ACCARDI CONTRA BELL (CUM MUNDI): THE IMPOSSIBLE COUPLING

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WITH AN APPENDIX BY

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An experimentally observed violation of Bell's inequality is supposed to show the failure of local realism to deal with quantum reality. However, finite statistics and the time sequential nature of real experiments still allows a loophole for local realism. We show that the randomised design of the Aspect experiment closes this loophole. Our main tool is van de Geer's (1995, 2000) martingale version of the classical Bernstein (1924) inequality guaranteeing, at the root n scale, a not-heavier-than-Gaussian tail of the distribution of a sum of bounded supermartingale differences. The results are used to specify a protocol for a public bet between the author and L. Accardi, who in recent papers (Accardi and Regoli, 2000a,b, 2001; Accardi, Imafuku and Regoli, 2002) has claimed to have produced a suite of computer programmes, to be run on a network of computers, which will simulate a violation of Bell's inequalites. At a sample size of twenty five thousand, both error probabilities are guaranteed smaller than about one in a million, provided we adhere to the sequential randomized design while Accardi aims for the greatest possible violation allowed by quantum mechanics.

1. Introduction

This paper is concerned with a celebrated paradox of quantum mechanics. Some keywords and phrases are locality, causality, counterfactuals, EPR (Einstein, Podolsky and Rosen, 1935) correlations, the singlet state, entanglement, Bell's (1964) inequalities, and the Aspect experiment (Aspect et al., 1982a,b). However the point of the paper is that almost the whole story can be told in terms of elementary classical probability and statistics. The only physics you should believe, is that the right mathematical model for the periodic, smooth, dependence of a certain correlation coefficient on a certain angle is given by the appropriate sine curve. It seems to me that this little example should be in every probability and statistics course as showing the power of probabilistic reasoning and the importance of statistics in modern day science (it is for instance in Williams, 2001, Chapter 10). Moreover, there is growing realisation that quantum physicists are up to interesting things these days (quantum information, quantum computation, quantum communication), and growing realisation that these things involve probability and

potentially statistics, and that we should get involved too. So why not take this as an aperatif, before consulting say Barndorff-Nielsen, Gill and Jupp (2001) or Gill (2001b) for a survey and a tutorial respectively, on quantum statistical inference: statistical inference for data coming from quantum experiments. Gill (2001a)—in another Festschrift—even introduces quantum asymptotic statistics.

The rest of the paper is structured as follows. Section 2 introduces the ongoing controversy around the application of Bell's (1964) inequality to quantum mechanics. The inequality is the elementary

$$(1.1) \qquad P\{X_1 = Y_2\} \leq P\{X_1 = Y_1\} + P\{X_2 = Y_1\} + P\{X_2 = Y_2\}$$

concerning coincidence probabilities between four 0/1-valued random variables. Its proof is postponed to Section 4. Though the inequality itself is trivial, the question of whether or not it should be applicable to certain real-world experiments is more subtle, and therein lies the controversy. The interesting fact is that the inequality is *apparently* violated by experimentally confirmed predictions of quantum mechanics.

In Section 3 we describe the celebrated Aspect experiment, which first confirmed the violation of Bell's inequality, predicted by Bell himself almost twenty years earlier. In each of a long sequence of runs or trials, a pair of photons are emitted from a source O and sent to two widely separated polarization filters X, Y. In each trial, each filter has one of two possible orientations (labelled 1, 2, supplied by independent agents A and B). Each photon either passes or does not pass its filter. We encode this with a 1 or a 0. We set down some notation and describe the empirical finding of the experiments, concerning the frequencies of various possible outcomes. We shall work with absolute frequencies rather than relative frequencies or empirical correlations. This will lead to a clean mathematical analysis without changing the conclusions.

Accardi and Regoli (2000a,b, 2001) and Accardi, Imafuku and Regoli (2002) claim to be able to reproduce these frequencies, replacing the source of the photons and the two polarization filters by three computers. A software package can be downloaded from `http://volterra.mat.uniroma2.it`, though in my opinion it does not respect the rules of the game, in particular, that outcomes are 0/1 valued, revealed at the proper moments, and there is no missing data. The author has publicly challenged Accardi to violate Bell's inequality in an Aspect-style experiment with a version of this software which verifiably satisfies these rules. In particular, the outcomes for each trial must be committed to, before each new trial. The challenge is provisionally accepted subject to finalising details of the protocol. The bet has been fixed at 3000 Euros. The bet will be settled by an independent jury who are only asked to verify the one-way connections between the com-

puters, and to observe if the empirical correlations violate Bell's inequality by a pre-agreed margin. The results of this paper allow the author to determine a protocol which will be acceptable for him. At the time of writing negotiations look set to continue indefinitely, Accardi having stated that my protocol is "perhaps mathematically interesting but physically irrelevant."

In the mean time, there have been more challenges to Bell (1964) in which an attempt is made to exploit time dependence and memory effects; see Hess and Philipp (2001a,b). I was unable to interest Walter Philipp in a bet: "our results are mathematically proven and a computer simulation is unnecessary." The present paper provides another mathematically proved theorem, which contradicts their results, see Gill et al. (2002), and see Barrett et al. (2002) for a related analysis of the potential memory loophole to Bell's theorem.

In Section 4 we prove (1.1). In Section 5 we write down the probabilities of the events of interest in a version of the Aspect experiment, which follow from a sine law and certain choices of experimental settings. It is not a priori clear that the set-up of Bell's inequality should apply in the Aspect experiment, but if it did, the results predicted by quantum mechanics and observed in the physics laboratory would be impossible. In Section 6 we argue why Bell's inequality *should* apply to Accardi's experiment, with everyday computers connected so as to mimic the possible communication lines between the calcium atom and the polarization filters. Since the behaviour of photons at distant filters cannot be simulated with classical computers, connected so as to respect the separation between the filters, it follows that quantum mechanics does make extraordinary predictions, namely, it predicts phenomena which for classical physical systems are impossible.

In order to test inequalities between expected values, one will in practice compute averages, and must take account of statistical variability of the outcome. Now quantum mechanics predicts the same results whether one does one trial in each of thousands of laboratories, or does thousands of trials, sequentially, in one laboratory. In the former case one might be prepared to assume independence from one trial to another, but in the latter case, it is harder to rule out. In the case of a computer network simulation, in which the software has been written by an opponent, one cannot rule out anything at all. In Section 7 we show, using the martingale Bernstein inequality of van de Geer (1995, 2000), see also Dzhaparidze and van Zanten (2001), that this does not provide a loophole for the Accardi experiment. Twenty five thousand trials carried out according to a simple protocol are sufficient that both Gill's and Accardi's error probabilities are much smaller than one in a million.

Recent research by this author has shown that the Hoeffding (1963) inequality, see Bentkus (2002) for the latest improvements, gives even better results, and this will be reported in a future paper.

Section 8 contains some closing remarks and further references.

2. Accardi contra Bell

Quantum mechanics makes statistical (or if you prefer, probabilistic) predictions about the world. Some of the strangest are connected to the phenomenon of *entanglement*, whereby two quite separate quantum systems (for instance, two distant particles) behave in a coordinated way which cannot be explained classically. Despite the fact that these properties are well known and experimentally verified (for instance, see Tittel et al. (1998), with pairs of photons passing below Lake Geneva through Swiss Telecom's glass-fibre cable network, between locations 10 Km distant from one another) controversy still surrounds them.

A popular explanation of entanglement runs something like this. "Paint one ping-pong ball red, another blue; put them in closed boxes and send them randomly to two distant locations. Before the boxes are opened either box could contain either ball. If one box is opened and turns out to contain a red ball, then far away and instantaneously, the state of the other box suddenly changes: it contains a blue ball." This is what Reinhard Werner calls the ping-pong ball test: to judge any popular explanation of some quantum mechanical paradox, replace the objects in the story by ping-pong balls, and check if it makes sense. Well, this ping-pong story does make sense, but misses the point. The behaviour we are trying to explain is a bit more complex (too complex for newspaper articles, but not too complex for mathematical statisticians). I will describe it precisely in the next section. Quantum mechanics would not have caused scientists of the calibre of Schrödinger, Bohr, and Einstein such intellectual discomfort if it were this easy to explain entanglement. The whole point which Bell was trying to make with his inequalities is that the dependence in the behaviour of distant but entangled particles is contradictory to 'local realism'. Loosely speaking, this phrase means a classical (though possibly probabilistic) explanation of the correlation in the behaviour of such particles, through their carrying information from the place where they were generated or 'born' to the places where they are measured or observed. In other words, a story like the ping-pong story will *not* explain it.

Repeatedly, elaborate and exotic theories have been put forward to explain away the problem. Non-measurable events (Pitowsky, 1989), *p*-adic probabilities (Khrennikov, 1995a,b, 1997, 1998, 1999), and most recently, the chameleon effect (Accardi et al., 2000a,b, 2001, 2002) have all been tried. In the mean time much of the physics community ignores the controversy, and

many have misunderstood or minimalised Bell's contribution, which goes back, via Bohm, to a celebrated thought experiment of Einstein, Podolsky and Rosen (1935). To give a local example, Nobel prize-winner G. 't Hooft learnt from his uncle N. van Kampen, a staunch adherant of the Copenhagen interpretation, that Bell's inequalities were not worth much attention, since they are derived by consideration of what would have happened if a different experiment had been performed, which according to Bohr's Copenhagen school is taboo. Counterfactuals have a bad name in quantum physics. Consequently 't Hooft (1999) was at first unaware, that a deterministic and classical hidden layer behind quantum mechanics—such as the one he is attempting to develop himself—is forced to be grossly non-local. He now has the onerous task of explaining why it is that, although every part of the universe is connected with invisible and instantaneous wiring to every other part, reality as we know it has that familiar 'local' look.

To return to the exotic explanations, Accardi in a number of papers has strongly argued that the randomness in quantum mechanics is not the randomness of urns, but of chameleons. By this he means that in classical probability, with the paradigm being choosing a ball out of an urn containing balls of different colours, the values of variables on the different outcomes are fixed in advance. A ball in an urn already has a particular colour, and this colour is not influenced by taking the ball out of the urn and looking at it. However the colour of a chameleon, let loose out of its cage, depends on its environment. Moreover if there is a chance that the chameleon is mutant, we will not be able to predict in advance what colour we will see. His image of the Aspect experiment has a pair of chameleons, one mutant and one normal, instead of the pair of ping-pong balls. There is some value in this imagery. Bell's findings reinforce Bohr's philosophy, that in quantum mechanics one should not think of the values of physical quantities as being fixed in advance of measurement, and independently of the total experimental set-up used to elicit the outcomes. However, in my opinion, if chameleons are to be thought of as classical physical objects (they may be mutant but not telepathic) it will not be possible to simulate quantum systems with them. But Accardi et al. (2000b, 2001, 2002) claim that they have simulated Accardi's chameleons on a network of PCs. The programme can be downloaded from `http://volterra.mat.uniroma2.it`. I have much respect for Accardi's many solid and deep contributions to quantum probability and quantum physics. On the other hand I cannot find fault with Bell's argument. I have therefore bet Luigi Accardi 1000 Euro (raised at his request to 3000 in view of the more stringent programming requirements which I have put down) that he cannot violate Bell's inequalities, in an experimental setup to be outlined below. Preparation of this bet required me to take a new look at the inequalities and in particular to study the effect of possible

time dependence in repeated trials. Most mathematical treatments consider
one trial and then invoke the law of large numbers and the central limit the-
orem, assuming independence. Now, quantum mechanics makes the same
predictions when one independently carries out one trial each in many lab-
oratories over the world, as when one makes many trials sequentially at one
location. Actual experiments, in particular Accardi's computer experiment,
are done sequentially in time. In order to show that sequentially designed
classical experiments (in particular, using computers or chameleons) cannot
simulate quantum systems, we are not able to assume independence. It will
become clear that it is essential that the experiment is randomised and the
randomization is disclosed sequentially, with the outcomes of the trials also
being disclosed sequentially, in step. We will see that a martingale structure
will prevent the computers from taking advantage of information gathered
in past trials. Put another way, the separation in time of consecutive trials
will play a similar role to the separation in space which is already central to
Bell's inequality.

3. The Aspect experiment

In an experiment carried out in Orsay, Paris, in 1982 by Alain Aspect and
his coworkers, a calcium atom O is excited by a lazer, and then returns
to its unexcited state by emitting a pair of photons in equal and opposite
directions. The photons always have equal polarization (in some versions
of the experiment, *opposite* rather than *equal*). In fact, their joint state of
polarization is a so-called quantum entangled state having rather remarkable
properties, as we will see. This is repeated many times (and there are many
calcium atoms involved), producing a long sequence of n pairs of photons.
We will refer to the elements of this sequence as 'trials.'

Each pair of photons speed apart until intercepted by a pair of polariza-
tion filters X and Y, at two locations several meters apart in the laboratory.
We will call these locations 'left' and 'right.' The orientations of the polar-
ization filters can be set, independently at the two locations, in any desired
direction. Aspect wanted that at each location a series of independent ran-
dom choices between two particular directions was made, independently at
the two locations, and each time in the short time span *while the photons
were in flight*. In 1982 it was not possible to achieve this ideal, and Aspect
made do with a surrogate. We will see that good randomization is absolutely
crucial. Recent experiments have neglected this, with the notable exception
of Weihs et al. (1998) who could claim to be the *only* people so far to have
actually carried out the Aspect experiment as Aspect intended; see Aspect
(2002).

Each photon either passes or does not pass through its filter. What
happens is registered by a photo-detector. The experiment thus produces,

in total, four sequences of binary outcomes: the filter-settings, both left and right, and the outcomes 'photon passes' or 'photon doesn't pass,' both left and right.

We will be particularly interested in the following event which either does or does not happen at each trial, namely, 'the two photons do the same': both pass or neither passes. Each trial is characterized by one of four possible combinations of settings of the two filters. We label these combinations by a pair of indices (i, j), $i = 1, 2$ for the left setting and $j = 1, 2$ for the right setting (we will be specific about the particular orientations later). Since at each trial, i and j are chosen independently and with equal probabilities, the four joint outcomes of the settings will occur approximately equally often, each approximately $n/4$ times. Let N_{ij} denote the number of times that the two photons do the same, *within* the subset of trials with joint setting (i, j). In Section 6 we will argue that in a 'local realistic' description of what is going on here, one will have

$$(3.1) \qquad\qquad N_{12} \lesssim N_{11} + N_{21} + N_{22}.$$

In fact one has four inequalites: each of the four random counts should be less than the sum of the other three, modulo random noise, which is what we indicate with the 'approximate inequality' symbol. Violation of the inequality, if at all, would be due to statistical variation and therefore at most of the order of $\sqrt{n}$, if one may assume independence between the trials. If we allow for sequential dependence then perhaps a worse violation could occur by chance, and it is the purpose of this paper precisely to quantify how large it could be.

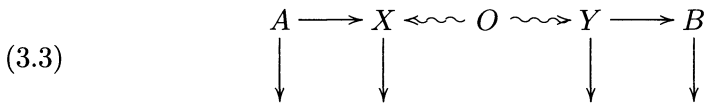Quantum mechanics predicts that, if the angles are chosen suitably, one can have

$$(3.2) \qquad\qquad N_{12} \gg N_{11} + N_{21} + N_{22},$$

and this is what Aspect et al. (1982a,b) experimentally verified; in particular the second paper introduced the randomly varying polarization filter settings. Nowadays this experiment can be done in any decent university physics laboratory, though twenty years ago the experiment was a tour de force. In fact one usually replaces the absolute frequencies in the equations (3.1) and (3.2) by relative frequencies. Since the denominators will be roughly equal, this does not make much difference, and working with absolute frequencies allows a much cleaner mathematical analysis below.

Actually I am simplifying somewhat and will not go into the major complications involved when one takes account of the fact that not all emitted photons are detected. To be honest it must be said that this still leaves a tiny, but rapidly dissappearing, loophole for local realism in ever more conspiratorial forms. For the latest theoretical progress in this area see Larsson

(2002), Larsson and Semitecolos (2001), Massar (2002); and for experimental progress, Weihs et al. (1998), Rowe et al. (2001).

Accardi et al. (2000b, 2001, 2002) claim that they can programme three computers, one representing the calcium atoms and sending information to two other computers, representing the polarization filters, to reproduce the predictions of quantum mechanics, or at the least, to satisfy (3.2). My bet is that their experiment will however reproduce (3.1). My protocol of the experiment stipulates that I provide two streams of binary outcomes to each of the two 'polarization filters', representing the choices of setting (orientation) of each filter. Graphically one trial of the experiment looks like this:

$$A \longrightarrow X \Longleftarrow\!\!\!\sim O \sim\!\!\!\Longrightarrow Y \longrightarrow B$$

(3.3)

$$\downarrow \qquad \downarrow \qquad\qquad \downarrow \qquad \downarrow$$

where X and Y denote the two polarization filters, O denotes the calcium atom, A and B are two operators (Alice, Bob) independently choosing the settings at X and Y. The downwards arrows coming from A and B represent exact copies of the settings sent by A and B to X and Y. The wiggly arrows emanating from O are supposed to suggest a quantum rather than a classical (straight) connection. Accardi claims he can replace them with straight arrows. The statistician must process four downward streams of binary data: the settings from A and B, and the outcomes from X and Y.

4. Bells' inequality

This little section derives Bell's inequality, which lies behind the prediction (3.1). For the time being treat this as a background fact from probability theory. Why it should be relevant to a local realistic version of the Aspect experiment, we will argue in Section 6. Actually, the inequality I prove is a form of the "CHSH", i.e., Clauser–Horne–Shimony–Holt (1969) version of Bell's inequality, better tuned to a stringent experimental distinction between quantum mechanical and classical systems. The way it will be proved here, as a probabilistic consequence of a deterministic inequality, is often attributed to Hardy (1993). In fact, others also earlier used this arguement, and its seeds are already in Bell's paper. Some trace the inequality back to the works of the nineteenth century logician Boole. I learnt it from Maassen and Kümmerer (1998). Bell himself, along with most physicists, gives a more involved proof, since the physics community does not make use of standard probabilistic notation and arguments. I also prefer, for transparency, an inequality in terms of probabilities of coincidences to one in terms of correlations (which however are what the physicists prefer to talk about).

Let X_1, Y_1, X_2, Y_2 denote four 0/1-valued random variables. Think of them positioned at the vertices of a square, with X_1 opposite to X_2, Y_1 opposite to Y_2. Each side of the square connects one of the X variables to one of the Y variables, and therefore represents an experiment one could possibly do with two photons and two polarization filter settings. Convince yourself, by following through the choice of a 0 or a 1 for X_1, that

$$(4.1) \qquad X_1 \neq Y_1 \& Y_1 \neq X_2 \& X_2 \neq Y_2 \implies Y_2 \neq X_1.$$

Taking the negation of each side and reversing the implication, it follows that

$$(4.2) \qquad X_1 = Y_2 \implies X_1 = Y_1 \text{ or } X_2 = Y_1 \text{ or } X_2 = Y_2.$$

Now use one of the first properties of probability:

$$(4.3) \qquad \mathrm{P}\{X_1 = Y_2\} \leq \mathrm{P}\{X_1 = Y_1\} + \mathrm{P}\{X_2 = Y_1\} + \mathrm{P}\{X_2 = Y_2\}.$$

If you are interested in correlations, by which the physicists mean *raw product moments*, note that (physicist's notation) $\langle X_1, Y_2 \rangle = \mathrm{E}(X_1 Y_2) = 2\,\mathrm{P}\{X_1 = Y_2\} - 1$.

5. Coincidence probabilities for entangled photons

The two photons in the Aspect experiment have in some sense exactly equal polarization. If the two polarization filters left and right are in perpendicular orientations, *exactly one* of the two photons will pass through the filter. For instance, if one filter is oriented horizontally, and the other vertically, one might imagine that the calcium atom either produces two horizontally polarized photons, or two vertically polarized photons, each with probability half. With probability half, both photons are horizontally polarized, and the one which meets the horizontal filter, passes through it, while the other meets a vertical filter and is absorbed. With probability half both photons are vertically polarized and again, exactly one passes the two filters. The same holds for any two perpendicularly oriented filters: the probability of coincidence—the two photons do the same—is zero. If however the two filters are oriented in the same direction, for instance, both horizontal, then *either* both photons pass, or both do not pass (each of these possibilities has probability half). The probability of coincidence is one. Now imagine keeping one filter fixed and slowly rotating the other. At zero degrees difference, the probability of coincidence is 1, at 90 degrees, it is 0, at 180 degrees it is back to one, and so on. It is a smooth curve (how could it not be smooth?), varying periodically between the values 0 and 1. Recalling that $\cos(2\theta) = 2\cos^2(\theta) - 1$, and that the cosine function is itself a shifted sine curve, we conclude that if the probability of coincidence is a sine curve, it

has to be the curve $\cos^2(\theta)$: it varies between 0 and 1, taking these values at $\theta = \pi/2$ and $\theta = 0$.

Quantum mechanics predicts precisely this probability of coincidence (in fact so does classical optics, but there light comes in continuous waves, not discrete particles, and the word "probability" has to be replaced by "intensity"). The quantum state involved, is the only pure state having the natural rotational invariance so this answer is pretty canonical. Recall that quantum mechanics is characterized by wave-particle duality: we know that photons are particles, when we look to see with a photo-dector if one is present or not. But we also know that light behaves like waves, exhibiting interference patterns. Waves are smooth but particles, especially deterministic particles, are discrete. However, random particles can have smoothly varying behaviour. It seems that randomness is a necessary consequence of the fundamental wave-particle duality of quantum mechanics, i.e., of reality.

Now suppose A chooses, for X, between the orientations $\alpha_1 = 0$ and $\alpha_2 = \pi/3$, while B chooses, for Y, between the orientations $\beta_1 = -\pi/3$ and $\beta_2 = 0$. The absolute difference between each α and each β is 0, $\pi/3$, or $2\pi/3 = \pi - \pi/3$. Since $\cos(\pi/3) = \frac{1}{2} = -\cos(\pi - \pi/3)$ the four probabilities of coincidence are $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, 1, and

$$(5.1) \qquad\qquad 1 \gg \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}.$$

Even better angles are $\alpha_1 = \pi/8$, $\alpha_2 = 3\pi/8$, $\beta_1 = -\pi/4$ and $\beta_2 = 0$ giving probabilities of coincidence approximately 0.15, 0.15, 0.15 and 0.85 and a difference of $\sqrt{2} - 1 \approx 4/10$ instead of our $\frac{1}{4}$.

6. Why Bell applies to Accardi's computers

Consider one trial. Suppose the computer O sends some information to X and Y. It may as well send the same information to both (sending more, does not hurt). Call the information λ. Operator A sends α_1 or α_2 to X. Computer X now has to do a computation, and output either a 0 or a 1 ('doesn't pass', 'does pass'). In our imagination we can perfectly clone a classical computer: i.e., put next to it, precisely the same apparatus with precisely the same memory contents, same contents of the hard disk. We can send α_1 to one of the copies and α_2 to the other copy; we can send λ to both (classical information can be cloned too). By the way, quantum systems cannot be cloned—that is a theorem of quantum mechanics! Therefore both copies of the computer X can do their work on both possible inputs from A, and the same input from O, and produce both the possible outputs. Similarly for Y.

Let us now suppose that this is actually the mth trial. I allow that computers O, X and Y use pseudo-random number generators and that I model the seeds of the generators with random variables. This means that I

now have defined four random variables X_{m1}, X_{m2}, Y_{m1} and Y_{m2}, the values of two of which are actually put on record, while the other two are purely products of your and my imagination. *Which* are put on record is determined by independent (of everything so far) Bernoulli trials, the choices of A between index 1 or 2 for the X variables, and of B between index 1 or 2 for the Y variables. Let me directly define variables U_{m11}, U_{m12}, U_{m21}, U_{m22} which are indicator variables of the four possible joint outcomes. Thus the sum of these four 0/1 variables is identically 1, and each is Bernoulli($\frac{1}{4}$). I will allow Accardi's computers, at the mth trial, to use results obtained so far in its computations for the current trial. So we arrive at the following model: for each $m = 1, \ldots, n$, the vector $(U_{m11}, U_{m12}, U_{m21}, U_{m22})$ is multinomial($1; \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$), independent of all preceding U, X and Y variables, and independent of the current X and Y variables. The counts on which the bet depends are $N_{ij} = \sum_m U_{mij} 1\{X_{mi} = Y_{mj}\}$. I compute the expectation of this by first conditioning, within the mth term, on the current and preceding X and Y variables and on the preceding U variables. By conditional independence and by taking the expectation of a conditional expectation I find $\mathrm{E}\, N_{ij} = \frac{1}{4} \sum_m \mathrm{P}\{X_{mi} = Y_{mj}\}$. Therefore

$$(6.1) \quad \mathrm{E}(N_{12} - N_{11} - N_{21} - N_{22})$$
$$= \frac{1}{4} \sum_m \big(\mathrm{P}\{X_{m1} = Y_{m2}\} - \mathrm{P}\{X_{m1} = Y_{m1}\}$$
$$- \mathrm{P}\{X_{m2} = Y_{m1}\} - \mathrm{P}\{X_{m2} = Y_{m2}\} \big)$$
$$\leq 0,$$

by Bell's inequality (4.3). *In expectation* Accardi must lose. If each trial is independent of each other, the deviation can be at most of the order of $\sqrt{n}$. In the next section we will see that serial dependence cannot worsen this at all, because of the obvious (super)martingale structure in the variable of interest.

7. Supermartingales

Let us allow the choices of computers O, X and Y at the mth trial to depend arbitrarily on the past up to that time. Write $\vec{1} = (1, 1, 1, 1)$,

$$\vec{U}_m = (U_{m1}, U_{m2}, U_{m3}, U_{m4}) = (U_{m12}, U_{m11}, U_{m12}, U_{m22})$$

and

$$\vec{X}_m = (1\{X_{m1} = Y_{m2}\}, -1\{X_{m1} = Y_{m1}\}, -1\{X_{m2} = Y_{m2}\}, -1\{X_{m1} = Y_{m2}\}).$$

Define $\Delta_m = \vec{U}_m \cdot \vec{X}_m$ and $S_m = \sum_{m=1}^{r} \Delta_r$. Define $\Delta_m^* = \frac{1}{4} \vec{1} \cdot \vec{X}_m$. Let $\mathcal{F}_m$ denote the σ-algebra of all X, Y and U variables up to and including

the mth trial. Let the smaller σ-algebra $\mathcal{A}_m$ be the σ-algebra generated by $\mathcal{F}_{m-1}$ together with X_{m1}, X_{m2}, Y_{m1}, Y_{m2}. Thus $\mathcal{F}_m$ is generated by $\mathcal{A}_m$ together with $\overrightarrow{U}_m$. Define $\tilde{\Delta}_m = \mathrm{E}(\Delta_m \mid \mathcal{F}_{m-1})$ and $\tilde{S}_m = \sum_{r=1}^m \tilde{\Delta}_r$. In the previous section we basically made the computation

$$\tilde{\Delta}_m = \mathrm{E}(\Delta_m \mid \mathcal{F}_{m-1}) = \mathrm{E}\big(\mathrm{E}(\Delta_m \mid \mathcal{A}_m) \mid \mathcal{F}_{m-1}\big) = \mathrm{E}(\Delta_m^* \mid \mathcal{F}_{m-1})$$

where surely, $-\frac{1}{2} \le \Delta_m^* \le 0$, therefore also $-\frac{1}{2} \le \tilde{\Delta}_m \le 0$ and $|\Delta_m - \tilde{\Delta}_m| \le \frac{3}{2}$. Define $\sigma_m^2 = \mathrm{Var}(\Delta_m - \tilde{\Delta}_m \mid \mathcal{F}_{m-1})$. Using the facts that the support of Δ_m is $\{-1, 0, 1\}$ with probabilities of the extreme values bounded by $\frac{3}{4}$ and $\frac{1}{4}$ one easily finds $0 \le \sigma_m^2 \le \frac{3}{4}$ almost surely. Define $V_m = \sum_{r=1}^m \sigma_r^2$.

To warm up, we investigate whether we can obtain a Chebyshev-like inequality in this situation. The answer will be yes, but the inequality will be too poor for practical use. After that we will make better use of the fact that all summands are bounded, and derive a powerful Bernstein-like inequality.

It follows from the computations above that $S_m - \tilde{S}_m$ is a martingale with respect to the filtration $(\mathcal{F}_m)_{m=1}^n$, and so is $(S_m - \tilde{S}_m)^2 - V_m$, while $\tilde{S}_m$ is a decreasing, negative, predictable process and V_m an increasing, positive, predictable process. By the inequality of Lenglart (1977) it follows that for any $\eta > 0$ and $\delta > 0$, $\mathrm{P}\{\sup_{m \le n}(S_m - \tilde{S}_m)^2 \ge \eta\} \le \delta/\eta + \mathrm{P}\{V_n \ge \delta\}$. Choosing $\eta = k^2 n$ and noting that $V_n \le 3n/4$, we find the inequality

$$(7.1) \qquad \mathrm{P}\{S_n \ge k\sqrt{n}\} \le \frac{\delta}{k^2 n} + \mathrm{P}\{V_n \ge \delta\} \le \frac{\delta}{k^2 n} + \frac{3n}{4\delta},$$

by Chebyshev's inequality. The right hand side is minimal at $\delta = \sqrt{3}n/2k$ giving us the inequality

$$(7.2) \qquad \mathrm{P}\{S_n \ge k\sqrt{n}\} \le \frac{\sqrt{3}}{k}.$$

This is nowhere as good as the result of applying Chebyshev's inequality when all trials are independent,

$$(7.3) \qquad \mathrm{P}\{S_n \ge k\sqrt{n}\} \le \frac{1}{k^2},$$

but it would allow us to choose a (huge) sample size and critical value to settle my bet with Luigi Accardi. Note that I can for free replace S_n by $\sup_{m \le n} S_m$ in these inequalities, so there is no chance that Accardi can win by stopping when things are looking favourable for him (they won't). However the sample size is prohibitively large, for the rather small error probabilities which we would like to guarantee.

In fact we can do much better, using exponential bounds for martingales, generalizing the well-known Bernstein (1924), Hoeffding (1963), or Bennett

(1962) inequalities for sums of bounded, independent random variables, and more generally for independent random variables with bounded exponential moment. From van de Geer (1995), or van de Geer (2000, Lemma 8.11), applied to the martingale $S_m - \widetilde{S}_m$ whose differences are bounded in absolute value by $\frac{3}{2}$ with conditional variances bounded by $\frac{3}{4}$ we obtain:

$$(7.4) \qquad \mathrm{P}\{\sup_{m \leq n} S_m \geq \frac{\sqrt{3}}{2} k\sqrt{n}\} \leq \exp\left(-\frac{1}{2}k^2\left(\frac{1}{1 + (1/\sqrt{3})k/\sqrt{n}}\right)\right).$$

More precisely, van de Geer (1995, 2000) gives us the stronger result obtained by replacing S_m with $S_m - \widetilde{S}_m$ in (7.4), but we also know that $\widetilde{S}_m \leq 0$. Thus at the root n scale, the tail of our statistic can be no heavier than Gaussian; though for much larger values (at the scale of n) it can be as heavy as exponential. This behaviour is no worse than for sums of independent random variables. In fact if S_n denotes the sum of n independent random variables each with mean zero, bounded from above by $\frac{3}{2}$, and variance bounded by $\frac{3}{4}$, the classic Bernstein inequality is simply (7.4) with $\sup_{m \leq n} S_m$ replaced by S_n. The discrete time martingale maximal Bernstein inequality goes back to Steiger (1969) and Freedman (1975), while Hoeffding (1963) already had a martingale maximal version of his, related, exponential inequality. A continuous time martingale version of the Bernstein inequality can be found in Shorack and Wellner (1986). A recent treatment of the inequality for independent random variables can be found in Pollard (2001, Chapter 11).

Note that if we had been working with the relative instead of the absolute frequencies, we could have treated the four denominators in the same way, used Bonferroni, and finished with a very similar but messier inequality.

We can now specify precisely a protocol for the computer experiment, which must settle the bet between Accardi and the author. In order that the supermartingale structure is present, it suffices that the settings and the outcomes are generated sequentially: Gill provides settings for trial 1, then Accardi provides outcomes for trial 1, then Gill provides settings for trial 2, Accardi outcomes for trial 2, and so on. Between subsequent trials, computers X, O and Y may communicate with one another in any way they like. Within each trial, the communications are one way only, from O to X and from O to Y; and from A to X and from B to Y. A very rough calculation from (7.4) shows that if both accept error probabilities of one in a million, Accardi and Gill could agree to a sample size of sixty five thousand, and a critical value $+n/32$, half way between the Bell expectation bound 0 and the Aspect experiment expectation $+n/16$. I am supposing here that Accardi plans not just to violate the Bell inequality, but to simulate the Aspect experiment with the filter settings as specified by me. I am also supposing that he is happy to rely on Bernstein's inequality, in the opposite direction. Only twenty five thousand trials are needed when Accardi

aims for the greatest violation allowed under quantum mechanics, namely an expectation value of approximately $+n/10$ and critical value $+n/20$.

The experiment will be a bit easier to perform, if Accardi does not want to exploit the allowed communication between his computers, *between* trials. In that case one might as well store the entire initial contents of memory and hard disk, of computer O, within computers X and Y. Now computers X and Y can each simulate computer O, without communicating with one another. Now we just have computers A and X, connected one-way, and completely separately, B and Y, also connected one-way. We carry out n sequential trials on each pair of computers.

It would be even more convenient if these trials could be done simultaneously, instead of sequentially. Thus computers A and B would deliver to X and Y, in one go, all the settings for the n trials. We now lose the martingale structure. For the mth trial, one can condition on all preceding *and* subsequent settings. Conditioning also on the intial contents of computers X and Y, we see that the outcomes of the mth trial are now deterministic functions of the *random* settings for the mth trial. Thus we still have Bell's inequality: in expectation, nothing has changed. But the martingale structure is destroyed; instead, we have something like a Markov field. Is there still a Bernstein-like inequality for this situation? It is not even clear if a Chebyshev inequality is available, in view of the possible correlations which now exist between different outcomes. However, since we have the Bell inequality in expectation, one could put the onus on keeping the variance small, on the person who claims they can simulate quantum mechanical correlations on a classical computer. For instance, Accardi might believe that he can keep the second decimal digit of N_{ij}/n fixed, when n is as large as, say, ten thousand. Then one could do the experiment in ten times four batches of ten thousand, sending files by internet forty times. Within each group of four batches, I supply a random permutation of the four joint settings (i, j). We settle on a critical value halfway between our two expectations, but Accardi must also agree to lose, if the second decimal digits of each group of 10 N_{ij}/n, n being the size of the batch now, ever vary. Am I safe? I feel uneasy, without Bernstein behind me.

In the actual Aspect experiment, the alternative set of angles mentioned above are used, so as to achieve, by an inequality of Cirel'son (1980), the most extreme violation of the Bell inequality which is allowed within quantum mechanics. Thus if an even larger violation had been observed, one would not just have had to reject the specific quantum mechanical calculations for this particular experiment, but more radically have to reject the accepted rules of quantum mechanics, altogether. Many authors have therefore considered those settings as providing "the most strong violation of local realism, possible." However, we would say that the strongest violation oc-

curs, when one is able to reject local realism, with the smallest possible number of samples. Thus concepts of efficiency in statistical testing, should determine "the strongest experiment."

Van Dam, Gill and Grunwald (2002) study this problem from a game-theoretic point of view, in which the believer in quantum mechanics needs to find the experimental set-up which provides the maximal "minimum Kullback-Leibler distance between the quantum mechanical predictions and any possible prediction subject to local realism." Such results can be reformulated in terms of size, power, and sample size, using Bahadur efficiency (large deviations).

Many authors discuss the Aspect experiment and Bell inequalities in a version appropriate for spin half particles (for instance, electrons) rather than photons. The translation from photons to electrons is: double the angles, and then rotate the settings in one wing of the experiment by 180°. To explain the doubling: a polarization filter behaves oppositely after rotating 90°, and identically after rotating 180°. A Stern-Gerlach magnet behaves oppositely after rotating 180°, identically after rotating 360°. As for the rotation: the photons in our version of the Aspect experiment are identically polarized while the spin of the spin half particles in the companion experiment are equal and opposite. The quantum state used in the spin half version is the famous Bell or singlet state, $|01\rangle - |10\rangle$, while for photons one uses the state $|00\rangle + |11\rangle$, where the 0 and 1 stands for "spin-up," "spin-down" for electrons, and "horizontal polarization," "vertical polarization" for photons. There are also photon experiments with oppositely polarized rather than equally polarized photons, and the state $|01\rangle + |10\rangle$.

8. A different kind of probability, or nonlocality?

The relation between classical and quantum probability and statistics has been a matter of heated controversy ever since the discovery of quantum mechanics. It has mathematical, physical, and philosophical ingredients and much confusion, if not controversy, has been generated by problems of interdisciplinary communication between mathematicians, physicists, philosophers and more recently statisticians. Authorities from both physics and mathematics, perhaps starting with Feynman (1951), have promoted vigorously the standpoint that 'quantum probability' is something very different from 'classical probability'. Most recently, Accardi and Regoli (2000a) state "the real origin of the Bell's inequality is the assumption of the applicability of classical (Kolmogorovian) probability to quantum mechanics" which can only be interpreted as a categorical statement that classical probability is *not* applicable to quantum mechanics. Accardi et al.'s (2002) aim is "to show that Bell's statement ... is theoretically and experimentally unjustified," and they diagnose Bell's error as an incorrect use of Kolmogorov probability

and conditioning. Malley and Hornstein (1993) conclude from the perceived
conflict between classical and quantum probability that 'quantum statistics'
should be set apart from classical statistics.

We disagree. In our opinion, though fascinating mathematical facts and
physical phenomena lie at the root of these statements, cultural preconcep-
tions have also played a role. Probabilistic and statistical problems from
quantum mechanics fall definitely in the framework of classical probability
and statistics, and the claimed distinctions have retarded the adoption of
statistical science in physics. The phenomenon of quantum entanglement in
fact has far-reaching technological implications, which can only be expressed
in terms of classical probability; their development will surely involve clas-
sical statistics too. Emerging quantum technology (entanglement-assisted
communication, quantum computation, quantum holography and tomogra-
phy of instruments) aims to capitalise on precisely those features of quantum
mechanics which in the past have often been seen as paradoxical theoretical
nuisances.

Our stance is that the predictions which quantum mechanics makes of
the real world are stochastic in nature. A quantum physical model of a
particular phenomenon allows one to compute probabilities of all possible
outcomes of all possible measurements of the quantum system. The word
'probability' means here: relative frequency in many independent repetitions.
The word 'measurement' is meant in the broad sense of: macroscopic results
of interactions of the quantum system under study with the outside world.
These predictions depend on a summary of the state of the quantum system.
The word 'state' might suggest some fundamental property of a particular
collection of particles, but for our purposes all we need to understand under
the word is: a convenient mathematical encapsulation of the information
needed to make any such predictions.

Now, at this formal level one can see analogies between the mathemat-
ics of quantum states and observables—the physical quantities of quantum
mechanics—on the one hand, and classical probability measures and random
variables on the other. This analogy is very strong and indeed mathemat-
ically very fruitful (also very fruitful for mathematical physics). Note that
collections of both random variables and operators can be endowed with al-
gebraic structure (sums, products, ...). It is a fact that from an abstract
point of view a basic structure in probability theory—a collection of random
variables X on a countably generated probability space, together with their
expectations $\int X \, dP$ under a given probability measure P—can be repre-
sented by a (commuting) subset of the set of self-adjoint operators Q on
a separable Hilbert space together with the expectations $\text{tr}\{\rho Q\}$ computed
using the trace rule under a given state ρ, mathematically represented by

another self-adjoint operator having some special properties (nonnegative and trace 1).

'Quantum probability,' or 'noncommutative probability theory' is the name of the branch of mathematics which studies the mathematical structure of states and observables in quantum mechanics. It is a fact that a *basic* structure in classical probability theory is isomorphic to a *special case* of a basic structure in quantum probability. Brief introductions, of a somewhat ambivalent nature, can be found in the textbooks, on classical probability, of Whittle (1970) and Williams (2001). Kümmerer and Maassen (1998), discussed in Gill (1998), use the "quantum probabilistic modelling" of the Aspect experiment—which just involves some simple linear algebra involving 2 × 2 complex matrices—to introduce the mathematical framework of quantum probability, giving the violation of the Bell inequalities as a motivation for needing "a different probability theory." From a mathematical point of view, one may justly claim that classical probability is a special case of quantum probability. The claim does entail, however, a rather narrow view of classical probability. Moreover, many probabilists will feel that abandoning commutativity is throwing away the baby with the bathwater, since this broader mathematical structure has no analogue of the sample outcome ω, and hence no opportunity for a probabilist's beloved probabilistic arguments.

Many authors have taken the probabilistic predictions of quantum theory, as exemplified by those of the Aspect experiment, as a defect of classical probability theory and there have been proposals to abandon classical probability in favour of exotic alternative theories (negative, complex or p-adic probabilities; nonmeasurable events; noncommutative probability; ...) in order to 'resolve the paradox'. However in our opinion, the phenomena are real and the defect, if any, lies in believing that quantum phenomena do not contradict *classical, deterministic, physical thinking.* This opinion is supported by the recent development of (potential) technology which acknowledges the extraordinary nature of the predictions and exploits the discovered phenomena (teleportation, entanglement-assisted communication, and so on). In other words, one should not try to explain away the strange features of quantum mechanics as some kind of defect of classical probabilistic thinking, but one should use classical probabilistic thinking to pinpoint these features.

The violation of the Bell inequalities show that any deterministic, underlying, theory intending to explain the surface randomness of quantum physical predictions, has to be grossly non-local in character. For some philosophers of science, for instance Maudlin (1994), this is enough to conclude that "locality is violated, *tout court.*" He goes on to analyse, with great clarity, precisely what kind of locality is violated, and he investigates possible conflicts with relativity theory. Whether or not one says that locality

is violated, depends on the meaning of the word "local". In our opinion, it can only be given a meaning relative to some model of the physical world, whether it be implicit or explicit, primitive or sophisticated.

Since quantum randomness is possibly the only *real* randomness in the world—all other chance mechanisms, like tossing dice or coins, can be well understood in terms of classical deterministic physics—there is justification in concluding that "quantum probability is a different kind of probability." And all the more worth studying, with classical statistical and probabilistic tools, for that.

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APPENDIX

A. What went wrong?

This appendix is provided by J.-Å. Larsson (`jalar@mai.liu.se`), Linköping, Sweden. It points out the error in the Accardi and Regoli construction.

In Accardi and Regoli (2001), it is argued that the Bell inequality can be violated by a classical system after a local dynamical evolution. After a dynamical evolution, in the Schrödinger picture an expectation is obtained as

$$(A.1) \qquad E(F) = \iint F(\lambda_1, \lambda_2)\, (\psi_0 \circ P)(d\lambda_1, d\lambda_2),$$

while in the Heisenberg picture,

$$(A.2) \qquad E(F) = \iint P(F)(\lambda_1, \lambda_2)\, \psi_0(d\lambda_1, d\lambda_2).$$

Perhaps it should be underlined here that the two above expressions are *equivalent representations of the same physical system.* This means among other things that the possible values of the observables (values of the random variables, outcomes of the experiment) in the right-hand sides should be equal, regardless of the representation. In mathematical language, $R(F) = R\big(P(F)\big)$.

In Accardi and Regoli (2001), it is claimed that $P(F)$ in the Heisenberg picture is of a certain form:

$$(A.3) \qquad P(F)(\lambda_1, \lambda_2) = F(T_{1,a}\lambda_1, T_{2,b}\lambda_2)T'_{1,a}(\lambda_1)T'_{2,b}(\lambda_2)$$

For the physical system in question in Accardi and Regoli, $R(F) = \{\pm 1\}$, so the only T_is that can be used if (A.3) holds are those for which

$$(A.4) \qquad T'_{1,a}(\lambda_1)T'_{2,b}(\lambda_2) = 1 \quad \text{a.e.}$$

The model (18) in Accardi and Regoli (2001) does not follow this requirement, but instead, the measurement results in the Heisenberg picture lie in the interval $[-\sqrt{2\pi}, \sqrt{2\pi}]$. The Bell inequality (the CHSH inequality) is only valid for systems for which the results are in $\{\pm 1\}$ ($[-1, 1]$), and for such systems, the correlation is less exciting.

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