On Itô's Complex Measure Condition[1]

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Abstract

The complex measure condition was introduced by Itô (1965) as a sufficient condition on the potential term in a one-dimensional Schrödinger equation and/or corresponding linear diffusion equation to obtain a Feynman-Kac path integral formula. In this paper we provide an alternative probabilistic derivation of this condition and extend it to include any other lower order terms, i.e. drift and forcing terms, that may be present. In particular, under a complex measure condition on the lower order terms of the diffusion equation, we derive a representation of mild solutions of the Fourier transform as a functional of a jump Markov process in wavenumber space.

Keywords: Duality, multiplicative cascade, multi-type branching random walk

1 Introduction

The *complex measure condition* was introduced by Itô (1965) as a sufficient condition on the potential term $\theta(x)$ in the one-dimensional Schrödinger equation

$$\frac{h}{i}\frac{\partial \phi}{\partial t} = \frac{h^2}{2m}\frac{\partial^2 \phi}{\partial x^2} - m\,\theta(x)\phi \tag{1.1}$$

for the so-called Feynman principle of quantization. More specifically, under the condition that $\theta(x)$ is the Fourier transform of a complex measure of bounded variation on $(-\infty, \infty)$, Itô (1965) establishes the validity of the Feynman-Kac path integral formula appropriate to (1.1). Throughout "complex measure" will imply a *regular measure* with *finite total variation* without further mention. Itô (1965) further notes that his method is also applicable to the linear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{a^2}{2}\frac{\partial^2 u}{\partial x^2} + c(x)u. \tag{1.2}$$

Our first encounter with the complex measure condition arose in efforts to better understand the branching random walk associated with incompressible Navier-Stokes equations that was originally developed by LeJan and Sznitman (1997) and elaborated upon in Bhattacharya et al (2002). From this point of view the binary branching tree structure associated with the nonlinear Navier-Stokes equation is replaced by a unary tree structure for the linear diffusion equation. However in preparing the present article we learned about variants of these results for the Schrödinger equation recently given by Kolokoltsov (2000, 2002) and in references therein. We have not found a specific reference to the extension to lower order terms given here, but given the rather sizeable physics

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interests in this topic, it may be known. In any case, a main point of emphasis for us is the apparent wide scope of applicability of the recursive branching techniques in Fourier space for both linear and non-linear partial differential equations.

In this paper we shall consider the n-dimensional linear diffusion equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\sum_{1\leq i,j\leq n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(b_j(x)u)+c(x)u+g(t,x) \tag{1.3}$$

with a view toward the complex measure condition. In particular we present an approach which yields a natural probabilistic understanding of Itô's condition for (1.2). Moreover, we will see that this condition extends to a general condition on the lower order terms of (1.3). This is achieved by a "Fourier dual Feynman-Kac formula" which also includes a Fourier dual to Duhamels' principle for the equation with source term $g(t,x)$. In place of the continuous Markov diffusion process associated with (3) via the Feynman-Kac formula and Duhamel principle in real-space time solutions, we derive a representation of mild solutions of the Fourier transformed equation as a functional of a jump Markov process in wave-number space. Although not explicitly treated in this paper, the reader may note that extensions to certain higher order and/or fractional differential equations of the type described in Podlubny (1999) are also possible by this approach.

The organization is as follows. In the next section we consider a simple example that reveals the basic idea. This also includes an alternative probabilistic derivation of Itô's complex measure condition for (1.2). The third section includes the more general result which extends the complex measure condition to each of the lower order coefficients with a source term $g(t,x)$ given by (1.3). Finally, we explore more general conditions on non-constant diffusion coefficient wherein mild solutions continue to have a representation as a multiplicative functional of a Markov jump process in wave-number space.

While it is a special pleasure for the authors to submit this paper in celebration of Rabi Bhattacharya's mathematical career, Rabi could easily have been listed as a co-author in connection with his continued inspiration and working relationship with this group at Oregon State University.

2 Itô's Condition: A basic example

To understand the nature of Itô's complex measure condition for (1.2) and to prepare the way for extension to (1.3), consider the specific example

$$\frac{\partial u}{\partial t}=\frac{a^2}{2}\frac{\partial^2 u}{\partial x^2}+\cos(x)u, \quad u(0^+,x)=u_0(x), \tag{2.4}$$

for positive constant $a^2>0$.

For an integrable function f on $\mathbf{R}^n$ the Fourier transform, as well as its corresponding distributional extension, is defined with

$$\hat{f}(\xi)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n} e^{-i\xi\cdot x}f(x)dx, \quad \xi\in\mathbf{R}^n. \tag{2.5}$$

Letting $\mathcal{S}$ denote the Schwartz space, a tempered distribution $F\in\mathcal{S}'$ has corresponding Fourier transform $\hat{F}\in\mathcal{S}'$ defined by $\langle\hat{F},\phi\rangle=\langle F,\hat{\phi}\rangle, \phi\in\mathcal{S}$. Similarly, the inverse Fourier transform is defined by $\langle \check{F}, \phi \rangle = \langle F, \check{\phi} \rangle, \phi \in \mathcal{S}$, where for integrable f, $\check{f}(x) = \hat{f}(-x), x \in \mathbf{R}^n$. In particular the Fourier inversion formula is simply $(\hat{F})\check{} = (\check{F})\hat{} = F$ for $F \in \mathcal{S}'$. Thus for Dirac point mass at the origin $\delta_0 \in \mathcal{S}'$, one has $\langle \hat{\delta}_0, \phi \rangle = \langle \delta_0, \hat{\phi} \rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} \phi(x)dx = \langle (2\pi)^{-\frac{n}{2}}, \phi \rangle, \phi \in \mathcal{S}$, i.e. $\hat{\delta}_0 = (2\pi)^{-\frac{n}{2}}$. Taking $n = 1$, the respective translates to point masses at ± 1, $\delta_{\pm 1} = \tau_{\pm 1}\delta_0$, therefore have inverse Fourier transforms $\check{\delta}_{\pm 1} = \frac{1}{\sqrt{2\pi}} e^{\pm ix}$. Thus for the special choice of $\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \frac{\sqrt{2\pi}}{2}(\check{\delta}_{+1}(x) + \check{\delta}_{-1}(x))$ one has

$$\widehat{\cos} = \frac{\sqrt{2\pi}}{2}(\delta_{+1} + \delta_{-1}). \tag{2.6}$$

For reasons that will become clear and without loss of generality let us rewrite (2.4) as

$$\frac{\partial u}{\partial t} = \frac{a^2}{2}\frac{\partial^2 u}{\partial x^2} + (\cos(x) + \frac{1}{2})u - \frac{1}{2}u, \quad u(0^+, x) = u_0(x). \tag{2.7}$$

By standard Fourier transform operational calculus, a solution $u \in \mathcal{S}'$ to (2.7) will have Fourier transform satisfying the Fourier transformed evolution

$$\begin{aligned} \frac{\partial \hat{u}(t,\xi)}{\partial t} &= -\lambda(\xi)\hat{u}(t,\xi) + \frac{1}{2}(\delta_{+1} + \delta_{-1} + \delta_0) * \hat{u}(t,\xi) \\ &= -\lambda(\xi)\hat{u}(t,\xi) + \frac{1}{2}\hat{u}(t,\xi+1) + \frac{1}{2}\hat{u}(t,\xi-1) + \frac{1}{2}\hat{u}(t,\xi) \end{aligned} \tag{2.8}$$

where

$$\lambda(\xi) = \frac{a^2}{2}\xi^2 + \frac{1}{2}, \quad \xi \in \mathbf{R}. \tag{2.9}$$

Integrating with the help of the integrating factor $e^{\lambda(\xi)t}$ gives the following so-called "mild form" of the Fourier transformed equation

$$\begin{aligned} &\hat{u}(t,\xi) \\ &= e^{-\lambda(\xi)t}\hat{u}_0(\xi) + \int_0^t e^{-\lambda(\xi)s}\frac{1}{2}(\hat{u}(t-s,\xi+1) + \hat{u}(t-s,\xi-1) + \hat{u}(t-s,\xi))ds \\ &= e^{-\lambda(\xi)t}\hat{u}_0(\xi) + m(\xi)\int_0^t \lambda(\xi)e^{-\lambda(\xi)s}\{\frac{1}{2}(\frac{1}{3}\hat{u}(t-s,\xi+1) + \frac{1}{3}\hat{u}(t-s,\xi-1) \\ &+\frac{1}{3}\hat{u}(t-s,\xi))\}ds \end{aligned} \tag{2.10}$$

where the multiplicative factor $m(\xi) = 3/\lambda(\xi)$ is introduced to write the recursion (2.10) in the form of an expected value. Namely,

$$\hat{u}(t,\xi) = E_{\xi_\theta=\xi}\{\hat{u}_0(\xi)1[S_\theta > t] + m(\xi)\kappa_\theta\hat{u}(t-S_\theta,\xi_{<1>})1[S_\theta \le t]\} \tag{2.11}$$

where

i. S_θ is exponentially distributed with parameter $\lambda(\xi_\theta) = \frac{a^2\xi_\theta^2}{2} + \frac{1}{2}$.

ii. Conditionally given $\xi_\theta = \xi$, $\xi_{<1>}$ is ξ or $\xi \pm 1$ with equal probabilities $\frac{1}{3}$ each, independently of S_θ.

iii. κ_θ is 0-1 valued symmetric Bernoulli (coin tossing) random variable, independent of S_θ, ξ_θ.

iv. $m(\xi)\lambda(\xi) = 3$ for all $\xi \in \mathbf{R}$.

In view of this structure one is naturally led to the jump Markov process $\{\xi(t) : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ starting at $\xi_\theta = \xi$ in Fourier frequency space $\mathbf{R}$ having simple symmetric random walk $\xi_\theta, \xi_{<1>}, \ldots$ as discrete spatial structure and positive infinitesimal rates $\lambda(\xi)$; see Blumenthal and Getoor (1968) for detailed construction of the strong Markov process $\{\xi(t) : t \geq 0\}$ so specified. Additionally, $\kappa_\theta, \kappa_{<1>}, \kappa_{<2>}, \ldots$ is an i.i.d. Bernoulli 0-1 valued sequence independent of the jump process $\{\xi(t) : t \geq 0\}$. Now consider the multiplicative random functional defined recursively by

$$\chi(t,\theta) = \begin{cases} \hat{u}_0(\xi_\theta), & \text{if } S_\theta > t \\ m(\xi_\theta)\hat{u}_0(\xi_{<1>}), & \text{if } S_\theta \leq t < S_\theta + S_1, \kappa_\theta = 1, \\ m(\xi_\theta)m(\xi_{<1>})m(\xi_{<2>})\hat{u}_0(\xi_{<2>}) & \text{if } S_\theta + S_1 \leq t < S_\theta + S_1 + S_2, \\ & \text{and if } \kappa_\theta = \kappa_{<1>} = 1, \\ \ldots \end{cases}$$

This stochastic recursion may be expressed more succinctly as

$$\chi(t,\theta) = \begin{cases} \hat{u}_0(\xi_\theta), & \text{if } S_\theta > t \\ \kappa_\theta m(\xi_\theta)\chi(t - S_\theta, <1>) & \text{if } S_\theta \leq t. \end{cases}$$

Thus defining

$$\hat{u}(t,\xi) = E_{\xi_\theta=\xi}\chi(t,\theta) = E_{\xi_\theta=\xi}(\hat{u}_0(\xi_{N_t})\exp\{-\sum_{j=0}^{N_t-1}\log(\lambda(\xi_j)/3)\}\prod_{i=0}^{N_t-1}\kappa_{<i>}), \tag{2.12}$$

where N_t counts the number of jumps in $\{\xi(t) : t \geq 0\}$ by time t, and using the strong Markov property of the jump process $\{\xi(t) : t \geq 0\}$ one has

$$\begin{aligned} \hat{u}(t,\xi) &= E_{\xi_\theta=\xi}\chi(t,\theta) \\ &= E_{\xi_\theta=\xi}\{\chi(t,\theta)1[S_\theta > t] + \chi(t,\theta)1[S_\theta \leq t]\} \\ &= e^{-\lambda(\xi)t}\hat{u}_0(\xi) + \int_0^t e^{-\lambda(\xi)s}(\frac{1}{6}E_{\xi_{<1>}=\xi+1}\chi(t-s,<1>) \\ &+ \frac{1}{6}E_{\xi_{<1>}=\xi-1}\chi(t-s,<1>) + \frac{1}{6}E_{\xi_{<1>}=\xi}\chi(t-s,<1>))ds \\ &= e^{-\lambda(\xi)t}\hat{u}_0(\xi) + m(\xi)\int_0^t \lambda(\xi)e^{-\lambda(\xi)s}\frac{1}{2}(\frac{1}{3}\hat{u}(t-s,\xi+1) \\ &+ \frac{1}{3}\hat{u}(t-s,\xi-1) + \frac{1}{3}\hat{u}(t-s,\xi))ds \end{aligned} \tag{2.13}$$

That is, $\hat{u}(t,\xi)$ defined by (2.12) solves the mild form of the Fourier transformed equation. The complex measure condition was designed in this example to provide the simple random walk as the "complex measure". However, more generally, under Itô's complex measure condition, by decomposing $\hat{u}$ into its real and imaginary parts and then by applying respective Hahn decompositions of these into positive and negative parts, up to appropriate normalizations to a mixture of probability measures which can be absorbed into the multiplicative factors $m(\xi)$, one may obtain a random walk distribution. These details and generalizations to additional lower order terms, including a Fourier-transformed Duhamel principle, are the subject of the next section.

3 A General Complex Measure Condition

Let L denote the second order elliptic differential operator defined by

$$Lf(x) = \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} f(x) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (b_j(x) f)(x) + c(x) f(x) - \epsilon f(x), \quad (3.14)$$

where $A = ((a_{jk} : 1 \leq j, k \leq n))$ is a positive-definite matrix of real numbers (constants), $\epsilon > 0$, $b = (b_j(x) : 1 \leq j \leq n)$, and $c(x)$ have the property that the Fourier transform of each term is a complex measure. We will also permit an additional forcing term $g(t, x)$ for which the Fourier transform $\hat{g}(t, \xi)$ is assumed to exist as a function.

Precise conditions characterizing when a function is the Fourier transform of a complex measure are not known to us, though various examples and sufficient conditions are relatively easy to provide. For example, Bochner's theorem may be used to get a sufficient condition for this in terms of non-negative definiteness.

We consider the Cauchy problem

$$\frac{\partial u}{\partial t} = Lu + g, \quad u(0, x) = u_0(x). \quad (3.15)$$

In view of the linearity of the equation, the term $-\epsilon u, \epsilon > 0$, appearing in (3.15), for L defined by (3.14), causes no loss in generality for applications to equations with $\epsilon = 0$. Let

$$\lambda(\xi) = < A\xi, \xi > + \epsilon, \quad \xi \in \mathbf{R}^n, \quad (3.16)$$

where $< \cdot, \cdot >$ is ordinary dot product. Then taking Fourier transforms in (3.15) one has by the integrating factor method that

$$\begin{aligned} \hat{u}(t, \xi) = e^{-\lambda(\xi)t} \hat{u}_0(\xi) + \int_0^t \lambda(\xi) e^{-\lambda(\xi)s} \{ \\ \frac{1}{2} [\frac{1}{n+1} \sum_{j=1}^{n} \frac{2i(n+1)\xi_j}{(2\pi)^{\frac{n}{2}} \lambda(\xi)} \int_{\mathbf{R}^n} \hat{u}(t-s, \xi - \eta) \hat{b}_j(d\eta) \\ + \frac{1}{n+1} \int_{\mathbf{R}^n} \frac{2(n+1)}{(2\pi)^{\frac{n}{2}} \lambda(\xi)} \hat{u}(t-s, \xi-\eta) \hat{c}(d\eta)] + \frac{1}{2} \frac{2\hat{g}(t-s, \xi)}{\lambda(\xi)} \} ds. \quad (3.17) \end{aligned}$$

A solution of the integral equation version (3.17) of the Fourier transformed differential equation is referred to as a *mild solution* of the Fourier transform.

The hypothesis that the lower order coefficients each contributes a complex measure provides a set of up to four probability measures by considering the positive and negative parts of each of the real and imaginary parts. To obtain a random walk distribution we proceed as follows. Define positive measures q, Q on $\mathbf{R}^n$ by

$$q(B) = |\hat{c}|(B) + \sum_{j=1}^{n} |\hat{b}_j|(B), \quad B \in \mathcal{B}^n, \quad (3.18)$$

and assuming $q(\mathbf{R}^n) > 0$,

$$Q(B) = \frac{q(B)}{q(\mathbf{R}^n)}, \quad B \in \mathcal{B}^n; \tag{3.19}$$

we leave the case $q(\mathbf{R}^n) = 0$ as a simple but illuminating exercise for the reader. Then Q is a probability distribution which dominates each of the complex measures $\hat{b}_j, \hat{c}, j = 1, 2, \dots n$. Let the corresponding Radon-Nikodym derivatives be denoted by

$$r_0(\eta) = \frac{d\hat{c}}{dQ}(\eta) \tag{3.20}$$

and

$$r_j(\eta) = \frac{d\hat{b}_j}{dQ}(\eta), \quad j = 1, \dots n. \tag{3.21}$$

Now let $\{\xi_n \equiv \xi_{<n>} : n = 0, 1, 2, \dots\}$ be the random walk on $\mathbf{R}^n$ with distribution of i.i.d. displacements $\eta_1, \eta_2, \dots$ given by Q. That is, $\xi_0 = \xi$ and $\xi_n = \xi_{n-1} - \eta_n$ for $n \geq 1$. Also let $\{\xi(t) : t \geq 0\}$ denote the corresponding pure jump Markov process on $\mathbf{R}^n$ with holding times $S_\theta, S_1, \dots$ defined by the positive rates $\lambda(\xi_n) =< A\xi_n, \xi_n > +\epsilon, n = 0, 1, 2, \dots$, respectively. Let $\{N_t : t \geq 0\}$ denote the corresponding counting process of the number of jumps by time t, and let $\kappa_\theta, \kappa_{<1>}, \dots$ be i.i.d. Bernoulli 0-1 valued random variables on $(\Omega, \mathcal{F}, P)$ independent of the jump process $\{\xi(t) : t \geq 0\}$. The Bernoulli coin tossing sequence will induce "virtual states" upon the occurence of $\kappa_j = 0$. Let

$$m_0(\xi) = \frac{2(n+1)}{(2\pi)^{\frac{n}{2}}\lambda(\xi)} \tag{3.22}$$

and

$$m_j(\xi) = i\xi_j m_0(\xi), \quad j = 1, 2, \dots, n. \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.17) gives

$$\begin{aligned}
&\hat{u}(t, \xi) \\
&= e^{-\lambda(\xi)t}\hat{u}_0(\xi) + \int_0^t \lambda(\xi)e^{-\lambda(\xi)s}[\frac{1}{2}\frac{1}{n+1}\int_{\mathbf{R}^n}\sum_{j=0}^n m_j(\xi)r_j(\eta) \\
&\cdot\hat{u}(t-s, \xi-\eta)Q(d\eta) \\
&\quad +\frac{1}{2}\frac{2\hat{g}(t-s,\xi)}{\lambda(\xi)}]ds \\
&= \mathbf{E}_{\xi_\theta=\xi}\{\mathbf{1}[S_\theta \geq t]\hat{u}_0(\xi_\theta) + [m_J(\xi_\theta)r_J(\eta_1)\hat{u}(t-S_\theta, \xi_{<1>})\kappa_\theta \\
&\quad +(1-\kappa_\theta)2\frac{\hat{g}(t-S_\theta,\xi_\theta)}{\lambda(\xi_0)}]\mathbf{1}[S_\theta < t]\}.
\end{aligned} \tag{3.24}$$

Recursively define a times functional by

$$\chi(t, \theta) = \begin{cases} \hat{u}_0(\xi_\theta), & \text{if } S_\theta \geq t \\ \varphi(t - S_\theta, \xi_\theta), & \text{if } S_\theta < t, \kappa_\theta = 0, \\ r_j(\eta_1)m_j(\xi_\theta)\chi(t - S_\theta, <1>) & \text{if } S_\theta < t, \kappa_\theta = 1, J_\theta = j \in \{0, \dots, n\} \end{cases} \tag{3.25}$$

where $\{J_i : i = 1, 2, \dots\}$, $\{\kappa_i : i = 0, 1, 2, \dots\}$ are respectively i.i.d. uniformly distributed over $\{0, 1, \dots, n\}$, and are i.i.d. symmetric Bernoulli 0-1, mutually independent and independent of the jump process, and

$$\varphi(t,\xi) = \frac{2\hat{g}(t,\xi)}{\lambda(\xi)}. \tag{3.26}$$

Define the number of generations to the first "virtual state" by

$$K = \inf\{n \geq 0 : \kappa_n = 0\}, \quad K_t = N_t \wedge K. \tag{3.27}$$

Iteration of the stochastic recursion leads to

$$\begin{aligned} \hat{u}(t,\xi) = \mathbf{E}_{\xi_\theta=\xi} X(t,\theta) &= \mathbf{E}_{\xi_\theta=\xi}\{\prod_{i=1}^{N_t} r_{J_{i-1}}(\eta_i) m_{J_{i-1}}(\xi_{i-1})\kappa_{i-1} \cdot \hat{u}_0(\xi_{N_t}) \\ &+ \prod_{i=1}^{K_t} r_{J_{i-1}}(\eta_i) m_{J_{i-1}}(\xi_{i-1}) 1[N_t > K]\varphi(t - S_0 - \dots - S_{K_t}, \xi_{K_t})\} \end{aligned} \tag{3.28}$$

where an empty product is assigned value one. Note that $[\kappa_i = 1, i = 1, 2, \dots, N_t - 1] = [N_t \leq K]$ so that the two terms in (3.28) are complementary.

Theorem 3.1. *Assume that there is a number B such that $|\hat{u}_0(\xi)| \leq B$, and $|\hat{g}(t,\xi)| \leq B\lambda(\xi)/2, \xi \in \mathbf{R}^n, t \geq 0$. Then the expectation*

$$\hat{u}(t,\xi) = \mathbf{E}_{\xi_\theta=\xi} X(t,\theta)$$

is finite for each $\xi \in \mathbf{R}^n, t \geq 0$.

The proof will follow from (3.28) via a series of lemmas. Throughout f_Y denotes the probability density of the designated random variable Y.

For $\xi \in \mathbf{R}^n$, let T_ξ be an exponentially distributed random variable with parameter $\lambda(\xi)$; i.e.

$$f_{T_\xi}(s) = \lambda(\xi)e^{-\lambda(\xi)s} \quad \text{for } s > 0. \tag{3.29}$$

Lemma 3.1. *For any $\xi \in \mathbf{R}^n$ and $s > 0$*

$$m_0(\xi) f_{T_\xi}(s) \leq m_0(0) f_{T_0}(s). \tag{3.30}$$

Proof. The matrix A is positive definite, giving $\lambda(0) = \epsilon \leq \epsilon + < A\xi, \xi >= \lambda(\xi)$ for any $\xi \in \mathbf{R}^n$. Then

$$\begin{aligned} m_0(\xi) f_{T_\xi}(s) &= \frac{2(n+1)}{(2\pi)^{n/2}} e^{-\lambda(\xi)s} \\ &\leq m_0(0) f_{T_0}(s). \end{aligned} \tag{3.31}$$

□

For $r > 0$ let $g(r, \cdot)$ denote the density of a gamma random variable having shape parameter r and scale parameter ϵ; that is $g(r,s) = \epsilon^r s^{r-1} e^{-\epsilon s}/\Gamma(r)$ for $s > 0$. Also define

$$\alpha := \max_{1 \leq j \leq n} \sup_{\xi \in \mathbf{R}^n} \frac{|\xi_j|}{\sqrt{< A\xi, \xi >}} > 0. \tag{3.32}$$

Since A is positive definite, α is finite.

Lemma 3.2. *For any $\xi \in \mathbf{R}^n$, $j = 1, \ldots, n$, and $s > 0$*

$$|m_j(\xi)| f_{T_\xi}(s) \le m_0(0)\alpha(\frac{\epsilon\pi}{2e})^{1/2} g(1/2, s). \tag{3.33}$$

Proof: First note that $\sup_{y>0} ye^{-y^2} = (2e)^{-1/2}$. Then for any $\xi \in \mathbf{R}^n$, $s > 0$, and $j = 1, \ldots, n$

$$\begin{aligned} |m_j(\xi)| f_{T_\xi}(s) &= m_0(0)\epsilon|\xi_j| e^{-\lambda(\xi)s} \\ &\le m_0(0)\alpha\, \epsilon\sqrt{< A\xi, \xi >} e^{-<A\xi,\xi>s-\epsilon s} \\ &\le m_0(0)\alpha\, \epsilon(2es)^{-1/2} e^{-\epsilon s} \\ &= m_0(0)\alpha\, (\frac{\epsilon\pi}{2e})^{1/2} g(1/2, s). \end{aligned} \tag{3.34}$$

□

Lemma 3.3. *For $\beta > 0$*

$$\sum_{k\ge 0} \frac{\beta^k}{\Gamma(k/2+1)} \le (1 + \frac{2\beta}{\sqrt{\pi}})e^{\beta^2}. \tag{3.35}$$

Proof: For $j \ge 1$

$$\Gamma(j + \frac{1}{2}) = (j - \frac{1}{2}) \cdots \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} \ge (j-1)!\, \frac{\sqrt{\pi}}{2} = \Gamma(j)\frac{\sqrt{\pi}}{2},$$

so

$$\begin{aligned} \sum_{k\ge 0} \frac{\beta^k}{\Gamma(k/2+1)} &= \sum_{j\ge 0} (\frac{\beta^{2j}}{\Gamma(j+1)} + \frac{\beta^{2j+1}}{\Gamma(j+\frac{3}{2})}) \\ &\le \sum_{j\ge 0} \frac{\beta^{2j}}{\Gamma(j+1)}(1 + \frac{2\beta}{\sqrt{\pi}}) \\ &= (1 + \frac{2\beta}{\sqrt{\pi}})e^{\beta^2}. \end{aligned} \tag{3.36}$$

□

Remark The function $E_\alpha(z) := \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha z+1)}$ is the Mittag-Leffler function, and $E_{\frac{1}{2}}(z) = e^{z^2}\{1 + \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt\}$, e.g. see Feller (1971), Podlubny (1999). However the simple bound in the above lemma is adequate for the present development.

Proof of Theorem 3.1: We can rewrite (3.28) as

$$\begin{aligned} \hat{u}(t,\xi) &= E_\xi \prod_{1}^{K\wedge N_t} r_{J_{i-1}}(\eta_i) m_{J_{i-1}}(\xi_{i-1}) [\hat{u}_0(\xi_{N_t}) \mathbf{1}_{[N_t \le K]} \\ &\quad + \varphi(t - S_0 - \cdots - S_K, \xi_K) \mathbf{1}_{[N_t > K]}] \end{aligned} \tag{3.37}$$

where K is a geometrically distributed random variable with parameter $1/2$, independent of the ξ_i's and the S_i's; that is $P(K = k) = 2^{-k-1}$ for $k \ge 0$. Observe that for any $\xi \in \mathbf{R}^n$, $j = 0, \ldots, n$

$$|r_j(\xi)| \le q(\mathbf{R}^n) := q < \infty. \tag{3.38}$$

The hypothesized bounds on $\hat{u}_0$ and $\hat{g}$ along with $P(K < \infty) = 1$ give

$$\begin{aligned} |\hat{u}(t,\xi)| &\leq B\, E_\xi \prod_1^{K\wedge N_t} q|m_{J_{i-1}}(\xi_{i-1})| \\ &= B\sum_{k\geq 0} E_\xi q^k \prod_1^k |m_{J_{i-1}}(\xi_{i-1})| \mathbf{1}_{[K\wedge N_t = k]}. \end{aligned} \tag{3.39}$$

For $k \geq 1$, set

$$\begin{aligned} A_k &:= E_\xi \prod_1^k |m_{J_{i-1}}(\xi_{i-1})| \mathbf{1}_{[K\wedge N_t = k]} \\ &= E_\xi \prod_1^k |m_{J_{i-1}}(\xi_{i-1})| (\mathbf{1}_{[N_t = k,\, K\geq k]} + \mathbf{1}_{[K=k,\, N_t > k]}) \\ &= E_\xi \prod_1^k |m_{J_{i-1}}(\xi_{i-1})| (2^{-k}\int_0^t f_{\sum_0^{k-1} S_m}(s) e^{-\lambda(\xi_k)(t-s)} ds \\ &\quad + 2^{-k-1}\int_0^t f_{\sum_0^k S_m}(s) ds) \end{aligned} \tag{3.40}$$

It is helpful to introduce the binomial random variable

$$Y_k = \sum_{i=0}^{k-1} \mathbf{1}_{[J_i = 0]} \tag{3.41}$$

with parameters k and $\frac{1}{n+1}$. Setting $m_0(0) = M$, and $\beta = \alpha(\pi/2e)^{1/2}$, Lemmas 3.1 and 3.2 give

$$|m_{J_{i-1}}(\xi_{i-1})| f_{S_{i-1}}(s) \leq \begin{cases} M\, g(1,s) & \text{if } J_{i-1} = 0 \\ \beta\epsilon^{1/2} M\, g(1/2,s) & \text{if } J_{i-1} \neq 0. \end{cases} \tag{3.42}$$

Combining this with the convolution properties of gamma densities gives

$$\prod_1^k |m_{J_{i-1}}(\xi_{i-1})| f_{\sum_0^{k-1} S_m}(s) \leq M^k \beta^{k-Y_k} \epsilon^{\frac{k-Y_k}{2}} g(\frac{k+Y_k}{2}, s). \tag{3.43}$$

Thus

$$\begin{aligned} A_k &\leq 2^{-k} M^k E_\xi \beta^{k-Y_k} \epsilon^{\frac{k-Y_k}{2}} \Big(\int_0^t g(\frac{k+Y_k}{2}, s) e^{-\lambda(\xi_k)(t-s)} ds \\ &\quad + 2^{-1} \int_0^t \int_0^r g(\frac{k+Y_k}{2}, r) \lambda(\xi_k) e^{-\lambda(\xi_k)(s-r)} dr\, ds \Big). \end{aligned} \tag{3.44}$$

This is bounded as follows:

$$\begin{aligned}
&\int_0^t g(\frac{k+Y_k}{2}, s)e^{-\lambda(\xi_k)(t-s)}ds \\
&\leq \frac{1}{\epsilon}\int_0^t g(\frac{k+Y_k}{2}, s)g(1, t-s)ds \\
&= \frac{1}{\epsilon}g(\frac{k+Y_k}{2}+1, t) \\
&\leq \frac{(\epsilon t)^{\frac{k+Y_k}{2}}}{\Gamma(\frac{k+Y_k}{2}+1)} \\
&\leq \frac{2(\epsilon t)^{\frac{k+Y_k}{2}}}{\pi^{1/2}\Gamma(\frac{k}{2}+1)}
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
&\int_0^t \int_{r=0}^s g(\frac{k+Y_k}{2}, r)\lambda(\xi_k)e^{-\lambda(\xi_k)(s-r)}dr\, ds \\
&= \int_{r=0}^t g(\frac{k+Y_k}{2}, r)e^{\lambda(\xi_k)r}\int_{s=r}^t \lambda(\xi_k)e^{-\lambda(\xi_k)s}ds\, dr \\
&= \int_{r=0}^t g(\frac{k+Y_k}{2}, r)(1-e^{-\lambda(\xi_k)(t-r)})dr \\
&\leq \int_{r=0}^t \frac{\epsilon^{\frac{k+Y_k}{2}}r^{\frac{k+Y_k}{2}-1}}{\Gamma(\frac{k+Y_k}{2})}dr \\
&= \frac{(\epsilon t)^{\frac{k+Y_k}{2}}}{\Gamma(\frac{k+Y_k}{2}+1)} \leq \frac{2(\epsilon t)^{\frac{k+Y_k}{2}}}{\pi^{1/2}\Gamma(\frac{k}{2}+1)}.
\end{aligned} \tag{3.46}$$

Combining the above bounds one has

$$\begin{aligned}
A_k &\leq \frac{3M^k 2^{-k}}{\pi^{1/2}\Gamma(\frac{k}{2}+1)}E_\xi \beta^{k-Y_k}\epsilon^{\frac{k-Y_k}{2}}(\epsilon t)^{\frac{k+Y_k}{2}} \\
&= \frac{3}{\pi^{1/2}}(\frac{M\epsilon}{2(n+1)})^k\frac{(\beta n t^{1/2}+t)^k}{\Gamma(\frac{k}{2}+1)}.
\end{aligned} \tag{3.47}$$

Then (3.39) together with Lemma 3.3 gives

$$\begin{aligned}
|\hat{u}(t,\xi)| &\leq B(1+\sum_{k\geq 1}q^k A_k) \\
&\leq \frac{3B}{\pi^{1/2}}(1+\frac{M\epsilon(\beta n t^{1/2}+t)}{\pi^{1/2}(n+1)})e^{\frac{M^2\epsilon^2}{4(n+1)}t(\beta n+t^{1/2})^2} \\
&= \frac{3B}{\pi^{1/2}}(1+2^{-\frac{n-2}{2}}\pi^{-\frac{n+1}{2}}(\beta n t^{1/2}+t))e^{(2\pi)^{-n}t(\beta n+t^{1/2})^2} < \infty.
\end{aligned} \tag{3.48}$$

□

It is convenient to introduce the Fourier dual operator $\hat{L}$ defined for $\xi \in \mathbf{R}^n$ by

$$\hat{L}f(\xi) = -(< A\xi, \xi > +\epsilon)\hat{f}(\xi) + \frac{i\xi}{\sqrt{2\pi}}\cdot\hat{b}*\hat{f}(\xi) - \frac{1}{\sqrt{2\pi}}\hat{c}*\hat{f}(\xi), \tag{3.49}$$

where the middle convolution is componentwise, $\hat{b} * \hat{f}(\xi) = (\int_{\mathbf{R}^n} f(\xi - \eta)\hat{b}_j(d\eta) : 1 \leq j \leq n)$.

Corollary 3.1. *Under the conditions of the Theorem 3.1 one has that $\hat{u}(t,\xi)$ is a mild solution of (3.17).*

Proof. Conditioning (3.28) one has by independence of $\{(\xi_n, \kappa_n, S_n) : n \geq 1\}$ and S_θ, κ_θ, using the substitution lemma for conditional expectations, that

$$\begin{aligned} \hat{u}(t,\xi) \quad &= \mathbf{E}_{\xi_\theta = \xi}(\mathbf{1}[S_\theta > t]X(t,\theta) + \mathbf{1}[S_\theta \leq t]X(t,\theta)) \\ &= e^{-\lambda(\xi)t}\hat{u}_0(\xi) \\ &+ \mathbf{E}_{\xi_{<1>}}(\mathbf{1}[S_\theta \leq t, \kappa_\theta = 1] r_{J_\theta}(\eta_1) m_{J_\theta}(\xi) X(t - S_\theta, <1>)) \\ &+ \mathbf{E}_{\xi_{<1>}}(\mathbf{1}[S_\theta \leq t, \kappa_\theta = 0]\varphi(t - S_\theta, \xi_{<1>})X(t - S_\theta, <1>)). \end{aligned} \tag{3.50}$$

□

4 Nonconstant Diffusion Coefficient and the Complex Measure Conditions

In this section we explore the complex measure condition on the lower order coefficients when the highest order diffusion coefficient is not necessarily constant. Since the results are somewhat exploratory, we restrict to one dimension and consider the Cauchy problem

$$\frac{\partial u}{\partial t} = Lu + g, \quad u(0,x) = u_0(x) \tag{4.51}$$

for

$$Lf(x) = (a(x)f(x))_{xx} + (b(x)f(x))_x + c(x)f(x) - \epsilon f(x). \tag{4.52}$$

Letting $\lambda(\xi) = \epsilon + \frac{\hat{a}(\{0\})}{\sqrt{2\pi}}\xi^2$, the Fourier transformed equation may be expressed in the mild form

$$\begin{aligned} \hat{u}(t,\xi) \quad &= \hat{u}_0(\xi)e^{-\lambda(\xi)t} + \int_0^t \lambda(\xi)e^{-\lambda(\xi)s} \\ &\{\frac{\xi^2}{\lambda(\xi)}\int_R \hat{u}(t-s,\xi-\eta)\alpha(d\eta) + \frac{i\xi}{\sqrt{2\pi}\lambda(\xi)}\int_R \hat{u}(t-s,\xi-\eta)\hat{b}(d\eta) \\ &\frac{1}{\sqrt{2\pi}\lambda(\xi)}\int_R \hat{u}(t-s,\xi-\eta)\hat{c}(d\eta) + \hat{g}(t-s,\xi)\}ds, \end{aligned} \tag{4.53}$$

where α is the complex measure defined by

$$\alpha := \frac{1}{\sqrt{2\pi}}(\hat{a}(\{0\})\delta_0 - \hat{a}). \tag{4.54}$$

For later notational convenience let

$$\gamma = \frac{\hat{a}(\{0\})}{\sqrt{2\pi}}. \tag{4.55}$$

To make the probabilistic construction in Fourier frequency space under the complex measure condition on the lower order coefficients we will require a condition of the following form on the leading order coefficient $a(x)$.
CONDITION A: *Assume that $\hat{a}$ is a complex measure and*

$$\hat{a}(\{0\}) > |\hat{a}|(\mathbf{R}\backslash\{0\}), \tag{4.56}$$

where $|\hat{a}|$ denotes the corresponding total variation measure. One may note that in the case of constant coefficient $a(x) = a$, Condition A is merely the condition that $a > 0$.

The stochastic jump Markov process $\{\xi(t) : t \geq 0\}$ and multiplicative functional χ in this setting are defined as follows. First let q, Q denote the measure and probability distribution defined by the coefficients b, c exactly as in (3.18)-(3.19) with dimension $n = 1$. Similarly, one defines

$$r_0 = \frac{d\hat{c}}{dQ}, \qquad r_1 = \frac{d\hat{b}}{dQ} \tag{4.57}$$

precisely as in (3.20)-(3.21). In addition, let $\alpha_0 = \frac{|\alpha|}{|\alpha|(\mathbf{R})}$ be the probability defined by normalizing the total variation measure of the complex measure α defined above in (4.55). Now define

$$r_2 = \frac{d\alpha}{d\alpha_0}. \tag{4.58}$$

Next let $\{J_i : i \geq 1\}$ and $\{\kappa_i : i \geq 1\}$ be mutually independent sequences of i.i.d. symmetric Bernoulli $0-1$ random variables as defined earlier for (3.25) with $n = 1$. Additionally, let $\{\sigma_i : i \geq 1\}$ be a sequence of independent Bernoulli $0-1$ random variables, independent of $\{J_i : i \geq 1\}$ and $\{\kappa_i : i \geq 1\}$, and distributed according to the law

$$P(\sigma_i = 1) = p = \frac{|\hat{a}|(\mathbf{R}\backslash\{0\})}{\hat{a}(\{0\})} \in (0, 1). \tag{4.59}$$

For future reference, one should also note that $p = \gamma^{-1}|\alpha|(\mathbf{R})$.

Now the increments $\{\eta_i : i \geq 1\}$ of the jump Markov process are i.i.d. and independent of the above coin tossing sequences $\{J_i\}, \{\kappa_i\}$ with

$$P(\eta_i \in d\eta|\sigma_{i-1}) = \sigma_{i-1}\alpha_0(d\eta) + (1 - \sigma_{i-1})Q(d\eta). \tag{4.60}$$

Accordingly the skeletal jump process starting at $\xi_0 = \xi$ is given by

$$\xi_k = \xi - \sum_{i=1}^{k} \eta_i, \quad k \geq 1. \tag{4.61}$$

Conditionally on the spatial random walk $\{\xi_k\}$ the holding times $\{S_k : k \geq 1\}$ for the jump Markov process may be defined by specifying infinitesimal rates $\lambda(\xi_k)$, (e.g. see Blumenthal and Getoor (1968), Bhattacharya and Waymire (1990)), where

$$\lambda(\xi) = \epsilon + \frac{\hat{a}(\{0\})}{\sqrt{2\pi}}\xi^2 = \epsilon + \gamma\xi^2. \tag{4.62}$$

Recall that $\epsilon > 0$ is a parameter of (4.52).

Finally, the multiplicative times functional χ is recursively defined with scale factors

$$m_j(\xi) = \begin{cases} \frac{4}{(1-p)\sqrt{2\pi}\lambda(\xi)}, & \text{if } j = 0, \\ \frac{4i\xi}{(1-p)\sqrt{2\pi}\lambda(\xi)}, & \text{if } j = 1 \\ \frac{\xi^2}{p\lambda(\xi)} & \text{if } j = 2, \end{cases} \tag{4.63}$$

and rescaled forcing term $\varphi(t,\xi) = 2((1-p)\lambda(\xi))^{-1}\hat{g}(t,\xi)$, by the following stochastic recursion

$$\chi(t,\theta) = \begin{cases} \hat{u}_0(\xi_\theta), & \text{if } S_\theta \geq t \\ \varphi(t - S_\theta, \xi), & \text{if } S_\theta < t, \kappa_\theta = 0, \\ r_j(\eta_1)m_j(\xi_\theta)\chi(t - S_\theta, < 1 >) & \text{if } S_\theta < t, \kappa_\theta = 1, \sigma_\theta = 0, J_\theta = j \in \{0,1\} \\ r_2(\eta_1)m_2(\xi_\theta)\chi(t - S_\theta, < 1 >) & \text{if } S_\theta < t, \kappa_\theta = 1, \sigma_\theta = 1, J_\theta = j \in \{0,1\} \end{cases} \tag{4.64}$$

We are now ready to state the theorem in this setting.

Theorem 4.1. *Assume that the diffusion coefficient a satisfies Condition A. If $\hat{b}$ and $\hat{c}$ are complex measures and if there is a number B such that $|\hat{u}_0(\xi)| \leq B$, and $|\hat{g}(t,\xi)| \leq B\lambda(\xi)/2, \xi \in \mathbf{R}^n, t \geq 0$. Then a mild solution $\hat{u}(t,\xi)$ for the Fourier transformed equation is given by the stochastic representation*

$$\hat{u}(t,\xi) = \mathbf{E}_{\xi_\theta=\xi}\chi(t,\theta), \quad \xi \in \mathbf{R}^n, t \geq 0.$$

Proof. The focus is on the implied convergence of the expectation. We leave it to the reader to use the strong Markov property of the underlying jump process to verify the equation from the stochastic recursion defined by the times functional χ. To establish integrability first note that

$$|r_2(\eta)| \leq |\alpha|(\mathbf{R}) = |\hat{a}|(\mathbf{R}\backslash\{0\})$$

and

$$0 \leq m_2(\xi) \leq (p\gamma)^{-1} = \frac{1}{|\hat{a}|(\mathbf{R}\backslash\{0\})}.$$

Thus

$$|r_2(\eta)m_2(\xi)| \leq 1, \quad \forall \eta, \xi. \tag{4.65}$$

Now with counting random variables K, N_t, and K_t defined exactly as in (3.27), where N_t is again the number of jumps by time t, one has upon iteration of the stochastic recursion that

$$\begin{aligned} \hat{u}(t,\xi) &= \mathbf{E}_{\xi_\theta=\xi}\chi(t,\theta) = \mathbf{E}_{\xi_\theta=\xi}\prod_{i=1}^{K_t} r_{J_{i-1}}(\eta_i)m_{J_{i-1}}(\xi_{i-1}))^{1-\sigma_{i-1}} \\ &\quad \cdot (r_2(\eta_i)m_2(\xi_{i-1}))^{\sigma_{i-1}}\kappa_{i-1}\{\hat{u}_0(\xi_{N_t})1[N_t \leq K] \\ &\quad + \varphi(t - S_0 - \cdots - S_{K_t}, \xi_{K_t})1[N_t > K]\} \end{aligned} \tag{4.66}$$

where an empty product is assigned value one. Therefore, letting $q = q(\mathbf{R})$, one has

$$|\hat{u}(t,\xi)| \leq B\sum_{k\geq 0} E_{\xi_\theta=\xi}\prod_0^k |qm_{J_{i-1}}(\xi_{i-1})|^{1-\sigma_{i-1}}1[K \wedge N_t = k].$$

For each k it is helpful to introduce the mutually dependent pair of binomial distributed random variables

$$X_k = \sum_{i=0}^{k-1}(1-\sigma_i)1[J_i = 1], \quad Y_k = \sum_{i=0}^{k-1}(1-\sigma_i)1[J_i = 0]. \tag{4.67}$$

Also in the case that $\sigma_0 = \cdots = \sigma_{k-1} = 0$ set $h_k = 1$, else let h_k denote the density of $\sum_0^{k-1}\sigma_i S_i$ conditional on the σ_i's, J_i's, and ξ_i's. Then, proceeding similarly as in the proof of Theorem 3.1, consider

$$\begin{aligned}
A_k := & E_{\xi_\theta=\xi}\prod_0^{k-1}|qm_{J_i}(\xi_i)|^{1-\sigma_i}1[K\wedge N_t = k] \\
\leq \ & E_{\xi_\theta=\xi}\prod_0^{k-1}|qm_{J_i}(\xi_i)|^{1-\sigma_i}1[N_t \geq k]P(K\geq k) \\
= \ & 2^{-k}E_{\xi_\theta=\xi}\prod_0^{k-1}|qm_{J_i}(\xi_i)|^{1-\sigma_i}\int_0^t f_{\sum_0^{k-1}S_m}(s)ds \\
\leq \ & 2^{-k}E_{\xi_\theta=\xi}|qm_0(0)|^{X_k+Y_k}(\frac{\epsilon\pi}{2e\gamma})^{\frac{X_k}{2}}\int_0^t g(\frac{X_k}{2}+Y_k,s)h_k(t-s)ds \\
& \cdot(1[\frac{X_k}{2}+Y_k\geq 1]+1[X_k=1,Y_k=0]+1[X_k+Y_k=0]) \\
\leq \ & 2^{-k}E_{\xi_\theta=\xi}\{|qm_0(0)|^{X_k+Y_k}(\frac{\pi}{2e\gamma})^{\frac{X_k}{2}}\frac{\epsilon^{\frac{X_k}{2}+Y_k}t^{\frac{X_k}{2}+Y_k-1}}{\Gamma(\frac{X_k}{2}+Y_k)}1[\frac{X_k}{2}+Y_k\geq 1] \\
& +qm_0(0)(\frac{\epsilon\pi}{2e\gamma})^{1/2}1[X_k=1,Y_k=0]+p^k\} \\
= \ & 2^{-k}E_{\xi_\theta=\xi}\{(\frac{4q^2}{(1-p)^2e\gamma})^{\frac{X_k}{2}}(\frac{4q}{(1-p)\sqrt{2\pi}})^{Y_k}\frac{t^{\frac{X_k}{2}+Y_k-1}}{\Gamma(\frac{X_k}{2}+Y_k)}1[\frac{X_k}{2}+Y_k\geq 1] \\
& +\frac{2q}{(1-p)\sqrt{e\pi\gamma}}1[X_k=1,Y_k=0]+p^k\}
\end{aligned}$$

Setting $\beta = \frac{4q}{(1-p)}\max\{\frac{q}{(1-p)e\gamma},\frac{1}{\sqrt{2\pi}}\}$, we have

$$A_k \leq \frac{2^{-k}}{t}E_{\xi_\theta=\xi}\frac{(\beta t)^{\frac{X_k}{2}+Y_k}}{\Gamma(\frac{X_k}{2}+Y_k)}1[\frac{X_k}{2}+Y_k\geq 1]+\frac{kq}{2\sqrt{e\gamma}}(\frac{p}{2})^{k-1}+(\frac{p}{2})^k.$$

Thus we obtain

$$\begin{aligned}
|\hat{u}(t,\xi)| &\leq B\{\sum_{k\geq 0}\frac{2^{-k}(t\wedge 1)}{t}E_{\xi_\theta=\xi}\frac{(\beta t)^{\frac{X_k}{2}+Y_k}}{\Gamma(\frac{X_k}{2}+Y_k)}1[\frac{X_k}{2}+Y_k\geq 1] \\
&\quad +\frac{2q}{\sqrt{e\gamma}(2-p)^2}+\frac{2}{2-p} \\
&= Bt^{-1}(t\wedge 1)\sum_{k\geq 1}\sum_{2\leq n\leq 2k}\frac{2^{-k}(\beta t)^{\frac{n}{2}}}{\Gamma(n/2)}P(X_k+2Y_k=n) \\
&\quad +\frac{2B}{2-p}(\frac{q}{\sqrt{e\gamma}(2-p)}+1) \\
&\leq 2B(t\wedge 1)t^{-1}\sum_{n\geq 2}\frac{(\beta t/2)^{\frac{n}{2}}}{\Gamma(n/2)}+\frac{2B}{2-p}(\frac{q}{\sqrt{e\gamma}(2-p)}+1) \\
&\leq B(t\wedge 1)\beta(1+\sqrt{2\beta t/\pi})e^{\frac{\beta t}{2}}+\frac{2B}{2-p}(\frac{q}{\sqrt{e\gamma}(2-p)}+1).
\end{aligned}$$

This establishes the desired convergence. □

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