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EXTREME VALUES FOR A CLASS OF SHOT-NOISE PROCESSES

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Abstract

The distribution of the maximum of a shot-noise process based on amplitudes which are heavy tailed and follow a chain-dependent structure is analysed. Asymptotic results are obtained. The process is seen to have a strong local dependence and its extremal index is computed. A simulation study shows the finite sample size performance of an asymptotic approximation to the distribution of the maximum.

1 Introduction

In the paper we are concerned with the asymptotic behavior of the extreme values for a class of shot noise processes. Shot noise processes provide a wide class of stochastic models that are particularly well suited to modeling time series with sudden jumps. Such processes have been applied to modeling river flow data where a rise in the riverflow level could, for example, be attributed to rainfall, Lawrance and Kottegoda (1977) and Weiss (1973). Moreover, rainfall data, itself, has been modeled via shot noise processes, Waymire and Gupta (1981). The basic model under study here takes the form

$$X(t) = \sum_{\tau_k \le t} A_k h(t - \tau_k), \quad t \ge 0$$

where $\{A_k\}$ is a sequence of random amplitudes, $\{\tau_k\}$ forms a point process of event times and h is the impulse response function, typically, taken to be nonincreasing with support in $[0, \infty)$. In the current investigation, we take the $\{A_k\}$ to be a stochastic process of heavy-tailed random variables. In applications the sequence of shocks or amplitudes exhibits dependence. It may be that large shocks tend to occur in succession followed by periods of mild or small shocks. To model this dependence, we assume that $\{A_k\}$ form a sequence of chain dependent variables. A chain dependent process operates in the following way. The observed process $\{A_k\}$ is linked with a secondary Markov chain $\{\xi_k\}$. Conditional on the values of $\{\xi_k\}$, the A_k's are independent with the distribution of A_k depending on the value of ξ_{k-1}. Chain dependent processes form a useful class of stochastic models which, for example, have been applied with success in modeling extremes of precipitation data, Guttorp (1995) p. 74. Theorectical work on extremes for chain dependent models has been done in Resnick and Neuts (1970), Denzel and O'Brien (1975) and more recently in McCormick and Seymour (2001).

In section 2 we present an extreme value analysis of the shot noise model. Extremes for shot noise processes have been considered by several authors under various assumptions. However, all previous work have taken the $\{A_k\}$ sequence to be iid. On the other hand, in practice, the data often appear to contradict such an assumption. When the $\{A_k\}$ are constant and the $\{\tau_k\}$ form a homogeneous Poisson process, i.e. in the case of a filtered Poisson process, Hsing and Teugels (1989) have obtained results on the limiting distribution of the maximum. Doney and O'Brien (1991) provide an extension to the results of Hsing and Teugels (1989) while working under the assumption of constant amplitude. The case of light-tailed amplitudes, viz. Gamma or Weibull distribution, was considered in Homble and McCormick (1995) and the heavy-tailed amplitude case, e.g. Pareto distribution, was developed in McCormick (1997).

The process under consideration has a strong local dependence quantified by a value referred to as its extremal index. The method developed in Chernick et al. (1991) for calculating extremal index is conveniently applied here and represents an essential step in obtaining the asymptotics for the maxima. In section 3 the results of a simulation study are shown.

2 Asymptotics

Let $\{\xi_n\}$ be a stationary finite state space Markov chain with probability transition matrix $P = (p_{ij})$, $1 \le i, j \le r$. Further, let $\{\pi_j, 1 \le j \le r\}$ denote the stationary probability measure for the chain. Next, we define a chain-dependent sequence associated with the Markov chain $\{\xi_n\}$ as follows. Let $H_i(x)$, $1 \le i \le r$ be distribution functions and let $\{A_n\}$ be such that

$$P\{A_n \le x, \xi_n = j | A_k, \xi_k, k \le n-1\} = P\{A_n \le x, \xi_n = j | \xi_{n-1}\}$$

and

$$P\{A_n \le x, \xi_n = j | \xi_{n-1} = i\} = p_{ij} H_i(x).$$

Let $\{\tau_n, n \ge 0\}$ be a renewal process with a fixed renewal at $\tau_0 = 0$. We assume that the sequence $\{\tau_n\}$ is independent of the sequence $\{A_n, \xi_n\}$ and define a shot noise process X by

$$X(t)=\sum_{0\le\tau_k\le t} A_k h(t-\tau_k),\quad t\ge 0 \tag{2.1}$$

where h is the impluse response function for which we assume its support to be a compact subset of $[0,\infty)$.

To define the stationary process associated with X, we first introduce the stationary renewal process $\{\delta_j, j\in\mathbb{Z}\}$ where we label the points so that

$$-\infty<\cdots<\delta_{-1}<\delta_0\le 0<\delta_1<\cdots<\infty,$$

that is, δ_0 is the last nonpositive point. Further, let the interpoint distance d.f. for $\{\tau_n\}$ be $F(x)=P\{\tau_n-\tau_{n-1}\le x\}$. We assume $F(0^+)=0$ and $\mu=\int_0^\infty x dF(x)<\infty$. Then, take $\{\delta_j\}$ such that

$$\text{(i)}\quad \delta_j-\delta_{j-1}, j\ne 1 \text{ i.i.d. with } \delta_j-\delta_{j-1}\sim F, j\ne 1$$

and

$$\text{(ii)}\ \ \delta_1,-\delta_0\sim G(x)=\frac{1}{\mu}\int_0^x(1-F(t))dt \tag{2.2}$$

and

$$\text{(iii)}\ \ P\{\delta_0>x,\delta_1>y\}=\frac{1}{\mu}\int_{x+y}^\infty(1-F(t))dt.$$

With the choices made in (2.2), setting $N_s(A)=\sum_j \epsilon_{\delta_j}(A)$ where $\epsilon_x(A)=I_A(x)$, $x\in\mathbb{R}$, $A\subset\mathbb{R}$, we have for any $k\ge 1$ and sets $A_1,\ldots,A_k$ in $B(\mathbb{R})$ that

$$(N_s(A_1),\ldots,N_s(A_k))\stackrel{d}{=}(N_s(A_1+t),\ldots,N_s(A_k+t)),\quad t\in\mathbb{R}. \tag{2.3}$$

Now define

$$Y(t)=\sum_{\delta_j\le t} A_j h(t-\delta_j),\quad t\ge 0. \tag{2.4}$$

Note by letting $\mathcal{M}=\sum_{j=-\infty}^{\infty}\epsilon_{(\delta_j,A_j)}$, a point process with points $(\delta_j,A_j)\in E=\mathbb{R}\times[0,\infty)$, $j\in\mathbb{Z}$, we have the representation

$$Y(t)=\int_E g_t(s,x)d\mathcal{M} \tag{2.5}$$

with $g_t(s,x)=xh(t-s)$.

For $t\in\mathbb{R}$ and doubly infinite sequences $\boldsymbol{\delta}=(\ldots,\delta_{-1},\delta_0,\delta_1,\ldots)$ with $-\infty<\ldots<\delta_{-1}<\delta_0\le 0<\delta_1<\ldots<\infty$, define an integer m_t by

$$m_t=\max\{i:\delta_i+t\le 0\}.$$

Let $\boldsymbol{\delta}'$ be the doubly infinite sequence defined by

$$\delta_i' = \delta_{(i+m_t)} + t, \quad i \in \mathbb{Z}.$$

Then stationarity of the point process $N_s(\cdot) = \sum_i \epsilon_{\delta_i}(\cdot)$ is equivalent to $\boldsymbol{\delta}' \stackrel{d}{=} \boldsymbol{\delta}$ for any $t \in \mathbb{R}$.

Under the assumption that $\{\delta_n\}$ is independent of the sequence $\{A_n, \xi_n\}$, it is easily checked that for the doubly infinite sequences $\boldsymbol{A} = (\cdots, A_{-1}, A_0, A_1, \cdots$ and $\boldsymbol{A}' = (\cdots, A_{-1}' A_0', A_1', \cdots)$ with $A_i' = A_{i+m_t}$, we have

$$\boldsymbol{A} \stackrel{d}{=} \boldsymbol{A}'.$$

Hence, letting $\mathcal{M}' = \sum_{j=-\infty}^{\infty} \epsilon_{(\delta_j', A_j')}$, we have $\mathcal{M} \stackrel{d}{=} \mathcal{M}'$ so that

$$\begin{aligned} Y(t_1) &= \int_E g_{t_1}(s,x) d\mathcal{M} \\ &\stackrel{d}{=} \int_E g_{t_1}(s,x) d\mathcal{M}' \\ &= \sum_j A_j' h(t_1 - \delta_j') \\ &= \sum_j A_{(j+m_t)} h(t_1 - t - \delta_{(j+m_t)}) \\ &= Y(t_1 - t). \end{aligned}$$

The same argument shows

$$(Y(t_1), \ldots, Y(t_k)) \stackrel{d}{=} (Y(t_1 - t), \ldots, Y(t_k - t))$$

which establishes stationarity of $(Y(t), t \in \mathbb{R})$. Let us further observe that the marginal distribution for $X(t)$ is such that

$$Y(t) \stackrel{d}{=} Y(0) \stackrel{d}{=} \sum_{j=1}^{\infty} A_j h(\delta_j) \tag{2.6}$$

since we assumed that $cl\{x : h(x) \neq 0\} \subset [0, \infty)$.

In the following we make the assumptions that

$$\begin{aligned} &\text{(i)} \quad \text{support } (h) = cl\{x : h(x) \neq 0\} \subset [0,1] \\ &\text{(ii)} \quad h \text{ is strictly decreasing} \\ &\text{(iii)} \quad 0 \leq h(1) < h(0) = 1. \end{aligned} \tag{2.7}$$

Furthermore, we assume that the d.f.'s H_i, $1 \leq i \leq r$, are such that

$$H_i(x) = H(x/\kappa_i), \quad 1 \leq i \leq r, \tag{2.8}$$

where $\bar{H}(x) = x^{-\alpha}L(x)$ for some slowly varying function L and α a positive constant.

Our first lemma establishes the tail behavior of the stationary distribution for the shot noise process defined in (2.1). To that end define a variable

$$Y = \sum_{j=1}^{\infty} A_j h(\sigma_j) \tag{2.9}$$

where $\{A_j, \xi_j\}$ is a chain-dependent process independent of $\{\sigma_j\}$. Assume that the d.f.'s H_i associated with the chain dependent process satisfies (2.8) and that $\{\sigma_j\}$ forms a delayed renewal process.

Lemma 2.1 Let Y be given as in (2.9). Then

$$P\{Y > x\} \sim cx^{-\alpha}L(x) \quad \text{as} \quad x \to \infty$$

where $c = (\sum_{i=1}^{r} \kappa_i^\alpha \pi_i) E \sum_{j=1}^{\infty} h^\alpha(\sigma_j)$. In the special case that $\sigma_j = \delta_j$, $j \geq 1$, i.e. $\sigma_1 \sim G$ and $\sigma_j - \sigma_{j-1} \sim F$, $j \geq 2$ in (2.2),

$$c = (\sum_{i=1}^{r} \kappa_i^\alpha \pi_i) \frac{1}{\mu} \int_0^1 h^\alpha(s) ds.$$

Proof: Consider for $0 \leq s_1 < \ldots < s_n \leq 1$

$$\begin{aligned}
& P\{Y > x | \sigma_j = s_j, \quad j = 1, \ldots, n, \quad \sigma_{n+1} > 1\} \\
= \quad & P\{\sum_{j=1}^{n} A_j h(s_j) > x\} \\
= \quad & EP\{\sum_{j=1}^{n} A_j h(s_j) > x | \xi_{j-1}, \quad j = 1, \ldots, n\}.
\end{aligned}$$

Let $\kappa(i) = \kappa_i$ and let $\{X_j\}$ be iid with d.f. H and be independent of $\{\xi_j\}$. Then

$$P\{Y > x | \sigma_j = s_j, \quad 1 \leq j \leq n, \sigma_{n+1} > 1\} = EP_X\{\sum_{j=1}^{n} \kappa(\xi_{j-1}) h(s_j) X_j > x\}$$

where P_X denotes taking probability w.r.t. $\{X_j\}$ only. Since

$$P_X\{\sum_{j=1}^{n} \kappa(\xi_{j-1}) h(s_j) X_j > x\} \sim \sum_{j=1}^{n} \kappa^\alpha(\xi_{j-1}) h^\alpha(s_j) \bar{H}(x) \quad \text{as } x \to \infty,$$

we have

$$P\{Y > x | \sigma_j = s_j, 1 \le j \le n, \sigma_{n+1} > 1\} \sim E\sum_{j=1}^{n} \kappa^\alpha(\xi_{j-1}) h^\alpha(s_j)\bar{H}(x) \text{ as } x \to \infty. \tag{2.10}$$

Finally, it follows from (2.10) as in McCormick (1997) that

$$\begin{aligned} P\{Y > x\} &\sim E\sum_{j=1}^{\infty} \kappa^\alpha(\xi_{j-1}) h^\alpha(\sigma_j)\bar{H}(x) \\ &= (\sum_1^r \kappa_i^\alpha \pi_i) E\sum_{j=1}^{\infty} h^\alpha(\sigma_j)\bar{H}(x) \text{ as } x \to \infty. \end{aligned} \tag{2.11}$$

In the stationary point process case, i.e. $\sigma_1 \sim G$, we have

$$E\sum_{j=1}^{\infty} h^\alpha(\sigma_j) = \int_0^1 h(s)\frac{ds}{\mu}. \tag{2.12}$$

Thus the lemma holds from (2.11) and (2.12). □

Consider the stationary sequence

$$W_k = \sum_{j\le k} A_j h(\tau_k - \tau_j), \quad k = 0, 1, \ldots \tag{2.13}$$

where we define τ_j with $j < 0$ such that $\{-\tau_{-j}, j = 1, 2, \ldots\}$ is an independent copy of $\{\tau_j, j = 1, 2, \ldots\}$. By Lemma 2.1

$$P(W_0 > x) \sim cx^{-\alpha}L(x) \text{ as } x \to \infty \tag{2.14}$$

where $c = (\sum_1^r \kappa_i^\alpha \pi_i) \sum_{n=1}^{\infty} \int_0^1 h(s)F^{n*}(ds)$ with F^{n*} denoting the n-th fold convolution of F. As a first step in determining the asymptotic distribution for the maximum of the shot noise process, we obtain the asymptotic distribution for the maxima of the W_k. To that end define intermediary sequences

$$W_k^m = \sum_{j=k-m}^{k} A_j h(\tau_k - \tau_j), \quad k = 0, 1, \ldots \tag{2.15}$$

We will establish that the sequences $\{W_k^m, k \ge 0\}$, $m = 1, 2, 3, \ldots$, satisfy mixing conditions $D(u_n)$ and $D^{(m+1)}(u_n)$ for a suitable sequence u_n. By Corollary 1.3 in Chernick et al (1991), this implies that the sequence $\{W_k^m, k \ge 0\}$ has an extremal index θ_m for each m.

We recall the definition of $D(u_n)$ and $D^{(k)}(u_n)$. For a stationary sequence $\{W_n\}$ and sequence of constants $\{u_n\}$, set for $1 \le l \le n-1$

$$\alpha_{n,l} = \sup |P\{W_j \le u_n, j \in A \cup B\} - P\{W_j \le u_n, j \in A\}P\{W_j \le u_n, j \in B\}|,$$

where the supremum is taken over all A, B such that

$$A \subset \{1, \ldots, k\} \text{ and } B \subset \{k+l, \ldots, n\} \text{ for some } k \text{ with } 1 \leq k \leq n-l.$$

The condition $D(u_n)$ is said to hold for the stationary sequence $\{W_k\}$ if for some sequence $l_n = o(n)$, $\alpha_{n,l_n} \to 0$ as $n \to \infty$. If $D(u_n)$ holds for $\{W_k\}$, we say that condition $D^k(u_n)$ holds provided there exist sequences of integers $\{s_n\}$ and $\{l_n\}$ with $s_n \to \infty$, $s_n\alpha_{n,l_n} \to 0$, $s_n l_n/n \to 0$ and

$$\lim_{n\to\infty} nP\{W_1 > u_n \geq \bigvee_{i=2}^{k} W_i, \bigvee_{i=k+1}^{r_n} W_i > u_n\} = 0,$$

where $r_n = [n/s_n]$ and $\bigvee_i^j$ signifies the maximum as the index varies over i to j.

For the sequence $\{u_n\}$ consider for any $\beta > 0$, $u_n = u_n(\beta)$ satisfying for all n sufficiently large

$$u_n = H^{\leftarrow}(1 - \frac{\beta}{n}) \tag{2.16}$$

where we take $H^{\leftarrow}(x) = \inf\{y : H(y) \geq x\}$ and H is as given in (2.8). Note regular variation of $1 - H(x)$ implies $n(1 - H(u_n)) \to \beta$ as $n \to \infty$.

First, we note from Denzel and O'Brien (1975) that the chain-dependent process $\{A_n\}$ is strong mixing, and since the m-tuples $(h(\tau_k - \tau_{k-m}), \ldots, h(\tau_k - \tau_{k-1}))$, $k = 0, 1, \ldots$ are m-dependent, we see that $\{W_k^m, k \geq 0\}$ is strong-mixing. Thus condition $D(u_n)$ holds.

Next, we consider $D^{(m+1)}(u_n)$. Observe

$$nP\{W_1^m > u_n \geq \bigvee_{i=2}^{m+1} W_i^m, \bigvee_{i=m+2}^{r_n} W_i^m > u_n\}$$

$$\leq n \sum_{i=m+2}^{r_n} P\{W_1^m > u_n, W_i^m > u_n\}.$$

Now observe that for $i \geq m+2$

$$P\{W_1^m > u_n, W_i^m > u_n\} = EP\{W_1^m > u_n, W_i^m > u_n | \xi_k, k < i\}$$

$$= EP_{\tau,X}\{\sum_{j=1-m}^{1} \kappa(\xi_{j-1})X_j h(\tau_1 - \tau_j) > u_n\} P_{\tau,X}\{\sum_{j=i-m}^{i} \kappa(\xi_{j-1})X_j h(\tau_i - \tau_j) > u_n\}$$

where $P_{\tau,X}$ denotes taking the probability with respect to the τ and X sequences only. Thus if $\kappa^* = \bigvee_{i=1}^{r} \kappa_i$,

$$P\{W_1^m > u_n, W_i^m > u_n\} \leq P^2\{\sum_{j=1}^{m+1} X_j > u_n/\kappa^*\}.$$

Hence, $D^{(m+1)}(u_n)$ holds. We next turn our attention to computation of the extremal index for the W_k.

Lemma 2.2. Let the W_k, $k \geq 0$, be as defined in (2.13). Then $\{W_k\}$ has an extremal index θ given by

$$\theta = \{1 + \sum_1^\infty \int_0^1 h^\alpha(s) F^{n*}(ds)\}^{-1}.$$

Proof. We begin by computing the extremal index for $\{W_k^m\}$. Since conditions $D(u_n)$ and $D^{(m+1)}(u_n)$ hold for $\{W_k^m\}$, we have by Corollary 1.3 in Chernick et al. (1991) that the extremal index for the $\{W_k^m\}$ exists and is given by θ_m where

$$\theta_m = \lim_{n\to\infty} P\{\bigvee_{i=2}^{m+1} W_i^m \leq u_n | W_1^m > u_n\}.$$

Now observe that

$$\begin{aligned}
& nP\{W_1^m > u_n \geq \bigvee_{i=2}^{m+1} W_i^m\} \\
= \; & nEP\{W_1^m > u_n \geq \bigvee_{i=2}^{m+1} W_i^m | \tau_k, \xi_k, \quad k \leq m+1\} \\
= \; & nEP_X\{ \sum_{j=1-m}^{1} \kappa(\xi_{j-1}) h(\tau_1 - \tau_j) X_j > u_n \geq \bigvee_{i=2}^{m+1} (\sum_{j=i-m}^{i} \kappa(\xi_{j-1}) h(\tau_i - \tau_j) X_j
\end{aligned}$$

Following the development in Chernick et al (1991), it is checked that as $n \to \infty$,

$$\begin{aligned}
& nEP_X\{ \sum_{j=1-m}^{1} \kappa(\xi_{j-1}) h(\tau_1 - \tau_j) X_j > u_n \geq \bigvee_{i=2}^{m+1} (\sum_{j=i-m}^{i} \kappa(\xi_{j-1}) h(\tau_i - \tau_j) X_j \\
\sim \; & nE \sum_{j=1-m}^{1} P_X\{\kappa(\xi_{j-1}) h(\tau_1 - \tau_j) X_j > u_n > \bigvee_{i=2}^{j+m} \kappa(\xi_{j-1}) h(\tau_i - \tau_j) X_j\} \\
\sim \; & \beta \sum_{j=1-m}^{1} E[h^\alpha(\tau_1 - \tau_j) - \bigvee_{i=2}^{j+m} h^\alpha(\tau_i - \tau_j)]_+ E\kappa^\alpha(\xi_{j-1}) \\
= \; & \beta \sum_{j=1-m}^{1} E[h^\alpha(\tau_1 - \tau_j) - h^\alpha(\tau_2 - \tau_j)](\sum_1^r \kappa_i^\alpha \pi_i) \\
= \; & \beta[1 - Eh^\alpha(\tau_2 - \tau_{1-m})] \;\; (\sum_1^r \kappa_i^\alpha \pi_i). \qquad (2.17
\end{aligned}$$

Now by Lemma 2.1 and (2.16)

$$nP\{W_1^m > u_n\} \sim \beta(\sum_{i=1}^{r} \kappa_i^\alpha \pi_i)(1 + E\sum_{n=1}^{m} \int_0^1 h^\alpha(s) F^{n*}(ds)).$$

Thus we obtain

$$\theta_m = \frac{1 - Eh^\alpha(\tau_{m+1})}{E\sum_{n=0}^{m} h^\alpha(\tau_n)} \tag{2.18}$$

Finally, one argues as in McCormick (1997) that $\{W_k\}$ has extremal index θ given by

$$\theta = \lim_{m\to\infty} \theta_m = (1 + E\sum_{1}^{\infty} h^\alpha(\tau_n))^{-1} \tag{2.19}$$

completing the proof.

The following result is immediate from Lemma 2.2.

Theorem 2.3. Let $\{W_k\}$ be the stationary sequence defined in (2.13). Then with u_n given in (2.16),

$$\lim_{n\to\infty} P\{\max_{1\le k\le n} W_k \le u_n\} = \exp\{-\beta \sum_{1}^{r} \kappa_i^\alpha \pi_i\}$$

Following Hsing and Teugels (1989) we obtain the limit result for maxima of the shot noise process.

Corollary 2.4. Let $\{X(t)\}$ be the shot noise process defined in (2.1). Then

$$\lim_{T\to\infty} P\{\max_{0\le t\le T} X(t) \le u_T\} = \exp\{-(\beta/\mu) \sum_{1}^{r} \kappa_i^\alpha \pi_i\}$$

where $\mu = E\tau_1$.

Remark. We have that

$$\frac{T}{\mu}\sum_{i=1}^{r}[1 - H_i(u_T)]\pi_i \to \frac{\beta}{\mu}\sum_{i=1}^{r} \kappa_i^\alpha \pi_i \quad \text{as } T \to \infty.$$

Thus, we obtain the following asymptotic approximation

$$P\{\max_{0\le t\le T} X(t) \le x\} \approx \exp\{-\frac{T}{\mu}\sum_{i=1}^{r}[1 - H_i(x)]\pi_i\}$$

3 Simulation Study

The performance of the approximation given in the Remark above is now investigated via simulation. The steps needed to carry out this simulation are specified as follows.

First, the distribution of the maximum of T observations of a shot noise process $\{X(t)\}$ (2.1) must be simulated. This simulation will compare $T = 20$ and $T = 100$.

Generate an r-state Markov chain $\{\xi_n\}$ of length L with transition matrix P and stationary distribution π. The value of L depends stochastically on T, as indicated in the generation of the shot noise process below. Take $r = 3$, and let the probability transition matrix

$$P = \begin{bmatrix} 0.15 & 0.80 & 0.05 \\ 0.15 & 0.70 & 0.15 \\ 0.01 & 0.89 & 0.10 \end{bmatrix}$$

drive the Markov chain $\{\xi_n\}$. The stationary distribution is then given by

$$\pi = (0.1318 \ \ 0.7379 \ \ 0.1303).$$

Then, generate the chain-dependent process $\{A_n\}$ of length L, with chain-state distributions defined via $1 - H_i(x) = \kappa_i x^{-\alpha}$, $i = 1, 2, 3$. Take $\kappa_i = i$, and let α take the values 2.5 and 1.5 in order to compare the performance of the approximation under finite- and infinite-variance conditions, respectively.

Next, generate a renewal process $\{\tau_n\}$ of length L, with mean μ. Take a simple Poisson process with mean $\mu = 1$. Use $\{A_n\}$ and $\{\tau_n\}$ to generate the shot noise process $\{X(t)\}$ of length T, where $T \leq \tau_L$. Three impulse response functions $h(\cdot)$ are compared: $h(x) = 1$ for $0 \leq x \leq 1$ and 0 otherwise; $h(x) = 1 - x$ for $0 \leq x \leq 1$ and 0 otherwise; and finally, $h(x) = \exp(-x)$ where $x \geq 0$, just to test the assumptions made on $h(\cdot)$.

Finally, find the maximum of the shot noise process.

This procedure is followed 500 times in order to approximate the true underlying distribution. To complete the simulation study, the approximation is computed and then compared to the empirical cumulative distribution function (CDF) of the simulated process.

The results of this exercise are shown in Figure 1 (for $\alpha = 2.5$, or finite variance) and Figure 2 (for $\alpha = 1.5$, or infinite variance). Each figure is a table of graphs, with the cases indicated by the value of T across the top, and by the impulse response function $h(x)$ down the left-hand side. Each individual graph shows the approximation of the distribution of $\max_{0 \leq t \leq T} X(t)$ for that particular case (the solid line), along with 10 different realizations of the empirical CDF (the dotted lines) computed from simulations of the true underlying distribution of $\max_{0 \leq t \leq T} X(t)$ outlined above.

In both of Figure 1 and Figure 2, the approximation is doing well, and clearly better for $T = 100$ than for $T = 20$ (which is to be expected). In Figure 1, where the distributions $H_i(x)$, $1 \leq i \leq 3$, have finite variance, the approximation seems to do better when the impulse response function decreases rapidly. However, in Figure 2, where the distributions $H_i(x)$, $1 \leq i \leq 3$, have infinite variance, the approximation appears to do best when the impulse response function is constant. These are points worthy of further investigation in future research.

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Figure 1: Approximation Compared to 10 Simulated Distributions, $\alpha = 2.5$

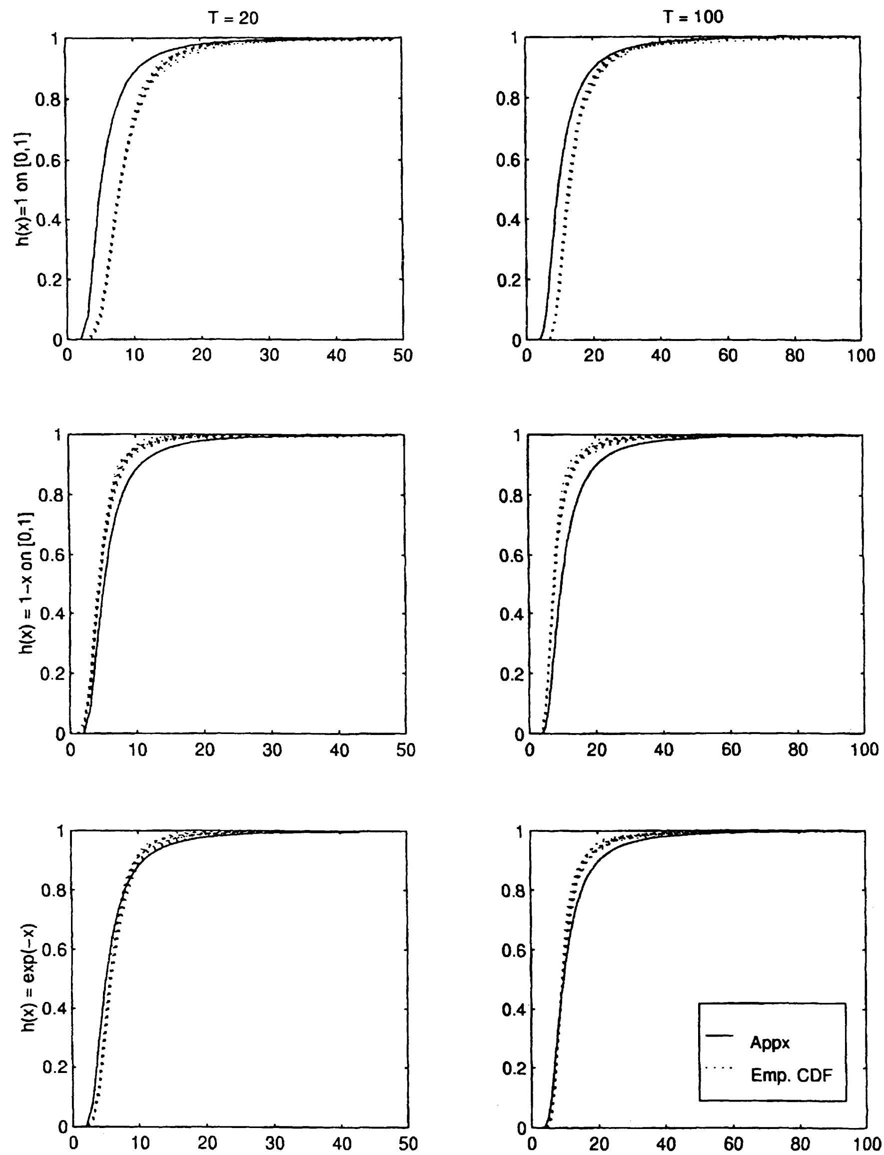

Figure 2: Approximation Compared to 10 Simulated Distributions, $\alpha = 1.5$