ALIGNED RANK TEST FOR THE BIVARIATE RANDOMIZED BLOCK MODEL

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An aligned rank test for treatment effects in the bivariate randomized block model is proposed. The test is easy to implement and its validity requires only minimal assumptions. Furthermore, the test statistic is affine-invariant and has a limiting χ^2 distribution under the null hypothesis when the number of blocks goes to infinity. If the number of blocks is not large enough, we show how to perform a permutation test and illustrate this method with an example. Finally, a simulation study indicates that the new test performs well compared to the likelihood ratio test, to a coordinate-wise aligned rank test and to a sign test based on the Oja measure of scatter.

1. Introduction

Consider the bivariate randomized block model with one observation in each cell. We wish to test the hypothesis that there are no treatment effects. If we assume that the observations follow a bivariate normal distribution, one sensible test is based on the likelihood ratio statistic described, for example, in Section 8.9 of Anderson (1984). This statistic is affine-invariant, that is, its value remains unchanged if a nonsingular linear transformation is applied to the observations. In practice, this important property means that the power of the test is not affected by the correlation structure or the scale of the variables.

For univariate data, there are two main approaches to construct tests based on ranks. The first one uses intra-block ranking which means that the observations are ranked separately within each block. The Friedman test is the most well-known example of this approach, Hollander and Wolfe (1999). This test is distribution-free but its efficiency relative to the classical variance-ratio test is quite low at the normal model when the number of treatment is small and this is essentially due to the fact that no inter-block comparisons are made. To alleviate this problem, Quade (1979) proposed a method that is still based on within-block ranking but where each block is given a data-driven weight. His test does have a better efficiency when the number of treatments is small and remains distribution-free. On the other hand, as opposed to the Friedman test, its efficiency decreases as the number of treatments increases, see Table 1 of Tardif (1987). Larocque

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and Tardif (1995) show how to overcome this problem by considering appropriate scores for the block weights. The second main approach uses overall ranking after aligning the observations to get rid of the block effects, Lehmann (1998). For instance, the linear score rank test that uses the block means for aligning is no longer distributon-free but is more efficient at the normal model than the tests based on within-block ranking, see Table 7.1 of Puri and Sen (1971) for example. This can be explained by the fact that this approach recovers some inter-block comparisons. This is the main reason why this approach is used in this paper to construct a test for bivariate data.

For multivariate observations, the earliest propositions uses coordinatewise ranks, that is, the ranking is done separately for each variable. Both the intra-block ranking and the ranking after alignment methods are described in Chapter 7 of Puri and Sen (1971). One drawback of this approach is that the statistics are scale-invariant but not affine-invariant.

A revival in interest for multivariate sign and rank methods has appeared since the late 80's, see Chapter 6 of Hettmansperger and McKean (1998). Despite this, very few articles discuss in details these new approaches for the randomized block model. One way to construct generalizations of sign and rank tests for multivariate data is to use the Oja's median, see Oja (1999) for a recent review. Using this idea, an affine-invariant test based on intra-block ranking for bivariate data is discussed briefly in Brown and Hettmansperger (1987). This test can be seen as a bivariate generalization of the Friedman test. An affine-invariant sign test based on aligned observations for multivariate data is mentioned briefly in Hettmansperger and Oja (1994).

Another approach for constructing affine-invariant sign and rank tests for bivariate observations is described in a series of paper by Larocque, Tardif and van Eeden (2000a,b,c) and is based on projections. In this paper we use this approach to construct an affine-invariant test for treatment effects based on the ranking after alignment method. The proposed test statistic is simple to describe, easy to implement, has good efficiency properties and its validity requires only minimal assumptions.

The model, the statistic and its asymptotic null distribution are presented in Section 2. An illustrative example and the results of a simulation study that compares the new test to three competitors are given in Section 3. Concluding remarks follow in Section 4 and the proofs are sketched in the appendix.

2. Model, test statistic and asymptotic null distribution

Suppose we want to compare the effects of p treatments. We have n blocks of p subjects that are randomly assigned to treatments within each block and we observe two characteristics (V, W) for each subjects. Consider the following model for the observations, let

$$(V_{ij}, W_{ij}) = (\mu_1, \mu_2) + (\alpha_{i1}, \alpha_{i2}) + (\beta_{j1}, \beta_{j2}) + (\epsilon_{ij1}, \epsilon_{ij2}) \quad i = 1, \ldots, n; \quad j = 1, \ldots, p \ (\geq 2) \tag{2.1}$$

where the error vectors $(\epsilon_{ij1}, \epsilon_{ij2})$ are independent and identically distributed from a continuous distribution. The block effects, that is, the $(\alpha_{i1}, \alpha_{i2})$'s can be fixed or random. The treatment effects, that is, the (β_{j1}, β_{j2})'s are fixed and satisfy without loss of generality $\sum_{j=1}^{p}(\beta_{j1}, \beta_{j2}) = (0, 0)$. The vector (μ_1, μ_2) is the overall mean. Let $N = np$ be the total sample size and let $M = N(N-1)/2$.

We wish to confront the hypotheses of no treatment effects

$$H_0 : (\beta_{11}, \beta_{12}) = \cdots = (\beta_{p1}, \beta_{p2}) \quad H_1 : (\beta_{j1}, \beta_{j2}) \neq (\beta_{k1}, \beta_{k2}) \text{ for at least one pair } (j, k). \tag{2.2}$$

Firstly, here is a motivation for the test statistic that will be introduced shortly. To keep things simple, suppose that we want to test that the location vector of a bivariate distribution is $(0, 0)$. Suppose that we have a random sample from this distribution and let $\text{WSR}(\theta)$ be the Wilcoxon-signed-rank statistic computed using the (univariate) projections of the observations on the directed line with angle θ. For a given vector (x, y), this projection is simply $x\cos(\theta) + y\sin(\theta)$. To test the location vector, a reasonable statistic can be constructed by "averaging" $\text{WSR}(\theta)$ over θ. One way of doing this is to consider statistics of the form $\int_{-\pi/2}^{\pi/2}(\text{WSR}(\theta))^2\, dw(\theta)$ where $w(\theta)$ is a weight function.

This idea, based on projections, is the starting point to construct the bivariate location test proposed in Larocque, Tardif and van Eeden (2000c). It has has also been used, in conjunction with the sign test instead of the Wilcoxon signed-rank test, in Larocque, Tardif and van Eeden (2000a). By exploiting the relationship between the univariate Wilcoxon signed-rank and the two-sample Wilcoxon statistics, this approach has been generalized to the two-sample problem and, moreover, to the p-sample and simple regression problems in Larocque, Tardif and van Eeden (2000b).

In this paper, the idea is to use the p-sample statistic of Larocque, Tardif and van Eeden (2000b) with aligned (block by block) observations. Here is the description of the test statistic.

In order to make the observations from different blocks comparable, we first align them by subtracting the block averages. Let, for $i = 1, \ldots, n$ and $j = 1, \ldots, p$,

$$(X_{ij}, Y_{ij}) = (V_{ij}, W_{ij}) - (\bar{V}_{i.}, \bar{W}_{i.})$$

where

$$(\bar{V}_{i.}, \bar{W}_{i.}) = \frac{1}{p}\sum_{j=1}^{p}(V_{ij}, W_{ij})$$

is the average of the i^{th} block. We will use the aligned vectors to construct the test statistic. By aligning the observations, we get rid of the block effects at the expense of introducing correlation between the observations from a given block. The problem then becomes a p-sample testing problem with correlated observations. Instead of using the block average, it is possible to align the observations by using another location estimator, one that is more robust for example. To ensure that the final test statistic is affine-invariant and that the asymptotic result remains valid, this estimator should be root n consistent and affine-equivariant.

Let

$$\theta_{ij,lk} = -\arctan\left(\frac{X_{ij} - X_{lk}}{Y_{ij} - Y_{lk}}\right) + \frac{\pi}{2}$$
$$1 \le i \le l \le n; \quad j = 1, \ldots, p; \quad k = 1, \ldots, p; \quad j < k \text{ if } i = l.$$

be the angle ($\in [0, \pi)$) between the vector $(X_{ij} - X_{lk}, Y_{ij} - Y_{lk})$ and the X-axis. Note that there are M such angles corresponding to each choice of two distinct observations. Define $R_{ij,lk}$ as the rank of $\theta_{ij,lk}$ among the M θ's. The test of the hypothesis H_0 is based on the $p(p-1)$ statistics

$$\begin{aligned} A_{jk} &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n} s(Y_{ij} - Y_{lk})\cos(\pi R_{ij,lk}/M) \\ B_{jk} &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n} s(Y_{ij} - Y_{lk})\sin(\pi R_{ij,lk}/M), \end{aligned} \tag{2.3}$$

$1 \le j < k \le p$, where $s(u) = 1$ or -1 as $u > 0$ or < 0 is the sign function.

A consistent estimator, $\widehat{\Sigma}$, of the null covariance matrix of (A_{jk}, B_{jk}) is needed and we describe it now. Let

$$\hat{\sigma}_{cc1} = \frac{1}{d_1}\sum_{I_1} s(Y_{ij} - Y_{kl})s(Y_{ij} - Y_{rt})\cos(\pi R_{ij,kl}/M)\cos(\pi R_{ij,rt}/M)$$

$$\hat{\sigma}_{ss1} = \frac{1}{d_1}\sum_{I_1} s(Y_{ij} - Y_{kl})s(Y_{ij} - Y_{rt})\sin(\pi R_{ij,kl}/M)\sin(\pi R_{ij,rt}/M)$$

$$\begin{aligned} \hat{\sigma}_{cs1} = \frac{1}{d_1}\sum_{I_1} s(Y_{ij} - Y_{kl})s(Y_{ij} - Y_{rt})\frac{1}{2}\big(&\cos(\pi R_{ij,kl}/M)\sin(\pi R_{ij,rt}/M) \\ &+ \cos(\pi R_{ij,rt}/M)\sin(\pi R_{ij,kl}/M)\big) \end{aligned}$$

where $I_1 = \{(i,k,r,j,l,t) : i,k,r = 1,\ldots n;\ j,l,t = 1,\ldots p;\ k \neq i; r \neq i,k\}$ and $d_1 = n(n-1)(n-2)p^3 = \text{Card}(I_1)$. Also, let

$$\hat{\sigma}_{cc2} = \frac{1}{d_2}\sum_{I_2} s(Y_{ij} - Y_{kl})s(Y_{iu} - Y_{rt})\cos(\pi R_{ij,kl}/M)\cos(\pi R_{iu,rt}/M)$$

$$\hat{\sigma}_{ss2} = \frac{1}{d_2}\sum_{I_2} s(Y_{ij} - Y_{kl})s(Y_{iu} - Y_{rt})\sin(\pi R_{ij,kl}/M)\sin(\pi R_{iu,rt}/M)$$

$$\hat{\sigma}_{cs2} = \frac{1}{d_2}\sum_{I_2} s(Y_{ij} - Y_{kl})s(Y_{iu} - Y_{rt})\frac{1}{2}\big(\cos(\pi R_{ij,kl}/M)\sin(\pi R_{iu,rt}/M) + \cos(\pi R_{iu,rt}/M)\sin(\pi R_{ij,kl}/M)\big)$$

where $I_2 = \{(i,k,r,j,u,l,t) : i,k,r = 1,\dots n;\ j,u,l,t = 1,\dots p;\ j \neq u;\ k \neq i;\ r \neq i,k\}$ and $d_2 = n(n-1)(n-2)p^3(p-1) = \text{Card}(I_2)$.

We define the estimator of the null covariance matrix of (A_{jk}, B_{jk}) as

$$\widehat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{cc1} - \hat{\sigma}_{cc2} & \hat{\sigma}_{cs1} - \hat{\sigma}_{cs2} \\ \hat{\sigma}_{cs1} - \hat{\sigma}_{cs2} & \hat{\sigma}_{ss1} - \hat{\sigma}_{ss2} \end{pmatrix}.$$

In practice, the number of terms in the sums needed to compute $\widehat{\Sigma}$ can be quite large. One simple way to proceed then is to select terms at random (10000 for example) and approximate the sums. This approach was used in the next section and it works very well.

The test then consists in rejecting the hypothesis H_0 for large values of the statistic

$$D_n = \frac{n}{p}\sum_{j=1}^{p-1}\sum_{k=j+1}^{p}(A_{jk}, B_{jk})\widehat{\Sigma}^{-1}(A_{jk}, B_{jk})^T. \tag{2.4}$$

It can easily be seen that D_N is affine-invariant by showing successively that it is invariant for rotations, for coordinatewise scale changes and for reflection about the X-axis as in Larocque, Tardif and van Eeden (2000c).

As mentioned earlier, D_n is essentially the statistic S_N, proposed in Larocque, Tardif and van Eeden (2000b) for the bivariate p-sample problem, computed using the aligned observations. The difference here is that, by aligning them, the observations are not independent within block while all of them are assumed independent in the p-sample problem. Consequently, the estimator $\widehat{\Sigma}$ now depends on three more quantities, $\hat{\sigma}_{cc2}$, $\hat{\sigma}_{ss2}$ and $\hat{\sigma}_{cs2}$ that appear because of this intra-block dependence and the proof of the asymptotic result (as the number of blocks goes to infinity) given next has to be modified and is sketched in the appendix.

Theorem 2.1. *Under H_0 and as $n \to \infty$,*

$$D_n \xrightarrow{D} \chi^2_{2(p-1)}.$$

When the number of blocks is not large enough to use the asymptotic distribution, a conditionally distribution-free permutation test can easily be performed. The permutation distribution of D_n can be obtained by considering its distribution over all permutations of vectors within blocks. The permutation p-value is then simply computed using this distribution as reference. In practice however the total number of permutations to consider can be quite large. For example, if there are 6 blocks and 5 treatments, the number of possible permutations is $(5!)^6 = 2985984000000$. If that is the case we can simply select a large number of permutations at random and find the corresponding estimated permutation p-value. An example of this approach is given in the next section.

3. Example and simulation study

In this section, we illustrate the use of the new test with an example but first, we compare the D_n test to three of its competitors based on aligned observations in a simulation study. The first one is the normal theory likelihood ratio statistic explained in Section 8.9 of Anderson (1984). The second one is the coordinate-wise rank test based on aligned observations with linear score function, Puri and Sen (1971). This test is scale-invariant but not affine-invariant and an unfortunate consequence of that fact is investigated in the simulation. The last one is the sign test based on aligned observations using the Oja's criterion function mentioned in Hettmansperger and Oja (1994). Under the null hypothesis, all those tests possess the same asymptotic distribution which is $\chi^2_{2(p-1)}$ but the the likelihood ratio test has also an exact F distribution under normality. Thus, the F distribution quantile was used for this test and the asymptotic χ^2 quantile was used for the others. All tests were performed at the 5% level. The number of replications is 5000. The computations were performed using Ox version 2.20; Doornik (1999). A program to compute D_n is available from the first author upon request.

Four distributions were used to generate bivariate error terms (ϵ_1, ϵ_2). The first one is the standard normal distribution with independent marginals. The second one is the standard bivariate t-distribution with 3 degrees of freedom. The third one is a symmetric but non-elliptical distribution generated the following way. First we generate an angle ($\in [0, 2\pi]$) by $angle = \pi(B + S)$ where B is distributed as a Beta random variable with parameters .2 and .2 and S is a Bernouilli random variable (independent of B) with probability of success .5. Second, a radius R is generated as a uniform random variable on the interval $[0, 10]$. The pseudo-observation is then $R(\cos(angle), \sin(angle))$. We denote this distribution by beta-angle for simplicity. The last distribution is a non-symmetric distribution. It is simply the uniform distribution inside the section of the unit circle that is over the X-axis. We denote this distribution by the term half-uniform.

The design used is $p = 3$ treatments and $n = 40$ blocks. Treatment effects were added to the error terms in the following manner. The observations from the first treatment are shifted by $(-c, c)$, the second treatment is left untouched (i.e., with shift $(0, 0)$) and the observations from the third treatment are shifted by $(c, -c)$. For each of the four distributions, four values of c were selected, specifically, the value $c = 0$ (corresponding to the null hypothesis) and three other positive values each producing a different alternative.

In addition from using the original set of observations, a second set was produced by applying a linear transformation to the original points. This was done in order to introduce higher correlation between the two components of the vector of observations and thus examine its impact on the coordinate-wise rank test that is not affine-invariant. Specifically, the original points were multiplied by the matrix $((.8100154, .5864086), (.5864086, .8100154))$. This transformation produces a correlation of .95 between the two components for the bivariate normal and t distributions, a correlation of almost .97 for the beta-angle distribution and a correlation of almost .96 for the half-uniform distribution. Obviously, the three other tests are not affected by this transformation since they are all affine-invariant so only the coordinate-wise aligned rank test has to be recomputed. Consequently, two different observed powers are reported for this test, one for the original observations and one for the transformed observations.

The results are reported in Tables 1 to 4. The well-known adverse effects caused by the fact that the coordinate-wise test is not affine-invariant is once again illustrated. This test is competitive when the original observations are used but its power is always (sometimes considerably) lower with the transformed observations. Consequently, the following discussion will concentrate on comparing the three affine-invariant tests.

For normal errors (Table 1), the likelihood ratio test is slightly more powerful that the test D_n and the Oja sign test comes in last position.

For the heavier-tailed t distribution (Table 2), the test D_n is the better one followed closely by the Oja sign test. This goes in accordance with the fact that sign based methods are more competitive for heavy-tailed distribu-

Table 1. Observed probability of rejecting H_0 for normal errors (5000 replications)

Value	Statistics			
of c[a]	Likelihood ratio	Coordinate-wise[b]	Oja sign	D_n
0	.050	.045 (.043)	.043	.044
.14	.244	.225 (.202)	.180	.222
.21	.524	.499 (.442)	.411	.496
.29	.821	.801 (.746)	.720	.797

[a]shifts of the 3 groups are $(-c, c)$, (0,0) and $(c, -c)$.

[b]results when correlation is added are given between parentheses.

Table 2. Observed probability of rejecting H_0 for bivariate t errors (5000 replications)

Value of c^a	Statistics			
	Likelihood ratio	Coordinate-wise^b	Oja sign	D_n
0	.043	.050 (.046)	.042	.047
.17	.157	.184 (.162)	.164	.187
.28	.392	.472 (.409)	.445	.472
.38	.638	.757 (.674)	.743	.762

[a]shifts of the 3 groups are $(-c, c)$, (0,0) and $(c, -c)$.
[b]results when correlation is added are given between parentheses.

Table 3. Observed probability of rejecting H_0 for beta-angle errors (5000 replications)

Value of c^a	Statistics			
	Likelihood ratio	Coordinate-wise^b	Oja sign	D_n
0	.046	.046 (.042)	.043	.043
.34	.154	.252 (.141)	.276	.195
.57	.393	.564 (.361)	.598	.487
.80	.695	.835 (.655)	.843	.787

[a]shifts of the 3 groups are $(-c, c)$, (0,0) and $(c, -c)$.
[b]results when correlation is added are given between parentheses.

Table 4. Observed probability of rejecting H_0 for half-uniform errors (5000 replications)

Value of c^a	Statistics			
	Likelihood ratio	Coordinate-wise^b	Oja sign	D_n
0	.051	.048(.050)	.045	.048
.037	.190	.182(.160)	.143	.171
.066	.542	.522(.466)	.425	.513
.092	.865	.844(.791)	.732	.836

[a]shifts of the 3 groups are $(-c, c)$, (0,0) and $(c, -c)$.
[b]results when correlation is added are given between parentheses.

tions. Despite this, the rank based test D_n is still superior in this particular case.

For the non-elliptical but symmetric beta-angle distribution (Table 3), the Oja sign test is the more powerful followed by the test D_n.

Finally, for the non-elliptical and non-symmetric half-uniform distribution (Table 4), the likelihood ratio test is the better one followed closely by the test D_n.

In summary, we can say that the new test D_n is very competitive and offers a stable performance in the cases considered as it always comes in first or second place among the affine-invariant tests. The likelihood ratio test is not very powerful for heavy-tailed distributions while the Oja sign test does a decent job but is less powerful for the normal and half-uniform distributions. The coordinate-wise test should be used with caution because its performance depends on the correlation structure of the data.

An illustrative example is now given to conclude this section. The data are taken in Table 9.6 of Seber (1984) and are the results from an experiment conducted in the Cook Islands. Six different treatments were randomly assigned to plots of bean plants infested by the serpentine leaf miner insect in each of four different blocks. Three variables were measured on each plot but we retain only the first two here, namely, V = the number of miners per leaf and W = the weight of beans per plot (in kilograms). The data are reproduced in the form (V, W) in Table 5 for completeness.

The same tests compared in the simulation are used here to test the hypothesis of no treatment effects. It is not justified to use the asymptotic distribution for the sign and rank tests since we only have $n = 4$ blocks. Instead, for those three tests, the preferred approach is to perform a permutation test as described in the last section. The p-value were obtained using 1000000 random permutations. Also, midranks were used for the coordinate-wise test to handle tied observations. The results are given in Table 6 where, for comparison purposes, the asymptotic p-values of the sign and rank tests are also reported. For the likelihood ratio test, only the usual p-value using the F distribution is reported.

Table 5. Leaf miner insect data

	Treatment					
Block	1	2	3	4	5	6
1	(1.7, 0.4)	(1.7, 1.0)	(1.8, 0.8)	(0.1, 0.8)	(1.3, 1.0)	(1.7, 0.5)
2	(1.2, 1.4)	(1.2, 0.6)	(1.5, 0.8)	(0.2, 1.2)	(1.4, 1.2)	(2.1, 1.0)
3	(1.3, 0.6)	(1.7, 0.1)	(1.1, 0.7)	(0.3, 1.2)	(1.3, 0.8)	(2.3, 0.4)
4	(1.7, 1.1)	(1.1, 0.0)	(1.1, 0.9)	(0.0, 0.4)	(1.2, 0.6)	(1.3, 0.9)

Table 6. Results for the leaf miner insect data

	Statistic			
	likelihood ratio	Coordinate-wise	Oja sign	D_n
Value of the statistic	6.118	16.263	16.000	19.324
Permutation p-value	–	.054	.060	.008
Asymptotic p-value	.000[a]	.092	.100	.036

[a]for the likelihood ratio test, the F distribution p-value is reported.

We see that the likelihood ratio test rejects easily (p-value = .000) the hypothesis of no treatment effects. Looking at the permutation p-values, we see that the test D_n also rejects the null hypothesis (p-value = .008) at the 5% level. The other two tests are borderline with p-values just above .05. Once again, it is not justified to use the asymptotic p-values here but nevertheless, we see that the test D_n is the only one rejecting the null hypothesis among the three sign and rank tests if we look at those.

4. Concluding remarks

In this paper we have proposed a new test for treatment effects in the bivariate randomized block model. The test statistic is affine-invariant, easy to implement and valid under minimal assumptions. Its asymptotic distribution is convenient when we have a large number of blocks and we described how to perform a permutation test when we only have a small number of blocks. This last approach was illustrated with an example. The results of the simulation study showed that the new test does very well for a whole range of distributions.

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APPENDIX

Proof of Theorem 2.1

The proof follow the same lines as in Larocque, Tardif and van Eeden (2000b). Consequently, we only sketch here and focus only on the modifications that need to be made. The idea of the proof is the following: we first define approximating random variables, then we show that D_n and an approximate version of it named D_n^* are asymptotically equivalent under H_0 and we conclude by finding the asymptotic null distribution of D_n^*.

For $1 \leq j < k \leq p$, define

$$(.1)\qquad \begin{aligned} A_{jk}^* &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n} s(Y_{ij} - Y_{lk})\cos(\pi F(\theta_{ji,lk})) \\ B_{jk}^* &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n} s(Y_{ij} - Y_{lk})\sin(\pi F(\theta_{ij,lk})), \end{aligned}$$

where F is the null cumulative distribution function of $\theta_{11,21}$. Under H_0, if $i \neq l$, it is easily shown that $s(Y_{ij} - Y_{lk})$ and $\theta_{ij,lk}$ are independent. Furthermore, $E[s(Y_{ij} - Y_{lk})] = 0$ in that case. This entails that, for $1 \leq j < k \leq p$, both $E[A_{jk}^*]$ and $E[B_{jk}^*]$ converge to 0 as $n \to \infty$ under H_0.

Define

$$\begin{aligned}
\sigma_{cc1} &= E_{H_0}[s(Y_{11} - Y_{21})s(Y_{11} - Y_{31})\cos(\pi F(\theta_{11,21}))\cos(\pi F(\theta_{11,31}))] \\
\sigma_{cc2} &= E_{H_0}[s(Y_{11} - Y_{21})s(Y_{12} - Y_{31})\cos(\pi F(\theta_{11,21}))\cos(\pi F(\theta_{12,31}))] \\
\sigma_{ss1} &= E_{H_0}[s(Y_{11} - Y_{21})s(Y_{11} - Y_{31})\sin(\pi F(\theta_{11,21}))\sin(\pi F(\theta_{11,31}))] \\
\sigma_{ss2} &= E_{H_0}[s(Y_{11} - Y_{21})s(Y_{12} - Y_{31})\sin(\pi F(\theta_{11,21}))\sin(\pi F(\theta_{12,31}))] \\
\sigma_{cs1} &= E_{H_0}[s(Y_{11} - Y_{21})s(Y_{11} - Y_{31})\sin(\pi F(\theta_{11,21}))\cos(\pi F(\theta_{11,31}))] \\
\sigma_{cs2} &= E_{H_0}[s(Y_{11} - Y_{21})s(Y_{12} - Y_{31})\sin(\pi F(\theta_{11,21}))\cos(\pi F(\theta_{12,31}))]
\end{aligned}$$

and let

$$\Sigma = \begin{pmatrix} \sigma_{cc1} - \sigma_{cc2} & \sigma_{cs1} - \sigma_{cs2} \\ \sigma_{cs1} - \sigma_{cs2} & \sigma_{ss1} - \sigma_{ss2} \end{pmatrix}.$$

Straightforward calculations show that under H_0 and as $n \to \infty$,

$$\begin{aligned}
V(A_{jk}^*) &\to \sigma_{cc1} - \sigma_{cc2}, \quad V(B_{jk}^*) \to \sigma_{ss1} - \sigma_{ss2}, \\
\mathrm{Cov}(A_{jk}^*, B_{jk}^*) &\to (\sigma_{cs1} - \sigma_{cs2}), \quad \mathrm{Cov}(A_{jk}^*, A_{jl}^*) \to \frac{1}{2}(\sigma_{cc1} - \sigma_{cc2}), \\
\mathrm{Cov}(B_{jk}^*, B_{jl}^*) &\to \frac{1}{2}(\sigma_{ss1} - \sigma_{ss2}), \quad \mathrm{Cov}(A_{jk}^*, A_{lj}^*) \to -\frac{1}{2}(\sigma_{cc1} - \sigma_{cc2}), \\
\mathrm{Cov}(B_{jk}^*, B_{lj}^*) &\to -\frac{1}{2}(\sigma_{ss1} - \sigma_{ss2}), \quad \mathrm{Cov}(A_{jk}^*, B_{jl}^*) \to \frac{1}{2}(\sigma_{cs1} - \sigma_{cs2})
\end{aligned}$$

and

$$\mathrm{Cov}(A_{jk}^*, B_{lj}^*) \to -\frac{1}{2}(\sigma_{cs1} - \sigma_{cs2})$$

for j, k, l all distinct.

Let

$$D_n^* = \frac{n}{p}\sum_{j=1}^{p-1}\sum_{k=j+1}^{p} (A_{jk}^*, B_{jk}^*)\Sigma^{-1}(A_{jk}^*, B_{jk}^*)^T.$$

With only minor adjustments, the argument for the remaining of the proof is basically the same as the one in Section 2 of Larocque, Tardif and van Eeden (2000b). Specifically, we can show that under H_0 and as $n \to \infty$,

$$\sqrt{n}|A_{jk} - A^*_{jk}| \xrightarrow{P} 0 \quad \text{and} \quad \sqrt{n}|B_{jk} - B^*_{jk}| \xrightarrow{P} 0$$

for $1 \leq j < k \leq p$, which entails that

$$|D_n - D^*_n| \xrightarrow{P} 0.$$

Then, basic U-statistics theory can be used to show the asymptotic normality of the $p(p-1)$-vector $(A^*_{12}, B^*_{12}, \dots, A^*_{(p-1)p}, B^*_{(p-1)p})$ and standard results on quadratic forms give $D^*_n \xrightarrow{D} \chi^2_{2(p-1)}$ which concludes the proof of Theorem 2.1.

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