see, for example, Raiffa (1970, p. 23)] by taking expectations at the random nodes and minimizing the expected loss at the decision nodes. For example, at node $\mathcal{D}_2$, it is well known that the posterior mean of $\hat{\eta}_u$ minimizes the quadratic loss function so $\hat{\eta}_u = E(\eta_u \mid D_m)$. Also, it can be shown that, at node $\mathcal{D}_1$, the optimal design is obtained by minimizing the preposterior risk over all possible values of m, S_i, and n_i, that is, the optimal design is given by

(2.2)

$$\min_{\substack{m, S_i, n_i \\ \forall i}} \{E[V(\eta_u \mid D_m)]\},$$

where $V(\eta_u \mid D_m)$ denotes the posterior variance of $\hat{\eta}_u$ given the test data and the expectation is taken with respect to D_m, the data.

The above formulation of the optimal design problem is valid for any life model, time transformation function, and failure characteristic of interest at the use stress. Furthermore, the failure frequencies $\{r_i \; i = 1, \; 2, \; \ldots, \; m\}$ displayed on the branch following node $\Re_1$ of the decision tree can be either random or specified, thus reflecting various testing scenarios including testing each item until failure ($r_i = n_i \; \forall i$), testing until a specific number of failures (r_i fixed $\forall i$), and testing until a specified time (r_i random).

In what follows, we will present an approach for identifying ALT designs that are optimal with respect to the criterion of minimum quadratic loss when the lifelengths of the items on test are exponentially distributed and all items are tested until failure.

3. The exponential life model. Assuming that the jth item on test at the ith stress level is assumed to have a constant failure rate, λ_i, the failure density for the lifelength Y_{ij} is given by the exponential model

(3.1)
$$f\,(y_{ij} \mid \lambda_i, S_i) = \lambda_i e^{-\lambda_i y_{ij}},$$

where the subscript i on the failure rate, λ_i, and the lifelength, y_{ij}, indicates that these quantities are dependent on the stress level, S_i. The relationship between the failure rate and the stress level is assumed to be given by the *power law*, as is common in both biometry and reliability,

(3.2)
$$\lambda_i = \theta_1 S_i^{\theta_2},$$

where θ_1 and θ_2 are unknown, positive-valued coefficients. It is assumed that (3.2) is valid over a particular range of stress levels and that θ_1 and θ_2 are constant over the range of stress levels for which (3.2) is valid. This range is denoted by $S_u \leq S_i \leq S_H$ where S_u is the use stress and S_H is the highest stress for which (3.2) is valid but is not so high as to cause instantaneous failures.

The time transformation function, (3.2) can be linearized by taking the natural logarithms of both sides and written as

(3.3)
$$\eta_i = \log(\lambda_i) = \mathbf{F}_i^{'}\boldsymbol{\theta},$$

where $\mathbf{F}_i^{'} = (1,\ \log(S_i))$ and $\boldsymbol{\theta}' = (\log(\theta_1),\ \theta_2)$. We assume that the test designer is interested in predicting the logarithm of the failure rate at the use stress, given by

(3.4)
$$\eta_u = \mathbf{F}_u^{'}\boldsymbol{\theta},$$

where $\mathbf{F}_u^{'} = (1,\ \log(S_u))$. The first step in finding the optimal design given by (2.2) is to obtain the posterior variance of η_u,

(3.5)
$$V(\eta_u \mid D_m) = \mathbf{F}_u^{'} V(\boldsymbol{\theta} \mid D_m)\,\mathbf{F}_u.$$

Assume complete testing, that is, let $r_i = n_i$ for all i, with the data relevant to $\boldsymbol{\theta}$ being the observed lifelengths. Under the assumption of the power law and exponentially distributed lifelengths, the joint posterior distribution for θ_1 and θ_2 cannot be obtained in closed form for any reasonable joint prior distribution of θ_1 and θ_2. Consequently, the variance-covariance matrix $V(\boldsymbol{\theta} \mid D_m)$ is not directly available. However, $V(\boldsymbol{\theta} \mid D_m)$ can be obtained in an approximate manner using a sequential procedure developed by West, Harrison and Migon (1985). Henceforth, we call this procedure WHM.

The WHM procedure is based on the linear Bayesian estimation (LBE) methods of Hartigan (1969) and allows for updating of the first two moments of $\boldsymbol{\theta}$ in a sequential manner from $(\boldsymbol{\theta} \mid D_{i-1})$ to $(\boldsymbol{\theta} \mid D_i)$ for $i = 1, 2, \ldots, m$. Prior to testing at stress level S_i, the distribution of $\boldsymbol{\theta}$ is partially described by the first and second-order moments, $\mathbf{m}_{i-1}$ and $\mathbf{C}_{i-1}$, respectively, and we denote this by

(3.6)
$$(\boldsymbol{\theta} \mid D_{i-1}) \sim (\mathbf{m}_{i-1}, \mathbf{C}_{i-1}).$$

Using (3.2) then yields the first two moments of the prior distribution of η_i:

$$E(\eta_i \mid D_{i-1}) = \mathbf{F}_i^{'}\mathbf{m}_{i-1};$$

(3.7)

$$V(\eta_i \mid D_{i-1}) = \mathbf{F}_i'\mathbf{C}_{i-1}\mathbf{F}_i.$$

At this point, a full distributional form for the prior of η_i can be specified to facilitate further analysis. As pointed out by West, Harrison and Migon (1985), the form of this prior distribution is arbitrary, providing (3.7) is satisfied. Analytical results for the posterior distribution of η_i can be obtained by using the conjugate prior for η_i which, when (3.2) holds, is the log-gamma density:

(3.8)

$$p(\eta_i \mid D_{i-1}) \;\alpha\; \exp\{a_i\eta_i - b_i e^{\eta_i}\},$$

where a_i and b_i are prior parameters selected such that

$$E(\eta_i \mid D_{i-1}) = \Psi(a_i) - \log(b_i),$$

(3.9)

$$V(\eta_i \mid D_{i-1}) = \Psi'(a_i),$$

where $\Psi(\cdot)$ and $\Psi'(\cdot)$ are the *digamma* and *trigamma* functions [see Abramowitz and Stegan (1965)], respectively. The prior parameters a_i and b_i are specified such that the first two moments of η_i agree with (3.7).

After testing at stress level S_i, the posterior distribution of η_i given D_i can be obtained by a standard application of Bayes' theorem. Under the scenario of a complete test, the sufficient statistic is the total time on a test at S_i, i.e., the sum of the observed lifelengths of the n_i items on test at S_i:

(3.10)

$$T_i \equiv \sum_{j=1}^{n_i} y_{ij},$$

and by Bayes' theorem, the posterior distribution of η_i is a log-gamma density, that is,

(3.11)

$$(\eta_i \mid D_i) \sim LG(a_i + n_i, b_i + T_i).$$

It follows from (3.11) that the posterior mean and variance of η_i are given by

$$E(\eta_i \mid D_i) = \Psi(a_i + r_i) - \log(b_i + t_i);$$

(3.12)

$$V(\eta_i \mid D_i) = \Psi'(a_i + r_i).$$

Posterior conditional moments of $\boldsymbol{\theta}$, $E(\boldsymbol{\theta} \mid \eta_i, D_i)$ and $V(\boldsymbol{\theta} \mid \eta_i, D_i)$ can be obtained in an approximate manner using the LBE method of WHM. Then by using (3.7) and (3.12) with $\mathbf{s}_i \equiv \mathbf{C}_{i-1}\mathbf{F}_i$, the posterior moments of $(\boldsymbol{\theta} \mid D_i)$ can be obtained as

$$\mathbf{m}_i \equiv E(\boldsymbol{\theta} \mid D_i) = \mathbf{m}_{i-1} + \mathbf{s}_i \frac{E(\eta_i \mid D_i) - E(\eta_i \mid D_{i-1})}{V(\eta_i \mid D_i)},$$

(3.13)

$$\mathbf{C}_i \equiv V(\boldsymbol{\theta} \mid D_i) = \mathbf{C}_{i-1} - \mathbf{s}_i\mathbf{s}_i' \left\{ \frac{1 - V(\eta_i \mid D_i)/V(\eta_i \mid D_{i-1})}{V(\eta_i \mid D_i)} \right\}$$

If the entire iteration is repeated for each of the m stress levels, inference about failure characteristics at the use stress can be made by obtaining the distribution of $(\eta_u \mid D_m)$. It follows from (3.7) that

$$E(\eta_u \mid D_m) = \mathbf{F}_u'\mathbf{m}_m;$$

(3.14)

$$V(\eta_u \mid D_m) = \mathbf{F}_u'\mathbf{C}_m\mathbf{F}_u.$$

Again a full distributional form can be specified for η_u given D_m as a log-gamma density with parameters a_u and b_u chosen to satisfy $\Psi(a_u) - \log(b_u) = E(\eta_u \mid D_m)$ and $\Psi'(a_u) = V(\eta_u \mid D_m)$.

4. Identification of optimal designs. We note that the optimal design given by (2.2) requires evaluation of $E[V(\eta_u \mid D_m)]$, the expectation of the posterior variance with respect to the distribution of D_m.

Considering the form of the posterior variance of $\boldsymbol{\theta}$, given by (3.13), it is evident that $V(\eta_u \mid D_m)$ is not a function of the data. As a result the optimal design (2.2) can be obtained by minimizing $\mathbf{F}_u'\mathbf{C}_m\mathbf{F}_u$ over n_i, m, and S_i for $i = 1, 2, \ldots, m$. This poses a formidable task due to the implicit nature of the trigamma function. Furthermore, the sequential nature of the procedure results in the expression of $\mathbf{C}_i$ in terms of $\mathbf{C}_{i-1}$ being a complicated function. However, the posterior variance in (3.14) can be simplified by using an approximation to the trigamma function, namely,

(4.1)
$$\Psi'(z) \approx \frac{1}{z},$$

whose accuracy increases with z.

Using approximation (4.1), and after a considerable amount of algebra, the posterior variance in (3.14), can be rewritten as

(4.2)
$$V(\eta_u \mid D_m) = \mathbf{F}_u' \left(\mathbf{C}_0^{-1} + \mathbf{F}\mathbf{F}'\right)^{-1} \mathbf{F}_u,$$

where the first n_1 columns of $\mathbf{F}$ are $(1, \ \log(S_1))'$, the next n_2 columns are $(1, \ \log S_2)'$, and so on, with the last n_m columns being $(1, \ \log(S_m))'$. The matrix $\mathbf{F}$ is referred to as the design matrix. As a result, the posterior variance given by (4.2) is a specific case of the more general preposterior risk analyzed at length by Chaloner (1982, 1984). Using results from Chaloner (1982), it can be shown that the optimal ALT design can be concentrated at a single point which implies that all n items can be tested at one stress, S_i^*. We note that the one-point optimal design can be justified when the approximation (4.1) is accurate. However, numerical investigations by Vopatek (1992) also indicate the existence of such one-point optimal designs without using the approximation. It can be shown that using the one-point optimal design, a series of alternative optimal designs can be generated involving more than one stress levels [see Vopatek (1992)]. Alternatively, the designs can be derived in an adaptive manner, namely, by testing $n_i < n$ items at S_i^* followed by a revision of uncertainties and the specification of another one-point design at S_{i+1}^*, and the process continues in a sequential manner, where for each of m stages the optimal one-point design for fixed sample size n is found. We note that such an adaptive design strategy can be useful in situations where there exists high uncertainty about model parameters. In what follows we will present one-point optimal designs for some special situations.

In a complete test, items are tested until all fail. Considering the one-point design where $m = 1$ and using our notation, D_m is written as $D_i = \{I_i, D_0\}$ to represent the information from testing at the single stress level S_i, as well as any relevant background information. The posterior variance (or the expected loss) can be rewritten as

(4.3)

$$\begin{aligned} V(\eta_u \mid D_i) &= \mathbf{F}_u'\mathbf{C}_i\mathbf{F}_u \\ &= \mathbf{F}_u'\mathbf{C}_0\mathbf{F}_u - \left(\mathbf{F}_u'\mathbf{C}_0\mathbf{F}_i\right)^2 \left\{\frac{1 - V[\eta_i \mid D_i]/V[\eta_i \mid D_{i-1}]}{V[\eta_i \mid D_{i-1}]}\right\}, \\ &= \mathbf{F}_u'\mathbf{C}_0\mathbf{F}_u - \left(\mathbf{F}_u'\mathbf{C}_0\mathbf{F}_i\right)^2 \left\{\frac{1 - \Psi'(a_i + n)/\left(\mathbf{F}_i'\mathbf{C}_0\mathbf{F}_i\right)}{\mathbf{F}_i'\mathbf{C}_0\mathbf{F}_i}\right\}. \end{aligned}$$

One immediate observation considering (4.3) together with the fact that the trigamma function $\Psi'(a_i + n)$ is a decreasing function of its argument is that, as n increases, the expected loss decreases over all stress levels. In addition, the expected loss is not dependent on specification of $\mathbf{m}_o$, the prior mean vector for $\boldsymbol{\theta}$. Further insight into the optimal design is made possible by considering various forms of $\mathbf{C}_0$, the prior variance-covariance matrix for $\boldsymbol{\theta}$.

The special case of (3.2) when $\theta_2 = 1$ yields the linear form of the power law, that is, $\lambda_i = \theta_1 S_i$. In this case it can be shown that the prior variance of the logarithm of the failure rate is

(4.4)

$$V(\eta_u \mid D_o) = \mathbf{F}_i'\mathbf{C}_0\mathbf{F}_i = V(\log(\theta_1) \mid D_o).$$

Also, the posterior variance-covariance matrix $\mathbf{C}_i$ for $\boldsymbol{\theta}$, does not depend on S_i, and therefore it does not matter what stress level is applied.

Another special case of the power law occurs when $\lambda_i = S_i^{\theta_2}$ (i.e., $\theta_1 = 1$ in (3.2)). In this case it can be shown that

(4.5)

$$V(\eta_u \mid D_i) = \mathbf{F}_u'\mathbf{C}_i\mathbf{F}_u = V(\theta_2 \mid D_o)(\log(S_u))^2 \left(\frac{\Psi'(a_i + n)}{\Psi'(a_i)}\right).$$

The expected loss (4.5) implies that the expected loss decreases as S_i increases due to the inverse relationship between S_i and a_i. Thus, the optimal design is to test all the items at the highest possible stress, that is, $S_i^* = S_H$.

When the prior variance-covariance matrix for $\boldsymbol{\theta}$ is diagonal, i.e.,

$$\mathbf{C}_0 = \begin{bmatrix} V(\log(\theta_1) \mid D_o) & 0 \\ 0 & V(\theta_2 \mid D_o) \end{bmatrix},$$

indicating that $\log(\theta_1)$ and θ_2 are assumed to be uncorrelated prior to testing, the optimal stress level is influenced by S_u, n, and the prior variance of θ_1. Using the approximation (4.1), the expected loss given by (4.3) can be written in the form of (4.2), where

(4.6)
$$\mathbf{FF}' = n \begin{bmatrix} 1 & x_i \\ x_i & x_i^2 \end{bmatrix}$$

and $x_i = \log(S_i)$. After considerable amount of algebra, it can be shown that there is only one point that satisfies the necessary first order conditions for a local minimum, and the second derivative of the expected loss with respect to S_i is positive when evaluated at the point

(4.7)
$$S_i^* = S_u^{\left[1+[nV(\log(\theta_1)|D_o)]^{-1}\right]}.$$

We note that (4.7) implies that the optimal stress level is close to the use stress when there is a large number of items on test. Also, increased prior uncertainty about $\log(\theta_1)$, as expressed by $V(\log(\theta_1) \mid D_o)$, results in an optimal stress level near S_u. As mentioned earlier, S_i^* is not affected by the prior mean vector $\mathbf{m}_0$ for $\boldsymbol{\theta}$. Finally, the optimal stress level is not dependent on prior uncertainty about the parameter θ_2. We note that (4.7) is obtained by using the approximation (4.1), and therefore, it may be more appropriate to refer to it as an "approximately" optimal design. However, using numerical methods, Vopatek (1992) obtained optimal one-point designs very similar to those given by (4.7). Numerical methods also indicated that the location of the optimal stress moves towards S_u as n and $V(\log(\theta_1) \mid D_o)$ are increased.

Derivation of optimal designs for Type I and Type II censored ALTs as well as for other types of time transformation functions and their dynamic forms were also considered in Vopatek (1992).

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