On a Binomial Admissibility Problem
In Honor of Jack Hall on his 70th Birthday

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In this paper, X has a binomial (n,p) distribution, where n is known and p is unknown, $0 \le p \le 1$. Furthermore, let f be a given real valued continuous function on $[0,1]$. We will be interested in the question exactly when the "natural" estimator $T(X) = f(X/n)$ of $f(p)$ is admissible, always under squared loss.

1. Introduction. In this paper, X has a binomial (n,p) distribution, where n is known and p is unknown, $0 \le p \le 1$. Furthermore, let f be a given real valued continuous function on $[0,1]$. We will be interested in the question exactly when the "natural" estimator $T(X) = f(X/n)$ of $f(p)$ is admissible, always under squared loss. Special attention will be paid to the function

$$f_0(p) = \max(p, 1-p), \tag{1.1}$$

with associated estimator

$$T_o(X) = f_o(X/n) = \max(X/n, 1 - X/n). \tag{1.2}$$

It was stated by Johnson [4, p. 1586], and is easily shown, that:

(1.3)

i. If $n = 2m$ is even and $n \ge 6$ then T_o is inadmissible for f_o.

ii. If $n = 2m$ is even and $n \le 4$ then T_o is admissible for f_o.

iii. If $n = 2m+1$ is odd and $n \le 7$ then T_o is admissible for f_o.

A main contribution of this paper is the following result.

THEOREM 1. *If $n = 2m+1$ is odd and $n \ge 9$ then T_o is inadmissible as an estimator of $f_o(p)$.*

There are many other results. Some related papers are included in the list of references .

2. Auxiliary results. The following result is due to Johnson [4]. Here and below n, j, k, r and s usually denote integers.

THEOREM 2. *With $T(X)$ as a proposed estimator, of the form $T(X) = f(X/n)$, the following properties (i), (ii) are equivalent:*

i. $T(X)$ is an admissible estimator of $f(p)$ relative to squared loss.

ii. $T(X)$ admits a representation of the form

(2.1) $$T(j) = f(0) \text{ if } 0 \le j < r;\ \ T(j) = f(1) \text{ if } s < j \le n;$$

(2.2) $$T(j) = \int_0^1 f(p)p^{j-r}(1-p)^{s-j}\mu(dp) \Big/ \int_0^1 p^{j-r}(1-p)^{s-j}\mu(dp) \text{ if } r \le j \le s.$$

Here $0 \le r, s \le n$, *and* $r \le s+1$, *while* μ *is a finite nonzero measure on* $[0,1]$ *with* $\mu((0,1)) > 0$. *Equivalently,* μ *is not entirely carried by the pair* $\{0,1\}$.

COMMENTS. Johnson [4] instead used the parameters $r' = r-1$ and $s' = s+1$. Note that the denominator in (2.2) is strictly positive if $r \le j \le s$, mainly due to the condition that $\mu((0,1)) > 0$.

The case $r = s+1$ will also be called the *trivial case.* Here T is always admissible and has the simple form

(2.3) $$T(j) = f(0) \text{ for } 0 \le j \le r-1 = s;\ \ T(j) = f(1) \text{ for } s+1 \le j \le n.$$

Here r is unique unless $f(0) = f(1)$.

Also rather simple is the case $r = s$. Here, one may as well assume that $T(r) \ne f(0)$, for, if not, then T is a trivial estimator of the form (2.3), but with r replaced by $r+1$. Similarly, one may assume that $T(r) \ne f(1)$. Condition (2.2) with $j = r = s$ obviously implies that

$$\min\{f(x) : 0 \le x \le 1\} \le T(r) \le \max\{f(x) : 0 \le x \le 1\}.$$

Hence, $0 \le p_o \le 1$ exists with $f(p_o) = T(r)$. Since $T(r) \ne f(0)$, one has $p_o > 0$. Similarly, $T(r) \ne f(1)$ implies $p_o \ne 1$; thus, $0 < p_o < 1$. Now observe that (2.2) with $j = r = s$ is satisfied by the 1-point (Dirac) measure $\mu = \partial_{p_o}$. It satisfies $\mu((0,1)) = 1 > 0$.

From now on, we assume that T is not of the trivial form (2.3). Choosing r as large as possible and s as small as possible, one can always achieve that $r \le s$; $f(r) \ne f(0)$ and $f(s) \ne f(1)$. In fact, such $r = r(T); s = s(T)$ are uniquely given by

(2.4) $$r = \min\{j : T(j) \ne f(0)\};\ \ s = \max\{j : T(j) \ne f(1)\},$$

where $j \in \{0,1,..,n\}$. Always $0 \le r \le s \le n$.

LEMMA 1. *Let* T *be a nontrivial admissible estimator of* $f(p)$ *and let* r, s *be as in (2.4). Further assume that* $j_o \in \{r, r+1, ..., s\}$ *exists such that*

(2.5) $$T(j_o) = \inf f := \min\{f(p) : 0 \le p \le 1\}.$$

Then necessarily

(2.6) $$T(j) = \inf f \text{ for all } j \in \{r+1, \dots, s-1\}.$$

An analogous result holds when $T(j_o) = \sup f$ *for some* $r \le j_o \le s$.

Proof of Lemma 1. Since T is admissible and nontrivial, there exists a measure μ as in Theorem 2, with $0 \le r \le s \le n$ as in (2.4). Consider the measures η and σ on [0.1] defined by

$$\eta(dp) := cp^{j_o - r}(1-p)^{s-j_o}\mu(dp); \quad \sigma(dp) := dp^{j-r}(1-p)^{s-j}\mu(dp).$$

Here c and d are uniquely defined positive constants. From (2.2),

$$T(j_o) = \int_0^1 f(p)\eta(dp); \quad T(j) = \int_0^1 f(p)\sigma(dp).$$

Thus the given property $T(j_o) = \inf f$ is equivalent to the probability measure η being supported by the compact set

$$S = \{p : 0 \le p \le 1; \ f(p) = \inf f\},$$

in the sense that $\eta(S^c) = 0$, while the desired property $T(j) = \inf f$ is equivalent to $\sigma(S^c) = 0$. It thus suffices to verify that σ is absolutely continuous with respect to η. In more detail, one even has that

$$\frac{d\sigma}{d\eta} = \frac{d}{c}\left(\frac{p}{1-p}\right)^{j-j_o} > 0 \text{ for all } 0 < p < 1.$$

Also observe that $\sigma(\{0\}) = 0$ and $\sigma(\{1\}) = 0$. The latter follow from $r < j < s$; thus $j - r > 0$ and $s - j > 0$.

COROLLARY. Thus a nontrivial estimator T of $f(p)$, which does not have the rather strange property (2.6), nor the analogous $\sup f$ property, can be admissible only when

$$\inf f < T(j) < \sup f, \text{ for all } j = r, r+1, \ldots, s. \tag{2.7}$$

APPLICATION. Consider the special case

$$T(X) = f(X/n). \tag{2.8}$$

Superficially, this might seem to be a "natural" estimator of $f(p)$. Note that $T(0) = f(0)$ and $T(n) = f(1)$. Assume further that

$$f\left(\frac{1}{n}\right) \ne f(0); \ f\left(\frac{n-1}{n}\right) \ne f(1), \tag{2.9}$$

Then T is nontrivial with $r = 1$ and $s = n-1$, from (2.4); (even when for instance $n = 3$; $f(0) \ne f(1)$; $f\left(\frac{1}{3}\right) = f(1)$ and $f\left(\frac{2}{3}\right) = f(0)$). Lemma 1 implies that T is **inadmissible** for $f(p)$ when

(2.10)

$$f\left(\frac{j}{n}\right) = \inf f \text{ for some } 1 \le j \le n-1; \ f\left(\frac{k}{n}\right) > \inf f \text{ for some } 2 \le k \le n-2.$$

And T is also inadmissible for $f(p)$ when

(2.11)

$$f\left(\frac{j}{n}\right) = \sup f \text{ for some } 1 \le j \le n-1; \ f\left(\frac{k}{n}\right) < \sup f \text{ for some } 2 \le k \le n-2.$$

For example, the criterion (2.10) (with $j = m$ and $k = m \pm 1$) immediately yields that the estimator

$$T_o(X) = f_o(X/n) = \max(X/n, 1 - X/n) \tag{2.12}$$

is inadmissible for $f_o(p) \equiv \max(p, 1-p)$ when $n = 2m$ is even and $n \geq 6$. This confirms part (i) of assertion (1.3).

Similarly, criterion (2.11) (again with $j = m$ and $k = m \pm 1$) implies that $T(X) = X(1 - X/n)$ is inadmissible for $f(p) \equiv np(1-p)$ when $n = 2m$ is even and $n \geq 6$, as was already observed by Brown, Chow and Fong [2].

In the sequel, $T = T(X)$ is a fixed nontrivial estimator of $f(p)$, (not necessarily of the form (2.8)), while $r = r(T)$, $s = s(T)$ are defined as in (2.4). Thus, $0 \leq r \leq s \leq n$ while $T(r) \neq f(0)$ and $T(s) \neq f(1)$.

Next, consider the continuous functions $g(j,p)$ on $[0,1]$ defined by

$$g(j,p) = (f(p) - T(j))p^{j-r}(1-p)^{s-j}, (j = r, r+1, \ldots, s). \tag{2.13}$$

It is important to note that

$$g(r,0) = f(0) - T(r) \neq 0;\ g(j,0) = 0 \text{ for } r < j \leq s; \tag{2.14}$$

$$g(s,1) = f(1) - T(s) \neq 0;\ g(j,1) = 0 \text{ for } r \leq j < s.$$

THEOREM 3. *Let T be a fixed nontrivial estimator of $f(p)$ as above. Then in order that T be admissible it is necessary and sufficient that there exists a finite nonzero measure μ on $[0,1]$ satisfying the moment conditions*

$$\int g(j,p)\,\mu(dp) = 0 \text{ for } j = r, r+1, \ldots, s. \tag{2.15}$$

We further assert that such a nonzero measure μ (if it exists) automatically satisfies $\mu((0,1)) > 0$.

Proof. In view of Theorem 2, it suffices to prove the last assertion. Thus, suppose μ is a nonzero measure on $[0,1]$ satisfying (2.15). In view of a remark following Theorem 1, one may as well assume that $r < s$. Now suppose that $\mu((0,1)) = 0$; that is, μ is of the form $\mu = \alpha\delta_o + \beta\delta_1$ with $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$. Applying (2.15) with $j = r$ and using (2.14), one obtains that

$$0 = \alpha g(r,0) + \beta g(r,1) = \alpha(f(0) - T(r)), \text{ and thus } \alpha = 0.$$

Similarly, $\beta = 0$ from (2.15) with $j = s$, and thus $\alpha + \beta = 0$, a contradiction.

EXAMPLE. If $n = 4$ then the estimator $T_o(X) = f_o(X/n)$ is admissible for $f_o(p) = \max(p, 1-p)$, as follows from Theorem 3 with $\mu = \delta_o + 4\delta_{1/2} + \delta_1$. This same measure implies admissibility if $n = 3$. If $n = 2$ (thus $r = s = 1$) then admissibility of T_o for f_o leads to the equation $\int (f_o(p) - 1/2)\mu(dp) = 0$. It is satisfied by $\mu = \delta_{1/2}$, and by no other probability measure on $[0,1]$.

THEOREM 4. *Let T be a fixed nontrivial estimator of $f(p)$ as above and let $r = r(T)$, $s = s(T)$ be as in (2.4) thus $0 \leq r \leq s \leq n$. Then the following five properties (i)-(v) are equivalent.*

i. T is inadmissible.

ii. There is no measure μ on $[0,1]$ satisfying (2.15) and $\mu((0,1)) > 0$.

iii. There is no nonzero measure μ on $[0,1]$ satisfying (2.15).

iv. There exist constants $c(j)$ $(j = r, r+1, \ldots, s)$ such that

$$\sum_{j=r}^{s} c(j) g(j,p) > 0, \text{ for all } 0 \le p \le 1. \tag{2.16}$$

v. There exist constants $c(j)$ $(j = r, r+1, \ldots, s)$ such that

$$\sum_{j=r}^{s} c(j) g(j,p) > 0, \text{ for all } 0 < p < 1. \tag{2.17}$$

REMARK. The results in the present section can also be developed by starting instead of from Theorem 2, from the equivalence of the properties (i) and (iv) of Theorem 4, which can easily be proved directly.

Proof of Theorem 4. Theorem 3 essentially says that (i), (ii) and (iii) are equivalent. Since it is obvious that (iv) $\Rightarrow$ (v), it suffices to show that (v) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv).

Proof that (v) $\Rightarrow$ (ii). Let the left hand side of (2.17) be denoted by $\phi(p)$. It satisfies $\phi(p) > 0$ for $0 < p < 1$. By continuity, $\phi(0) \ge 0$ and $\phi(1) \ge 0$. Now suppose μ were a measure on $[0,1]$ satisfying (2.15) and $\mu((0,1)) > 0$. Then it follows that

$$0 = \sum_{j=r}^{s} c(j) \int_0^1 g(j,p)\, \mu(dp) = \int_0^1 \phi(p)\, \mu(dp) > 0,$$

and we have a contradiction.

Proof that (iii) $\Rightarrow$ (iv). Here we assume that there exists no probability measure μ on $[0,1]$ satisfying (2.15). In other words, (letting $N = s - r + 1$), the origin 0 in $\mathbf{R}^N$ does not belong to the convex and compact subset K of $\mathbf{R}^N$ consisting of all points $y = (y_r, y_{r+1}, \ldots, y_s) \in \mathbf{R}^N$ that admit a representation

$$y_j = \int g(j,p)\mu(dp) \text{ for } j = r, r+1, \ldots, s,$$

for some probability measure μ on $[0,1]$. For each $0 \le \mathrm{p} \le 1$, taking $\mu = \delta_p$, we see that

$$h(p) \in K, \text{ where } h(p) = (g(r,p), g(r+1,p), \ldots, g(s,p)).$$

Since $0 \notin K$, there exists a hyperplane $\sum c(j) y_j = 0$ in $\mathbf{R}^N$ (passing through 0) such that $\sum_{j=r}^{s} c(j) y_j > 0$, for all $y \in K$, and hence for each point $y = h(p)$. In other words, $\sum_{j=r}^{s} c(j) g(j,p) > 0$ for all $0 \le p \le 1$, which is precisely condition (2.16).

3. Proof of Theorem 1. In the present section, we take $f = f_o$, that is

$$f(p) = f_o(p) = \max(p, 1-p) \text{ for } 0 \le p \le 1. \tag{3.1}$$

All admissibility statements below refer to $f = f_o$. Note that

$$f(1-p) = f(p); \quad f(p_1) > f(p_2) \text{ if } 0 \le p_1 < p_2 \le 1/2. \tag{3.2}$$

We further assume that $n = 2m + 1$ is odd. Our main goal is to prove that

$$T_o(X) = f(X/n) = \max(X/n, 1 - X/n) \tag{3.3}$$

is inadmissible as soon as $n \ge 9$. Let $T = T(X)$ be any estimator of $f(p)$ satisfying

$$T(j) = T(n-j), \text{ for } j = 0, \dots, n. \tag{3.4}$$

By the way, T can be shown to be inadmissible unless $f(0) \ge T(0) \ge \dots \ge T(m) \ge f(1/2)$. For convenience, we impose the slightly stronger condition that

$$1 = f(0) = T(0) > T(1) > \dots > T(m) > f(1/2) = 1/2; \tag{3.5}$$

(as happens for $T = T_o$). Thus, (2.4) and (3.4) yield that $r(T) = 1$ and $s(T) = n-1$. Further recall from (2.13) that

$$g(j,p) := (f(p) - T(j))p^{j-1}(1-p)^{2m-j}, (j = 1, \dots, n-1), \tag{3.6}$$

where $f(p) = \max(p, 1-p)$. From (3.2) and (3.4),

$$g(n-j, 1-p) = g(j,p), \text{ for } j = 1, \dots, n-1 = 2m. \tag{3.7}$$

From part (v) of Theorem 4, T is inadmissible if and only if there exist constants $c(j)$ $(j = 1, 2, ..., 2m)$ such that

$$\phi(p) := \sum_{j=1}^{n-1} c(j)g(j,p) \text{ satisfies } \phi(p) > 0 \text{ for all } 0 < p < 1. \tag{3.8}$$

Using (3.7), one has

$$\phi(p) + \phi(1-p) = \sum_{j=1}^{n-1} (c(j) + c(n-j))g(j,p).$$

Consequently, if (3.8) is possible at all, then it can even be attained in such a way that $c(n-j) = c(j)$ for all j. In which case (3.7) implies that $\phi(1-p) = \phi(p)$. Showing (3.8) is equivalent to

$$\phi(p) := \sum_{j=1}^{m} c(j)[g(j,p) + g(n-j,p)] > 0 \text{ for all } 0 < p \le 1/2. \tag{3.9}$$

Next, consider the 1:1 map of $[0, 1/2]$ onto $[0, 1]$ defined by

$$z = p/(1-p), \text{ thus, } p = z/(1+z) \text{ and } 1 - p = 1/(1+z).$$

In view of (3.4) and (3.6), and $f(p) = 1 - p = 1/(1+z)$, (if $p \le 1/2$), the inequality (3.9) is thus equivalent to

$$\sum_{j=1}^{m} c(j)[1/(1+z) - T(j)]\frac{z^{j-1} + z^{n-j-1}}{(1+z)^{n-2}} > 0 \text{ for } 0 < z \le 1. \tag{3.10}$$

Let

$$\alpha_j = 1/T(j) - 1, \text{ and thus, } T(j) = 1/(1+\alpha_j), \ (j = 1, ..., m). \tag{3.11}$$

It follows from (3.5) that

$$0 < \alpha_1 < \alpha_2 < ... < \alpha_m < 1. \tag{3.12}$$

Multiplying (3.10) by $(1+z)^{n-1} > 0$, and letting $c(j)/T(j) = d(j)$, we see that (3.10) in turn is equivalent to

$$\sum_{j=1}^{m} d(j)[\alpha_j - z](1 + z^{2(m-j)+1})z^{j-1} > 0 \text{ for } 0 < z \le 1. \tag{3.13}$$

In this way, we have proved the following result. Recall that always $n = 2m+1$ is odd while $T = T(X)$ is an estimator satisfying (3.4) and (3.5).

THEOREM 5. *In order that the estimator T be inadmissible (for $f = f_o$ and squared loss), it is necessary and sufficient that (3.13) holds for at least one choice of the constants $d(j)$, $(j = 1, .., m)$.*

COROLLARY. If $n = 5$ (thus $m = 2$) then T is always admissible for f_o.

Proof of Corollary. Let $m = 2$. It suffices to prove the impossibility of (3.13), which now is of the form

$$\psi(z) := c[\alpha_1 - z](1+z^3) + d[\alpha_2 - z](1+z)z > 0, \text{ for } 0 < z \le 1. \tag{3.14}$$

Here c and d are constants and $0 < \alpha_1 < \alpha_2 < 1$. It is impossible that $c = 0$ since the second term changes sign at $z = \alpha_2$. The first term dominates for z small, hence, $c > 0$. In order that $\psi(z) > 0$ for $\alpha_1 < z < \alpha_2$ we need that $d > 0$. But then $\psi(z) < 0$ for $\alpha_2 < z < 1$.

From now on, let $m \ge 3$; that is, $n \ge 7$.

DEFINITION. We will say that the estimator T is strongly inadmissible (for f_o) if and only if there exists constants d_1, d_2, d_3 such that (3.13) holds with

$$d(j) = 0 \text{ for } 1 \le j \le m-3, \tag{3.15}$$

and

$$d(m-2) = d_1; \ d(m-1) = d_2; \ d(m) = d_3. \tag{3.16}$$

If such constants d_1, d_2, d_3 do not exist then we will say that T is weakly admissible.

COMMENTS. One can show that weak admissibility of T is equivalent to ordinary admissibility of T relative to the new risk function

$$R_J(T,p) = \sum_{j \in J} \binom{n}{j} p^j (1-p)^{n-j} (T(j) - f(p))^2,$$

where $n = 2m + 1$ and $J = \{j : m - 2 \leq j \leq m + 3\}$. Equivalently, no better (in the ordinary sense) estimator T' exists with $T'(j) = T(j)$ for all $j \notin J$.

Note that the above admissibility properties depend only on the three values α_{m-2}, α_{m-1} and α_m, that is, the values $T(m-2)$, $T(m-1)$ and $T(m)$. In view of Theorem 5, strong inadmissibility is a sufficient condition for inadmissibility. Equivalently, weak admissibility is a necessary condition for admissibility.

Below, we will show, among other things, that T_o is strongly inadmissible (and hence inadmissible) if $m \geq 4$, that is, $n \geq 9$, thus proving Theorem 1. As a byproduct, the proof also shows that T_o is weakly admissible if $m = 3$. Since then the restriction (3.15) is void, this proves that T_o is admissible if $n = 7$. As was already asserted by Johnson [4].

To simplify the notation, we introduce

(3.17)

$$a = \alpha_{m-2} = 1/T(m-2) - 1; \; b = \alpha_{m-1} = 1/T(m-1) - 1; \; c = \alpha_m = 1/T(m) - 1.$$

Thus a, b, c depend on T and are such that $0 < a < b < c < 1$, in view of (3.5). In the spcial case $T = T_o$ these become

$$a = a_m = (m-2)/(m+3); b = b_m = (m-1)/(m+2); c = c_m = m/(m+1). \tag{3.18}$$

Namely, $T_o(j) = \max(j/n, 1 - j/n) = n - j/n$ if $0 \leq j \leq m$. A central role will be played by the polynomials

$$\begin{aligned} h_1(z) &= h_1(a,z) = (a-z)(1 - z + z^2 - z^3 + z^4); \\ h_2(z) &= h_2(b,z) = (b-z)(1 - z + z^2)z; \\ h_3(z) &= h_3(c,z) = (c-z)z^2, \end{aligned} \tag{3.19}$$

of degree 5, 4 and 3, respectively. They depend on a, b or c, respectively, and thus on T. Observe that $h_1(z) = h_1(a,z)$ changes sign at $z = a$ and similarly for $h_2(z)$ and $h_3(z)$.

First dividing (3.13) by $(1+z)z^{m-3} > 0$ (when $0 < z \leq 1$) and using (3.15), (3.16), (3.17), (3.19), one sees that T is strongly inadmissible if and only there exists constants d_1, d_2, d_3 such that

$$d_1 h_1(z) + d_2 h_2(z) + d_3 h_3(z) > 0, \text{ for all } 0 < z \leq 1. \tag{3.20}$$

Here, for $z > 0$ small, the first term $d_1 h_1(z)$ dominates while $h_1(z) > 0$, showing that necessarily $d_1 > 0$. We may and will assume that $d_1 = 1$. Also note that the LHS of (3.20) has at $z = 0$ the value $h_1(0) = a > 0$. Letting $s = -d_2$ and $t = d_3$, condition (3.20), for some constants s and t, is thus equivalent to the validity of

$$\phi(z) := h_1(z) - s h_2(z) + t h_3(z) > 0 \text{ for all } 0 \leq z \leq 1, \tag{3.21}$$

for some choice of s and t. Which in turn is equivalent to the estimator T being strongly inadmissible. By the way, (though we will not need this), it is easily seen that (3.21) can only hold when both $s > 0$ and $t > 0$.

LEMMA 2. *The estimator T is weakly admissible if and only if there exists a probability measure μ on $[0,1]$ such that*

$$\int_0^1 h_i(z)\mu(dz) = 0, \; \textit{for } i = 1,2,3. \tag{3.22}$$

Proof. If such a measure μ does exist then (3.21) is impossible thus T must be weakly admissible. Conversely, if μ does not exist then in the usual way, (see the last part of the proof of Theorem 4), one shows that (3.21) must hold for some choice of s and t, equivalently, T is strongly inadmissible.

In the sequel, we regard b and c as being fixed, $0 < b < c < 1$. In view of (3.17), this amounts to fixing the values $T(m-1)$ and $T(m)$. While the parameter a will be treated as a variable with $0 < a < b$; (equivalently $T(m-1) < T(m-2) < 1$).

Let $L = L(b,c)$ be the set of all values a, $0 < a < b$, for which T is weakly admissible. By Lemma 2, L can also be defined as the set of values $0 < a < b$, such that there exists a probability measure μ on $[0,1]$ satisfying (3.22), (where h_1 depends on a).

Let $R = R(b,c)$ denote the complement of $L = L(b,c)$ relative to the interval $(0,b)$. That is, R is the set of all values $0 < a < b$ for which T is strongly inadmissible. Equivalently, R is the set of values $0 < a < b$ such that (3.21) holds for at least one choice of the constants s and t. Since the continuous function $f(z) > 0$ on $[0,1]$ as in (3.21) is bounded away from 0, it follows easily that the set R is open.

In (3.21) only the term $h_1(z) = h_1(a,z)$ depends on the parameter a. By (3.19), $h_1(a,z)$ is itself a strictly increasing function of a, hence, so is the RHS of (3.21). Therefore, $0 < a < a' < b$ and $a \in R$ imply that $a' \in R$. Hence, $R = R(b,c)$ is an interval of the form $R = (a^*, b)$, with $a^* = a^*(b,c)$ such that $0 \le a^* \le b$. Accordingly, the complement $L = L(b,c)$ of R is a left interval of the form $L = (0, a^*]$. In this way we have proved the following.

LEMMA 3. *In order that T be strongly inadmissible (for f_o), it is necessary and sufficient that $a^*(b,c) < a < b$. Thus T is weakly admissible if and only if $0 < a \le a^*(b,c)$. Here a, b and c depend on T as in (3.17).*

The following Theorem 6 yields a rather explicit formula for the boundary value $a^*(b,c)$. In view of Lemma 3, that formula supplies us with a rather explicit test towards determining whether or not a given estimator T (satisfying (3.4) and (3.5)) is strongly inadmissible. The proof of Theorem 6 is based on Lemmas 4 and 5 below.

THEOREM 6. *For each choice of the numbers $0 < b < c < 1$, let*

$$z_o = z_o(b,c) = \frac{1 - bc - \sqrt{(1-bc)^2 - 4b(1-c)^2)}}{2(1-c)}. \tag{3.23}$$

and further

$$a_o = a_o(b,c) = \frac{cz_o^2 + (1-c)(1-z_o^3)z_o}{z_o^2 + (1-c)(1-z_o^3)}. \tag{3.24}$$

We assert that

$$0 < z_o < a_o < b < c < 1 \tag{3.25}$$

and further that

$$a^*(b,c) = a_o(b,c). \tag{3.26}$$

LEMMA 4. *Let $0 < b < c < 1$ be fixed. We claim that there is precisely one value $0 < a < b$, such that there exists a probability measure μ of the special form $\mu = \alpha\delta_\zeta + \beta\delta_1$ satisfying $0 < \zeta < 1$ and*

$$\int h_i(z)\mu(dz) = \alpha h_i(z) + \beta h_i(1) = 0, \text{ for } i = 1, 2, 3. \tag{3.27}$$

In fact, necessarily $a = a_o$ while μ is unique with $\zeta = z_o$. Here z_o and a_o are defined as in (3.23) and (3.24), and do satisfy (3.25).

COROLLARY. In view of Lemma 2 and the existence of the above special measure μ, it follows that for the special choice $a = a_o(b, c)$, the estimator T is weakly admissible. Equivalently, $a_o = a_o(b, c)$ belongs to $L(b, c) = (0, a^*(b, c)]$. In other words,

$$a^*(b, c) \geq a_o(b, c) > 0. \tag{3.29}$$

Proof of Lemma 4. Observe, from (3.19), that $h_i(1) < 0$ for all i and that h_1, h_2, h_3 have no common zero. Hence, (3.27) can only hold with $\alpha \neq 0$ and $\beta \neq 0$ in such a way that $\rho = h_i(z)/h_i(1)$ is independent of i. In fact, $\rho = \beta/\alpha$. The latter independence is equivalent to

$$h_2(\zeta)h_3(1) - h_3(\zeta)h_2(1) = 0; \ h_1(\zeta)h_3(1) - h_3(\zeta)h_1(1) = 0. \tag{3.30}$$

We will see that (3.30) is only possible when $\zeta = z_o$ and $a = a_o$, in which case $0 < \zeta < a$ and thus $\rho > 0$. Afterwards, one may as well assume that $\alpha > 0$, $\beta > 0$, and $\alpha + \beta = 1$, yielding the desired probability measure μ.

In view of these comments, we only need to satisfy condition (3.30). The first equality (3.30), divided by $\zeta(1 - \zeta)$, immediately leads to the quadratic equation $\psi(\zeta) = 0$, where

$$\psi(\zeta) := (1 - c)\zeta^2 - (1 - bc)\zeta + b(1 - c). \tag{3.31}$$

Since $\psi(0) > 0$, $\psi(b) = b(b - c) < 0$, and $\psi(1) = b - c < 0$, the two zeros ζ_1 and ζ_2 are such that $0 < \zeta_1 < b < 1 < \zeta_2$. In fact, $\zeta_1 = z_o$ with z_o as in (3.23). Note that $0 < z_o < b$; thus $0 < z_o < b < c < 1$.

From (3.19), the second equation (3.30) is linear in a. Taking $\zeta = z_o$, and solving for the parameter a, we find that necessarily $a = a_o$ with a_o exactly as in (3.24). Since $z_o < c$, we see from (3.24) that $z_o < a_o < c$.

In proving that $a_o < b$, we employ the equation

$$h_1(a_o, z_o)h_2(1) - h_2(z_o)h_1(a_o, 1) = 0,$$

which is linear in a_o, and is a consequence of (3.30). Solving for a_o, one finds that

$$a_o = b - \frac{(1 - b)(b - z_o)(1 - z_o^4)}{1 + z_o - b(1 - z_o^4)}, \tag{3.32}$$

clearly showing that $a_o < b$. The formulae (3.24) and (3.32) are naturally equivalent, because of (3.23). This completes the proof of Lemma 4.

LEMMA 5. *Let $0 < b < c < 1$ be fixed. Then there exist constants B, C, s, and t, such that, for all $\leq z \leq 1$,*

$$h_1(a_o, z) - sh_2(z) + th_3(z) \equiv (1-z)(z-z_o)^2(z^2 - Bz + C) \geq 0. \tag{3.33}$$

Here, $z_o = z_o(b,c)$ and $a_o = a_o(b,c)$ are as in (3.23), (3.24).

Proof of Lemma 5. Given $0 < b < c < 1$, let $\mu = \alpha\delta_{z_o} + \beta\delta_1$ be the unique probability measure described in Lemma 4; (it depends on b and c). Let further $\mathcal{H} = \mathcal{H}(b,c)$ denote the linear space consisting of all polynomials $h(z)$ of degree ≤ 5 that satisfy

$$\int h(z)\mu(dz) = \alpha h(z_o) + \beta h(1) = 0. \tag{3.34}$$

Clearly, $\dim(\mathcal{H}) = 5$. From Lemma 4, we know that h_1^o, h_2, $h_3 \in \mathcal{H}$, where $h_1^o(z) := h_1(a_o, z)$. Since (3.34) is true for instance if $h(z_o) = h(1) = 0$; also H_1, H_2, $H_3 \in \mathcal{H}$ where $H_i(z) = z^{3-i}(1-z)(z-z_o)^2$ $(i = 1,2,3)$. Hence, the six functions h_1^o, h_2, h_3, H_1, H_2, H_3 must be linearly dependent. In other words, there exist constants r, s, t, A, B, C not all zero such that

$$rh_1(a_o, z) - sh_2(z) + th_3(z) \equiv AH_1(z) - BH_2(z) + CH_3(z).$$

In more detail, we have the identity

$$r(a_o - z)(1 - z + z^2 - z^3 + z^4) - s(b-z)(1-z+z^2)z + t(c-z)z^2 \tag{3.35}$$

$$= (1-z)(z-z_o)^2(Az^2 - Bz + C), \text{ for all } z.$$

Comparing the coefficients of z^5, we see that $A = r$. We claim that $r \neq 0$.

For, suppose $r = 0$ and thus $A = 0$. Afterwards, taking $z = 0$, (3.35) yields that $0 = Cz_o^2$, and hence C $= 0$. The coefficients of z^4 and z yield next that

$$s = B \text{ and } -sb = -z_o^2 B; \text{ thus, } (b - z_o^2)B = 0.$$

Hence, $B = 0$ and thus $s = 0$. Finally, $t(c-z)z^2 \equiv 0$; thus $t = 0$ and we have a contradiction. Here and below we also use that $0 < z_o < a_o < b < c < 1$.

Knowing that $r \neq 0$, we may as well take $r = 1$; thus $A = 1$, so that (3.35) becomes the identity (3.33). It only remains to show that

$$\psi(z) := z^2 - Bz + C \geq 0, \text{ for all } 0 \leq z \leq 1. \tag{3.36}$$

Taking $z = 0$ in (3.35) (with $r = A = 1$) yields $a_o = Cz_o^2$; thus $C = a_o/z_o^2 > 1$. Hence, (3.36) is obvious when $B \leq 0$ and also when $B^2 \leq 4C$. It remains to consider the case that $B > \sqrt{4C} > 2$. Clearly $\psi(z)$ decreases for $z \leq B/2$ and hence for $z \leq 1$. It thus suffices to prove that $\psi(1) > 0$, or equivalently, that

$$B < 1 + C, \text{ that is, } B < 1 + a_o/z_o^2. \tag{3.37}$$

To prove this we need the formula

$$B = \frac{2a_o/z_o + a_o b - 2bz_o - 1}{b - z_o^2}. \tag{3.38}$$

For the moment, let us assume (3.38). Substituting (3.38), the desired inequality (3.37) takes the form

$$(2z_o + z_o^2 + bz_o^2 - b)a_o < bz_o^2 + 2bz_o^3 + z_o^2 - z_o^4, \tag{3.39}$$

as is obvious when the coefficient of a_o is nonpositive. Assuming that this coefficient is positive, and recalling that $a_o < b$, it suffices to prove the inequality obtained from (3.39) by replacing a_o by b. But then

$$\text{RHS} - \text{LHS} = (1 - z_o^2)(b - z_o)^2 > 0$$

and we are ready.

It only remains to prove (3.38). We will use the identity (3.35) where $r = A = 1$ and $C = a_o/z_o^2$. Taking the coefficients of z and z^4, one arrives at the equations

$$z_o(2C + Bz_o + Cz_o) = 1 + a_o + s; \; 1 + B + 2z_o = 1 + a_o + s.$$

Eliminating s and solving for B, one obtains (3.38).

Proof of Theorem 6. We must prove (3.26). In view of the Corollary following Lemma 4, it only remains to show that

$$a^*(b,c) \leq a_o(b,c). \tag{3.40}$$

Let $a_o = a_o(b,c)$. Since $h_1(a,z)$ is strictly increasing in a (see (3.19)), it follows from Lemma 5 that there exist constants s and t such that

$$h_1(a,z) - sh_2(z) + th_3(z) > 0 \text{ when } a_o < a < b \text{ and } 0 \leq z \leq 1.$$

It follows from criterion (3.20) that T is strongly inadmissible for all $a \in (a_o, b)$. In other words, (a_o, b) is a subset of (a^*, b), proving (3.40).

Proof of Theorem 1. Here, we restrict ourselves to the special estimator $T_o(X) = \max(X/n, 1 - X/n)$ with $n = 2m + 1$ odd. From (3.18), the associated parameters a, b, c are now given by

$$a_m = \frac{m-2}{m+3}; \; b_m = \frac{m-1}{m+2}; \; c_m = \frac{m}{m+1}. \tag{3.41}$$

Let further

$$a_m^* := a_o(b_m, c_m) = a_o(\frac{m-1}{m+2}, \frac{m}{m+1}). \tag{3.42}$$

Here the function $a_o(b,c)$ is defined by (3.23) and (3.24). It follows from Theorem 6, that T_o is strongly inadmissible (and thus inadmissible) if and only if

$$a_m > a_m^*. \tag{3.43}$$

We will show that (3.43) is true for all $m \geq 4$. The validity of (3.43) for small values $m \geq 4$ is obvious from Table 1. If $m \to \infty$ then $z_o \to 2 - \sqrt{3} = 0.26794919$. Since $a_3 < a_3^*$, we know that for $n = 7$ the estimator T_o is weakly admissible and thus admissible. See the remark preceding (3.17). Similarly, $a_m > a_m^*$ for $4 \leq m \leq 10$ implies that T_o is strongly inadmissible (and thus inadmissible) if n is odd, $9 \leq n \leq 21$.

Table 1: Numerical Assessment of the Validity of (3.43)

m	n	z_o	a_m^*	a_m
3	7	0.15100	0.20122	0.16667
4	9	0.17712	0.26199	0.28571
5	11	0.19376	0.31205	0.37500
6	13	0.20527	0.35475	0.44444
7	15	0.21370	0.39193	0.50000
8	17	0.22014	0.42475	0.54546
9	19	0.22521	0.45402	0.58333
10	21	0.22931	0.48033	0.61539
15	31	0.24184	0.58061	0.72222
20	41	0.24824	0.64811	0.78261
30	61	0.25472	0.73360	0.84849
100	201	0.26395	0.90120	0.95146
1000	2001	0.26755	0.98912	0.99502

In the general case $T = T_o$, employing (3.23), (3.24) and (3.42), together with some tedious but straightforward calculations, one arrives at the explicit formula

$$a_m^* = [P(m) + Q(m)\sqrt{3(1+m+m^2)}]/R(m), \tag{3.44}$$

where

$$P(m) = -21 - 74m - 135m^2 - 121m^3 - 58m^4 + 3m^5 + m^6;$$

$$Q(m) = (2m+1)(11 + 17m + 17m^2); \tag{3.45}$$

$$R(m) = (m+2)(3 + 5m^2 + 6m^3 + 12m^4 + m^5).$$

Because of (3.43), we are mainly interested in the sign of $\Delta(m) = a_m - a_m^*$. Eliminating the square root, (and possibly dividing by $R(m)$), one easily sees that $\Delta(m)$ has everywhere the same sign as the polynomial

$$S(m) = -37 - 46m - 43m^2 + 6m^3 + 3m^4. \tag{3.46}$$

Since $S(m)/m^3$ is strictly increasing, for $m > 0$, $S(m)$ has a single zero $m_o > 0$; in fact, $m_o = 3.54344$. Consequently,

$$a_m > a_m^* \text{ if } m > m_o; \quad a_m < a_m^* \text{ if } m < m_o. \tag{3.47}$$

Implying that T_o is strongly inadmissible (and thus inadmissible) for all odd $n = 2m+1$ with $m \geq 4$; that is, $n \geq 9$. This proves Theorem 1. By the way, the inadmissibility of T_o for all sufficiently large m already follows from $a_m^* = 1 - 11/m + O(1/m^2)$ and $a_m = \frac{m-2}{m+3} = 1 - 5/m + O(1/m^2)$.

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REFERENCES

[1] L. D. Brown. A complete class theorem for statistical problems with finite sample spaces. *Annals of Statistics*, 9:1289–1300, 1981.

[2] L. D Brown, Mosuk Chow, and Duncank. H. Fong. On the admissibility of the maximumlikelihood estimator of the binomial variance. *Canadian Journal of Statistics*, 20:353–358, 1992.

[3] J. T. Hwang. Improving upon standard estimators in discrete exponential families with applications to poisson and negative binomial cases. *Annals of Statistics*, 10:857–867, 1982.

[4] B. McK. Johnson. On the admissible estimators for certain fixed sample binomial problems. *Annals of Mathematical Statistics*, 42:1579–1587, 1971.

[5] A. Kozek. Toward a calculus for admissibility. *Annals of Statistics*, 10:825–837, 1982.

[6] G. Meeden and M. Ghosh. Admissibility in finite problems. *Annals of Statistics*, 9:846–852, 1981.

[7] F. C. Y. Suan. A stepwise bayesian procedure. *Annals of Statistics*, 7:860–868, 1979.

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