Ordered Triple Designs and Wreath Products of Groups

Cheryl E. Praeger and Csaba Schneider

Abstract

We explore an interesting connection between a family of incidence structures and wreath products of finite groups.

Keywords: ordered triple designs; permutation groups; wreath products; product action; innately transitive groups; maximal subgroups of symmetric groups

1 Introduction

The problem discussed in this paper arose from a study in [2] of the set of primitive maximal subgroups of a finite symmetric group $\mathsf{Sym}\,\Omega$ containing a given subgroup of $\mathsf{Sym}\,\Omega$. Application of group theoretic results, depending on the classification of finite simple groups, reduced the problem of describing one family of such maximal subgroups to a problem concerning a certain kind of incidence structures. We chose this topic because of the unexpected links between several types of mathematical objects.

For a finite set Ω the maximal subgroups of $\mathsf{Sym}\,\Omega$ may be divided into several disjoint families: intransitive maximal subgroups, imprimitive maximal subgroups, and several families of primitive maximal subgroups; see [6]. A given permutation group G on Ω may be contained in many maximal subgroups of $\mathsf{Sym}\,\Omega$. The intransitive and imprimitive maximal overgroups of G may be determined from the G-orbits and the G-invariant partitions of Ω. However, determining the primitive overgroups of G is a difficult problem in general. It has been essentially solved in [6] and [9] in the case where G itself is primitive, and even this case required significant use of the finite simple group classification. In [2] we were concerned with a more general situation: the groups G of interest were innately transitive, in other words, they contain a minimal normal subgroup that is transitive. The maximal overgroups of G studied in [2] were wreath products in product action (see Section 3 for the definition of wreath products and product actions). Investigating such overgroups led to a study of certain incidence structures discussed in Section 2. Their connection with overgroups of innately transitive groups is described in more detail in Section 3, and a construction is given in Section 4.

2 Suitable ordered triple designs

Describing and constructing a certain family of overgroups of innately transitive groups required incidence structures of the type introduced in the following definition. Note that a permutation group $H \leqslant \mathrm{Sym}\,\Omega$ acts naturally on the set

$$\Omega^{(3)} = \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1,\ \alpha_2,\ \alpha_3 \text{ are distinct points of } \Omega\}$$

of triples of distinct points of Ω via $h : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1^h, \alpha_2^h, \alpha_3^h)$ for all $h \in H$ and $(\alpha_1, \alpha_2, \alpha_3) \in \Omega^{(3)}$. We denote by S_3 the symmetric group on a set of size 3.

Definition 1

(a) An *ordered triple design* $\mathcal{H}$ is a pair $(\Omega, \mathcal{T})$ in which Ω is a finite set, and $\mathcal{T}$ is a subset of $\Omega^{(3)}$, and for each $i \in \{1,2,3\}$ and each $\alpha \in \Omega$, the number of triples in $\mathcal{T}$ containing the point α in position i is independent of α, namely it is $|\mathcal{T}|/|\Omega|$.

(b) An ordered triple design $(\Omega, \mathcal{T})$ is said to be *suitable* if there exists $H \leqslant \mathrm{Sym}\,\Omega$ that leaves $\mathcal{T}$ invariant and is transitive on both Ω and $\mathcal{T}$. For such a group H, the subgroup A of S_3 induced on $\{\alpha_1, \alpha_2, \alpha_3\}$ by the setwise stabiliser $H_{\{\alpha_1,\alpha_2,\alpha_3\}}$ is the same (up to isomorphism) for all triples $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}$. Thus we also say that $(\Omega, \mathcal{T})$ is *A-suitable relative to H*.

(c) If $H \leqslant \mathrm{Sym}\,\Omega$ and $A \leqslant \mathsf{S}_3$, such that H is transitive on Ω, then an H-orbit $\mathcal{T}$ in $\Omega^{(3)}$ is said to be *A-suitable* if, for $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}$, the setwise stabiliser in H of $\{\alpha_1, \alpha_2, \alpha_3\}$ induces a permutation group isomorphic to A on $\{\alpha_1, \alpha_2, \alpha_3\}$.

Definition 1(c) enables us to characterise A-suitable ordered triple designs group theoretically. In Section 3 we explain how ordered triple designs arose in [2], while in Section 4 we show that each suitable ordered triple design arises in relation to our problem.

The proof of the following lemma is easy and is omitted.

Lemma 1

Let H be a transitive permutation group on a finite set Ω, let $\mathcal{T}$ be an H-orbit in $\Omega^{(3)}$, and let $A \leqslant \mathsf{S}_3$. Then $\mathcal{T}$ is A-suitable if and only if $(\Omega, \mathcal{T})$ is an A-suitable ordered triple design relative to H.

The concepts of generously 2-transitive and almost generously 2-transitive permutation groups were introduced by Neumann [7]. In our terminology, a permutation group H acting on Ω is generously 2-transitive if and only if every H-orbit in $\Omega^{(3)}$ is S_3-suitable; and H is almost generously 2-transitive if and only if every H-orbit in $\Omega^{(3)}$ is A_3-suitable or S_3-suitable. It was shown in [7] that each almost generously 2-transitive group is 2-transitive with the single exception of A_3. The classification of 2-transitive groups is a consequence of the finite simple group classification, so the generously and almost generously 2-transitive groups can be regarded as known.

In our construction of innately transitive groups in Section 4, part of the input data is an A-suitable ordered triple design relative to H. It turns out that the structure of the group that is the result of our construction depends on A. We were interested in examples where A was either a cyclic group of order 2 or the trivial group. Hence the question arose as to how prevalent 1-suitable ordered triple designs might be. For some transitive permutation groups H on a set Ω, every H-orbit in $\Omega^{(3)}$ is 1-suitable. Such groups are characterised in Theorem 4 below. Here a permutation group G on a set Ω is said to be *semiregular* if the only element of G that fixes a point of Ω is the identity element.

Theorem 4
Let H be a transitive permutation group on a finite set Ω. Then all H-orbits in $\Omega^{(3)}$ are 1-suitable if and only if $|H|$ is not divisible by 3 and a Sylow 2-subgroup of H is semiregular.

PROOF. Let us assume that $|H|$ is not divisible by 3, and a Sylow 2-subgroup of H is semiregular. This implies that an element of H with even order has no fixed points in Ω. Let $\mathcal{T}$ be an H-orbit in $\Omega^{(3)}$ and $(\kappa_1, \kappa_2, \kappa_3) \in \mathcal{T}$. Suppose that $g \in H$ and g stabilises the set $\kappa = \{\kappa_1, \kappa_2, \kappa_3\}$, and consider the permutation g' induced by g on κ. Since the order $|g'|$ of g' divides the order of g, and hence divides $|H|$, we have that $|g'| \neq 3$. If $|g'| = 2$ then g has even order and g', and hence also g, fixes one element of the κ_i, which is a contradiction. Hence $g' = 1$ and it follows that $\mathcal{T}$ is 1-suitable.

Suppose now that every H-orbit in $\Omega^{(3)}$ is 1-suitable. If $|H|$ is divisible by 3, then there is an element $g \in H$ of order 3. If $\{\kappa_1, \kappa_2, \kappa_3\}$ is a $\langle g\rangle$-orbit of size 3, then $(\kappa_1, \kappa_2, \kappa_3)^H \subseteq \Omega^{(3)}$ is not 1-suitable. Hence $|H|$ is not divisible by 3. Suppose now that there is a non-identity 2-element g in H that fixes a point $\kappa_1 \in \Omega$. Then g^k is an involution, for some k, g^k fixes κ_1, and if $\{\kappa_2, \kappa_3\}$ is a $\langle g^k\rangle$-orbit in Ω with size 2, then $(\kappa_1, \kappa_2, \kappa_3)^H \subseteq \Omega^{(3)}$ is not 1-suitable. Hence a Sylow 2-subgroup of H is semiregular. □

The family of groups that satisfy the conditions of Theorem 4 contains some primitive and some insoluble examples, though most groups in this family are imprimitive and soluble.

Remark 1
Let H satisfy the conditions of Theorem 4.

(a) The only finite simple groups T for which 3 does not divide $|T|$ are the Suzuki groups $\mathsf{Sz}(q)$, where $q = 2^{2a+1} \geqslant 8$; see pages 8–9 of [5]. So if H is insoluble then the non-abelian composition factors of H are all isomorphic to $\mathsf{Sz}(q)$ for various q. There are certainly some insoluble examples H. For instance if $H = \mathsf{Sz}(q)$, and L is any subgroup of H such that $|L|$ is odd, then the action of H by right multiplication on $\{Lx \mid x \in H\}$ satisfies the conditions of Theorem 4.

(b) A transitive permutation group H is *primitive* on Ω if and only if the stabiliser H_α in H of a point $\alpha \in \Omega$ is a maximal subgroup. The conditions in Theorem 4 are equivalent to requiring $3 \nmid |H|$ and $2 \nmid |H_\alpha|$. Since all maximal subgroups of $\mathsf{Sz}(q)$ have even order (see [11]), none of the examples given in paragraph (a) are primitive.

Indeed, it is not difficult, using the O'Nan–Scott Theorem (see Theorem 4.1A of [4]), to show that if H is primitive and satisfies the conditions of Theorem 4, then H is a semidirect product $N \rtimes L$ where $N \cong \mathbb{Z}_p^d$, with p a prime, $p \neq 3$, and $L \leqslant \mathsf{GL}_d(p)$, such that $\gcd(6,|L|) = 1$ and L leaves no non-trivial, proper subspace of $\mathbb{Z}_p^d$ invariant. If $d = 1$ then any subgroup L with $\gcd(6,|L|) = 1$ gives an example. Also if $d \geqslant 2$ and $p^d - 1$ has a prime divisor $r \geqslant 5$ such that r does not divide $p^a - 1$ for any $a \leqslant d-1$ then $\mathsf{GL}_d(p)$ contains a cyclic subgroup of order r that satisfies these conditions. Such a prime divisor always exists unless $p^d = 64$ or $d = 2$ and p is of the form $2^a3^b - 1$ for some a, b; see [12].

Not every transitive group H gives rise to a 1-suitable orbit $\mathcal{T}$. If H is 3-transitive on Ω then the setwise stabiliser of each triple $\{\alpha,\beta,\gamma\}$ induces the symmetric group S_3 on $\{\alpha,\beta,\gamma\}$, and so 3-transitive groups have no 1-suitable orbits on triples. In [8] a transitive permutation group H on a set Ω was defined to be a *three-star group* if for all 3-subsets t of Ω the setwise stabiliser H_t does not fix t pointwise. Thus H is a three-star group if and only if it has no 1-suitable orbit in $\Omega^{(3)}$. Each 3-transitive group is a three-star group, and there are other examples, for example the group $H = \mathsf{PSL}_d(q)$ ($d \geqslant 3$ and q a prime-power) acting on the set Ω of 1-dimensional subspaces of the underlying d-dimensional vector space. An investigation of finite three-star groups by P. M. Neumann and the first author is in progress [8]. It has been shown in particular that primitive three-star groups have rank at most 3, but a complete classification of finite three-star groups is yet to be achieved.

3 Embedding permutation groups into wreath products

Let Γ be a finite set, $L \leqslant \mathsf{Sym}\,\Gamma$, $\ell \geqslant 2$ an integer, and $H \leqslant \mathsf{S}_\ell$. The *wreath product* $L\,\mathsf{wr}\,H$ is the semidirect product $L^\ell \rtimes H$ where for $(x_1,\dots,x_\ell) \in L^\ell$ and $\sigma \in \mathsf{S}_\ell$, $(x_1,\dots,x_\ell)^\sigma = (x_{1\sigma^{-1}},\dots,x_{\ell\sigma^{-1}})$. The product action of $L\,\mathsf{wr}\,H$ is the action of $L\,\mathsf{wr}\,H$ on Γ^ℓ defined by

$$(\gamma_1,\dots,\gamma_\ell)^{(x_1,\dots,x_\ell)\sigma} = (\gamma_{1\sigma^{-1}}^{x_{1\sigma^{-1}}},\dots,\gamma_{\ell\sigma^{-1}}^{x_{\ell\sigma^{-1}}})$$

for all $(\gamma_1,\dots,\gamma_\ell) \in \Gamma^\ell$, $(x_1,\dots,x_\ell)\sigma \in L\,\mathsf{wr}\,H$.

The following couple of remarks give a summary of the elementary properties of wreath products and product actions. The interested reader will find the proofs of these comments in Section 2.7 of the book by Dixon and Mortimer [4]. If $\gamma \in \Gamma$ then $(\gamma,\dots,\gamma) \in \Gamma^\ell$; set $\omega = (\gamma,\dots,\gamma)$. The stabiliser $(L\,\mathsf{wr}\,H)_\omega$ in $L\,\mathsf{wr}\,H$ of ω is the subgroup $(L_\gamma)^\ell \rtimes H = L_\gamma\mathsf{wr}\,H$, where L_γ is the stabiliser of γ in L. (It is easy to see that H normalises $(L_\gamma)^\ell$, and so $(L_\gamma)^\ell \rtimes H$ is indeed a subgroup of $L\,\mathsf{wr}\,H$.) The subgroup L^ℓ is normal in $L\,\mathsf{wr}\,H$ and is transitive on Γ^ℓ if and only if L is transitive on Γ. Moreover no non-identity element of $L\,\mathsf{wr}\,H$ stabilises all points of Γ^ℓ. In other words, the product action of $L\,\mathsf{wr}\,H$ on Γ^ℓ is faithful. Therefore $L\,\mathsf{wr}\,H$ can be considered as a permutation group on Γ^ℓ.

If $|\Gamma| \geqslant 5$ then $\mathsf{Sym}\,\Gamma \,\mathsf{wr}\, \mathsf{S}_\ell$ is a maximal subgroup of $\mathsf{Sym}\,(\Gamma^\ell)$, and is primitive on Γ^ℓ. The subgroups of a finite symmetric group $\mathsf{Sym}\,\Omega$ of the form $\mathsf{Sym}\,\Gamma \,\mathsf{wr}\, \mathsf{S}_\ell$, where Ω can be identified with Γ^ℓ in such a way that $\mathsf{Sym}\,\Gamma \,\mathsf{wr}\, \mathsf{S}_\ell$ acts on Γ^ℓ as above, form one of several classes of primitive maximal subgroups of $\mathsf{Sym}\,\Omega$, identified by the O'Nan–Scott Theorem; see [6]. Thus an important part of classifying the primitive maximal subgroups of $\mathsf{Sym}\,\Omega$ containing a given (innately transitive) subgroup G is finding all ways of identifying Ω with a Cartesian product Γ^ℓ with $\ell \geqslant 2$ and $|\Gamma| \geqslant 5$, so that G acts as a subgroup of $\mathsf{Sym}\,\Gamma \,\mathsf{wr}\, \mathsf{S}_\ell$ in product action.

For the rest of this section suppose that G is an innately transitive group on a finite set Ω and that M is a non-abelian, transitive, minimal normal subgroup of G. Let $\mathcal{W}_G$ be the set of primitive maximal subgroups W of $\mathsf{Sym}\,\Omega$ such that W is a wreath product in product action and $G \leqslant W$.

Let $W \in \mathcal{W}_G$. Then $W \cong \mathsf{Sym}\,\Gamma \,\mathsf{wr}\, \mathsf{S}_\ell$ for some Γ and $\ell \geqslant 2$, and we can identify Ω with the Cartesian product Γ^ℓ. It was proved in [3] that $M \leqslant (\mathsf{Sym}\,\Gamma)^\ell$. Let ω be a fixed element of Ω, say $\omega = (\gamma_1, \ldots, \gamma_\ell)$, and for $i = 1, \ldots, \ell$ let $K_i = M_{\gamma_i}$. It was shown in [3] that the set $\mathcal{K}_\omega(W) = \{K_1, \ldots, K_\ell\}$ is invariant under conjugation by G_ω, the K_i have the same size,

$$\bigcap_{i=1}^{\ell} K_i = M_\omega \quad \text{and} \quad K_i\left(\bigcap_{j\neq i} K_j\right) = M. \tag{1}$$

In general we say that a set $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ of subgroups of M is a *Cartesian system* of subgroups for M if $|K_i| = |K_j|$ for all $i,\ j \in \{1, \ldots, \ell\}$ and there is some $\omega \in \Omega$ such that (1) holds. Cartesian systems provide a way of identifying the subgroups in $\mathcal{W}_G$ from information internal to G.

Theorem 5 ([3])
For a fixed $\omega \in \Omega$ the map $W \mapsto \mathcal{K}_\omega(W)$ is a bijection between the set $\mathcal{W}_G$ and the set of G_ω-invariant Cartesian systems $\mathcal{K}$ of subgroups for M such that $\bigcap_{K\in\mathcal{K}} K = M_\omega$.

Fix $W \in \mathcal{W}_G$, say $W \cong \mathsf{Sym}\,\Gamma \,\mathsf{wr}\, \mathsf{S}_\ell$ for some Γ and $\ell \geqslant 2$, and let $\pi_W : W \to \mathsf{S}_\ell$ be the natural projection. Then $\pi_W(G)$ is a subgroup of S_ℓ. Moreover, since M is transitive on Ω, we have $G = MG_\omega$, and since $M \leqslant (\mathsf{Sym}\,\Gamma)^\ell = \ker \pi_W$, $\pi_W(G) = \pi_W(G_\omega)$. Thus π_W gives rise to an action of G_ω on $\{1, \ldots, \ell\}$. It was proved in [3] that the G_ω-actions on $\{1, \ldots, \ell\}$ and on the Cartesian system $\mathcal{K}_\omega(W)$ are equivalent. It can also be shown that $\pi_W(G)$ has at most 2 orbits in $\{1, \ldots, \ell\}$, and a description of the maximal subgroups $W \in \mathcal{W}_G$ for which $\pi_W(G)$ is intransitive on $\{1, \ldots, \ell\}$ is given in [2]. In this paper we are interested in the remaining case, namely in primitive maximal subgroups $W \in \mathcal{W}_G$ where $\pi_W(G)$ is transitive on $\{1, \ldots, \ell\}$. This is equivalent to requiring G_ω to be transitive on the corresponding Cartesian system $\mathcal{K}_\omega(W)$.

Suppose that $M = T^k$ where T is a finite, non-abelian, simple group and $k \geqslant 1$, and let $\sigma_i : M \to T$ denote the i-th coordinate projection map $(t_1, \ldots, t_k) \mapsto t_i$. Let $W \in \mathcal{W}_G$ and set $\mathcal{K}_\omega(W) = \{K_1, \ldots, K_\ell\}$. The properties of Cartesian systems imply that for all

$i \leqslant k$ and $j \leqslant \ell$

$$\sigma_i(K_j)\left(\bigcap_{j' \neq j} \sigma_i(K_{j'})\right) = T.$$

Thus it is important to understand the following sets of subgroups:

$$\mathcal{F}_i = \{\sigma_i(K_j) \mid j = 1, \ldots, \ell,\ \sigma_i(K_j) \neq T\}.$$

The set $\mathcal{F}_i$ is independent of i up to conjugation by G_ω, in the sense that for all i_1, $i_2 \in \{1, \ldots, k\}$ there is a $g \in G_\omega$ such that $\mathcal{F}_{i_2} = \mathcal{F}_{i_1}^g = \{L^g \mid L \in \mathcal{F}_{i_1}\}$. Moreover, using the finite simple group classification, it was shown in [1] that the number of indices j such that $\sigma_i(K_j) \in \mathcal{F}_i$ is at most 3. It is easy to see that this number is also independent of the choice of ω, and we denote this number by $c(G,W)$. In [2] the study of subgroups $W \in \mathcal{W}_G$ for which $\pi_W(G)$ is transitive is split into several cases corresponding to the value of $c(G,W) \in \{0,1,2,3\}$ and to the group theoretical structure of the Cartesian system elements. In the case when $c(G,W) = 3$ we prove the following theorem.

Theorem 6

Suppose that G is an innately transitive permutation group with a non-abelian, transitive, minimal normal subgroup M, and suppose that $W \in \mathcal{W}_G$ such that $\pi_W(G)$ is transitive. Let $\{K_1, \ldots, K_\ell\}$ be the corresponding Cartesian system $\mathcal{K}_\omega(W)$ for a fixed $\omega \in \Omega$, and suppose that $c(G,W) = 3$. Then the following hold.

(a) *The isomorphism type of the simple direct factor T of M and those of the subgroups A, B, and C in $\mathcal{F}_i$ are as in Table 1.*

(b) *For $j = 1, \ldots, \ell$, $\sigma_1(K_j)' \times \cdots \times \sigma_k(K_j)' \leqslant K_j$ and if T is as in row 1 or row 2 of Table 1 then $\sigma_1(K_j) \times \cdots \times \sigma_k(K_j) = K_j$.*

(c) *For $i = 1, \ldots, k$ let $a(i)$, $b(i)$, $c(i) \in \{1, \ldots, \ell\}$ be such that $\sigma_i(K_{a(i)}) \cong A$, $\sigma_i(K_{b(i)}) \cong B$, and $\sigma_i(K_{c(i)}) \cong C$, and set $\mathcal{T} = \{(K_{a(i)}, K_{b(i)}, K_{c(i)}) \mid i = 1, \ldots, k\}$. Then $(\mathcal{K}_\omega(W), \mathcal{T})$ is a suitable ordered triple design relative to the faithful action of $\pi_W(G)$ on $\mathcal{K}_\omega(W)$.*

PROOF. Suppose that $M = T^k$ for some k and for $i = 1, \ldots, k$ let $\sigma_i : M \to T$ be the i-th coordinate projection defined by $\sigma_i(x_1, \ldots, x_k) = x_i$. Also for i_1, $i_2 \in \{1, \ldots, k\}$ we define $\sigma_{\{i_1,i_2\}} : M \to T \times T$ by $\sigma_{\{i_1,i_2\}}(x_1, \ldots, x_k) = (x_{i_1}, x_{i_2})$.

(a) By (1), $K_j\left(\bigcap_{m \neq j} K_m\right) = M$ for all j, and hence

$$\sigma_i(K_j)\left(\bigcap_{m \neq j} \sigma_i(K_m)\right) = \sigma_i(M) = T \quad \text{for} \quad i = 1, \ldots, k \text{ and } j = 1, \ldots, \ell.$$

	T	A	B	C
1	$\mathsf{Sp}_{4a}(2)$, $a \geqslant 2$	$\mathsf{Sp}_{2a}(4).2$	$\mathsf{O}^-_{4a}(2)$	$\mathsf{O}^+_{4a}(2)$
2	$\mathsf{P}\Omega_8(3)$	$\Omega_7(3)$	$\mathbb{Z}_3^6 \rtimes \mathsf{PSL}_4(3)$	$\mathsf{P}\Omega_8(2)$
3	$\mathsf{Sp}_6(2)$	$\mathsf{G}_2(2)$	$\mathsf{O}^-_6(2)$	$\mathsf{O}^+_6(2)$
		$\mathsf{G}_2(2)'$	$\mathsf{O}^-_6(2)$	$\mathsf{O}^+_6(2)$
		$\mathsf{G}_2(2)$	$\mathsf{O}^-_6(2)'$	$\mathsf{O}^+_6(2)$
		$\mathsf{G}_2(2)$	$\mathsf{O}^-_6(2)$	$\mathsf{O}^+_6(2)'$

Table 1: Strong multiple factorisations $\{A,B,C\}$ of finite simple groups T

Thus if $\mathcal{F}_i = \{A,B,C\}$ for some i then the set $\{A,B,C\}$ is a strong multiple factorisation of T (see [1] for definitions), and, using [1, Table V], we obtain that T, A, B, and C must be as in one of the lines of Table 1.

(b) Suppose that $\mathcal{F}_1 = \{A,B,C\}$ for some subgroups A, B, and C of T. We see from Table 1 that A', B', and C' are perfect groups. Moreover for all $i \in \{1,\dots,k\}$ we have that either $\sigma_i(K_j) = T$, or $\sigma_i(K_j)$ is isomorphic to one of A, B, and C. We show next that

$$\sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \leqslant K_i \quad \text{for} \quad i = 1,\dots,\ell.$$

Since $\sigma_i(K_j)'$ is a perfect group, for all i and j, it follows from [10, Lemma 3.2] that we only have to prove that

$$\sigma_{i_1}(K_j)' \times \sigma_{i_2}(K_j)' \leqslant \sigma_{\{i_1,i_2\}}(K_j) \quad \text{for} \quad j = 1,\dots,\ell \text{ and } i_1,\ i_2 \in \{1,\dots,k\}. \tag{2}$$

Suppose that $j \in \{1,\dots,\ell\}$ and i_1, $i_2 \in \{1,\dots,k\}$ are such that (2) does not hold. If $\sigma_{i_1}(K_j) = T$ or $\sigma_{i_2}(K_j) = T$ then [10, Lemma 2.2] implies that $\sigma_{\{i_1,i_2\}}(K_j)$ is a diagonal subgroup of $T \times T$ isomorphic to T. Suppose that this is the case. By assumption, there are j_1, j_2, j_3, $j_4 \in \{1,\dots,\ell\} \setminus \{j\}$ such that $\sigma_{i_1}(K_{j_1}) \cong \sigma_{i_2}(K_{j_2}) \cong A$ and $\sigma_{i_1}(K_{j_3}) \cong \sigma_{i_2}(K_{j_4}) \cong B$. Now $K_j(K_{j_1} \cap K_{j_2} \cap K_{j_3} \cap K_{j_4}) = M$, and so applying the projection $\sigma_{\{i_1,i_2\}}$ gives

$$T \times T = \sigma_{\{i_1,i_2\}}(K_j)\sigma_{\{i_1,i_2\}}(K_{j_1} \cap K_{j_2} \cap K_{j_3} \cap K_{j_4}).$$

On the other hand,

$$\sigma_{\{i_1,i_2\}}(K_{j_1} \cap K_{j_2} \cap K_{j_3} \cap K_{j_4}) \leqslant (\sigma_{i_1}(K_{j_1}) \cap \sigma_{i_1}(K_{j_3})) \times (\sigma_{i_2}(K_{j_2}) \cap \sigma_{i_2}(K_{j_4}))$$

and so we obtain

$$T \times T = \sigma_{\{i_1,i_2\}}(K_j)\left((\sigma_{i_1}(K_{j_1}) \cap \sigma_{i_1}(K_{j_3})) \times (\sigma_{i_2}(K_{j_2}) \cap \sigma_{i_2}(K_{j_4}))\right). \tag{3}$$

Note that $(\sigma_{i_1}(K_{j_1}) \cap \sigma_{i_1}(K_{j_3})) \times (\sigma_{i_2}(K_{j_2}) \cap \sigma_{i_2}(K_{j_4})) \cong (A \cap B) \times (A \cap B)$. Therefore (3) is a factorisation of the characteristically simple group $T \times T$ in which one factor is a diagonal subgroup and the other factor is the direct product of two isomorphic subgroups. Therefore [10, Theorem 1.5] applies and we find that $\sigma_{i_1}(K_{j_1}) \cap \sigma_{i_1}(K_{j_3})$ has to be a maximal subgroup of T. On the other hand, $K_{j_1}K_{j_3} = M$ implies $\sigma_{i_1}(K_{j_1})\sigma_{i_1}(K_{j_3}) = T$, and so $\sigma_{i_1}(K_{j_1}) \cap \sigma_{i_1}(K_{j_3})$ is properly contained in $\sigma_{i_1}(K_{j_1})$ and $\sigma_{i_1}(K_{j_3})$. This is a contradiction, and so each of $\sigma_{i_1}(K_j)$, $\sigma_{i_2}(K_j)$ is isomorphic to one of A, B, or C.

Suppose without loss of generality that $\sigma_{i_1}(K_j) \cong A$. Then there are indices j_1, j_2, $j_3, j_4 \in \{1,\dots,\ell\} \setminus \{j\}$ such that $\sigma_{i_1}(K_{j_1}) \cong \sigma_{i_2}(K_{j_2}) \cong B$ and $\sigma_{i_1}(K_{j_3}) \cong \sigma_{i_2}(K_{j_4}) \cong C$. The defining properties of Cartesian systems imply that

$$\{\sigma_{\{i_1,i_2\}}(K_j), \sigma_{i_1}(K_{j_1}) \times \sigma_{i_2}(K_{j_2}), \sigma_{i_1}(K_{j_3}) \times \sigma_{i_2}(K_{j_4})\}$$

is a strong multiple factorisation of $T \times T$, as defined in [10]. However, [10, Theorem 1.7] implies that (2) holds, and we assumed that this was not the case. Thus $\sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \leqslant K_i$.

If T is as in row 2 then $\sigma_i(K_j)$ is a perfect group, for all $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,\ell\}$, and so $K_i = \sigma_1(K_i) \times \cdots \times \sigma_k(K_i)$. Let us now suppose that T is as in row 1 and set $\bar{K}_i = \sigma_1(K_i) \times \cdots \times \sigma_k(K_i)$ for all i. Since $\bigcap_i K_i = \bigcap_i \bar{K}_i$ (see [1] page 181), it follows that $\bar{\mathcal{K}} = \{\bar{K}_1,\dots,\bar{K}_\ell\}$ is a Cartesian system of subgroups for M. Therefore

$$\prod_{i=1}^{\ell} |M : K_i| = \left|M : \bigcap_{i=1}^{\ell} K_i\right| = \left|M : \bigcap_{i=1}^{\ell} \bar{K}_i\right| = \prod_{i=1}^{\ell} |M : \bar{K}_i|,$$

which forces $|M : K_i| = |M : \bar{K}_i|$, and hence $K_i = \bar{K}_i$ for all i.

(c) Finally let $\mathcal{T}$ be as in (c), and let us show that $(\mathcal{K}_\omega(W), \mathcal{T})$ is a suitable ordered triple design. It is clear that $\mathcal{T} \subseteq \mathcal{K}_\omega(W)^{(3)}$. Note that G_ω is transitive on $\mathcal{K}_\omega(W)$. Suppose that $T_1,\dots,T_k$ are the simple direct factors of M. If $g \in G_\omega$ and i_1, $i_2 \in \{1,\dots,k\}$ such that $T_{i_1}^g = T_{i_2}$, then $A \cong \sigma_{i_1}\left(K_{a(i_1)}\right) \cong \sigma_{i_1}\left(K_{a(i_1)}\right)^g = \sigma_{i_2}\left(\left(K_{a(i_1)}\right)^g\right)$, and so $K_{a(i_2)} = \left(K_{a(i_1)}\right)^g$. The same argument shows that $K_{b(i_2)} = \left(K_{b(i_1)}\right)^g$ and $K_{c(i_2)} = \left(K_{c(i_1)}\right)^g$. Hence $T_{i_1}^g = T_{i_2}$ implies $\left(K_{a(i_1)}, K_{b(i_1)}, K_{c(i_1)}\right)^g = \left(K_{a(i_2)}, K_{b(i_2)}, K_{c(i_2)}\right)$. For each $t \in \mathcal{T}$ let $I_t = \{T_i \mid (K_{a(i)}, K_{b(i)}, K_{c(i)}) = t\}$. Then $\{I_t \mid t \in \mathcal{T}\}$ is a G_ω-invariant partition of $\{T_1,\dots,T_k\}$ such that the G_ω-actions on $\mathcal{T}$ and $\{I_t \mid t \in \mathcal{T}\}$ are equivalent. Since G_ω is transitive on $\{T_1,\dots,T_k\}$, we obtain that G_ω is transitive on $\mathcal{T}$. Thus $(\mathcal{K}_\omega(W), \mathcal{T})$ is a suitable ordered triple design relative to the group $\pi_W(G_\omega)$ induced by G_ω on $\mathcal{K}_\omega(W)$. Since $\pi_W(G) = \pi_W(G_\omega)$, the proof is complete. □

Thus each $W \in \mathcal{W}_G$ such that $\pi_W(G)$ is transitive and $c(G,W) = 3$ gives rise to a suitable ordered triple design $\mathcal{H}$ relative to $\pi_W(G)$. In addition, for this to occur the simple direct factor T of M and the three subgroups A, B, C such that $A \cap B \cap C = \sigma_i(M_\omega)$ are restricted to those given in one of the rows of Table 1. In the next section we give a construction for such groups G to demonstrate that each T, A, B, C given in Table 1 and each suitable ordered triple design $(\Omega, \mathcal{T})$ relative to a subgroup H of $\mathrm{Sym}\,\Omega$ can occur in Theorem 6. The groups will be wreath products as defined above.

4 The construction

Let T, A, B, C be as in one of the rows of Table 1, let $(\mathcal{K},\mathcal{T})$ be a suitable ordered triple design relative to a subgroup H of $\mathrm{Sym}\,\mathcal{K}$, and set $\ell=|\mathcal{K}|$ and $k=|\mathcal{T}|$. We may assume without loss of generality that $\mathcal{K}=\{1,\dots,\ell\}$. Set $\Delta=\{(A\cap B\cap C)x \mid x\in T\}$, so T acts transitively on Δ by right multiplication. It follows from Definition 1 that H acts transitively and faithfully on $\mathcal{T}$, and so we can view H as a subgroup of S_k. Consider the wreath product $G=T\,\mathrm{wr}\,H=T^k\rtimes H$ defined with respect to this action of H. Set $\Omega=\Delta^k$. Then G acts on Ω in its product action. Let M denote the normal subgroup T^k of G.

Now we use the properties of $T\,\mathrm{wr}\,H$ discussed in the second paragraph of Section 3. The group G acts faithfully and transitively on Ω, and M is a transitive normal subgroup of G. Moreover, since H is transitive on $\mathcal{T}$, H permutes the k coordinate subgroups of M transitively, and hence M is a minimal normal subgroup of G. Thus G is innately transitive and M is a transitive, minimal normal subgroup of G. Let γ denote the trivial coset $A\cap B\cap C$ in Δ, and set $\omega=(\gamma,\dots,\gamma)$. Then $\omega\in\Omega$, $M_\omega=(A\cap B\cap C)^k$ and $G_\omega=M_\omega H$.

For each element $i\in\mathcal{K}$ set $K_i=\prod_{j\in\mathcal{T}}K_{ij}$ where

$$K_{ij}=\begin{cases} A & \text{if the first coordinate of } j \text{ is } i;\\ B & \text{if the second coordinate of } j \text{ is } i;\\ C & \text{if the third coordinate of } j \text{ is } i;\\ T & \text{otherwise.}\end{cases}$$

Let $\hat{\mathcal{K}}=\{K_1,\dots,K_\ell\}$. We claim that $\hat{\mathcal{K}}$ is a G_ω-invariant Cartesian system for M and $\bigcap_i K_i=M_\omega$. Let $\sigma_i:M\to T$ denote the i-th coordinate projection mapping $(x_1,\dots,x_k)\mapsto x_i$. First note that the K_i are direct products of their projections and for all i,

$$\sigma_i(K_1)\cap\cdots\cap\sigma_i(K_\ell)=A\cap B\cap C.$$

Therefore $K_1\cap\cdots\cap K_\ell=(A\cap B\cap C)^k=M_\omega$. The choice of A, B, and C implies that for $i=1,\dots,k$ and $j=1,\dots,\ell$

$$\sigma_i(K_j)\left(\bigcap_{j'\neq j}\sigma_i(K_{j'})\right)=T.$$

Since $\sigma_i(K_j)\leqslant K_j$ for each i, j, it follows that $K_j\left(\bigcap_{j'\neq j}K_{j'}\right)=M$ for all j. Thus (1) holds and $\hat{\mathcal{K}}$ is a Cartesian system for M. Let us prove that the set $\hat{\mathcal{K}}$ is invariant under conjugation by H. Let $i\in\mathcal{K}$ and $g=H$. Then $K_i^g=\prod_{j\in\mathcal{T}}K_{ij}^g$. If $K_{ij}=A$ then $j=(i,i',i'')$ for some i', $i''\in\mathcal{K}$ and $j^g=(i^g,i'^g,i''^g)$, and so $K_{i^g j^g}=A$. Similarly, if $K_{ij}=B$, C, T then $K_{i^g j^g}=B$, C, T, respectively. Therefore $K_i^g=K_{i^g}\in\hat{\mathcal{K}}$. Hence $\hat{\mathcal{K}}$ is H-invariant, and, since $M_\omega=(A\cap B\cap C)^k\leqslant K_i$ for all $i\in\{1,\dots,\ell\}$, the set $\hat{\mathcal{K}}$ is invariant under conjugation by $G_\omega=M_\omega H$.

Therefore the conditions of Theorem 5 hold, and there is a wreath product $W \in \mathcal{W}_G$ such that $\hat{\mathcal{K}} = \mathcal{K}_\omega(W)$. It follows from the definition of $\hat{\mathcal{K}}$ that $c(G,W) = 3$. Finally, transitivity of H on $\mathcal{K}$ implies that G_ω is transitive on $\hat{\mathcal{K}} = \mathcal{K}_\omega(W)$, and it follows from our comments above that $\pi_W(G)$ is transitive. Thus all conditions of Theorem 6 hold.

The group G constructed above has the very interesting property that there are two different maximal subgroups in $\mathcal{W}_G$. The first is the subgroup W in the previous paragraph, and it is of the form $W = \mathsf{Sym}\,\Gamma\,\mathsf{wr}\,\mathsf{S}_\ell$ where $|\Gamma| = |M : K_i|$ $(1 \leqslant i \leqslant \ell)$. It follows from the definition of G that G is contained in $\mathsf{Sym}\,\Delta\,\mathsf{wr}\,\mathsf{S}_k$, and so also $\mathsf{Sym}\,\Delta\,\mathsf{wr}\,\mathsf{S}_k \in \mathcal{W}_G$. These are necessarily different subgroups, for example $c(G,W) = 3$, while $c(G, \mathsf{Sym}\,\Delta\,\mathsf{wr}\,\mathsf{S}_k) = 1$. Thus the set Ω can be identified with both Δ^k and Γ^ℓ, and G preserves both of these Cartesian decompositions of Ω.

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Dedication

Exploring links between different areas of mathematics is something the first author valued in her research collaboration with Terry Speed, and we both wish Terry a happy 60th birthday and many more years of productive mathematical research.

Cheryl E. Praeger, Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, 6009 Crawley, Western Australia,
praeger@maths.uwa.edu.au

Csaba Schneider, Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, 6009 Crawley, Western Australia,
csaba@maths.uwa.edu.au

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